ON p-ADIC GENERALIZED WHITTAKER FUNCTIONS ON $\mathrm{GSp}(4)$

SATOSHI MORIMOTO

Mathematical Institute Tohoku University, Sendai, Miyagi 980–8578, Japan.
Email address: `sa4m27@math.tohoku.ac.jp`

Abstract. Let G be the general symplectic group of degree 4 defined over a non-Archimedean local field. We define p-adic generalized Whittaker functions for the Siegel parabolic subgroup of G, which naturally appear as the Euler factors of the Fourier coefficients of the Eisenstein series for the Borel subgroup of G. We will show an explicit formula and estimate its Euler product.

Key words: Whittaker functions; Eisenstein series
2000 MSC: 22E50; 11F30; 11F70

1. Introduction

Let G be a connected reductive algebraic group defined over a non-Archimedean local field F. The study of Whittaker functions on G has a long history. For example, if $G = \mathrm{GL}(2)$, an explicit formula for the Whittaker function is well-known (See [5]). Shintani proved an explicit formula for the case $G = \mathrm{GL}(n)$ and expressed it by Schur polynomials in [16]. Casselman and Shalika generalized Shintani's result to the case where G is split in [3]. Li further generalized it to the case where G is quasi-split in [8]. Reeder [14] etc. also studied p-adic Whittaker functions. However, the Whittaker functions we will consider may be a little different from theirs because our interest is the functions as the Fourier coefficients of the Eisenstein series described as follows.

In this paper, we mainly consider the local theory of Whittaker functions. However, we review the global theory of the Fourier expansion of the Eisenstein series for the Borel group of $G = \mathrm{GSp}(4)$ to describe our motivation of this study. Though the notion of the Fourier expansion has yet to be established for general reductive groups, it is known that there is such a notion for $G = \mathrm{GSp}(4)$. Let k be a number field and $\mathbb{A}$ its adele ring. We use the notation $G_k, G_{\mathbb{A}}$ for the group of k-rational points and the adelization of G respectively. Let B be the Borel subgroup of G whose unipotent radical N is generated by matrices in the form

$$n_1(u_1) = \begin{pmatrix} 1 & & & \\ u_1 & 1 & & \\ & & 1 & -u_1 \\ & & & 1 \end{pmatrix}, n_2(u_2, u_3, u_4) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ u_2 & u_3 & 1 & \\ u_3 & u_4 & & 1 \end{pmatrix}.$$

Let P be the Siegel parabolic subgroup of G and U the unipotent radical of P. Then U consists of matrices in the form $n_2(u_2, u_3, u_4)$. Let

$$S = \{z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \mid z_1 + z_3 = z_2 + z_4 = 0\}$$

and $E_B(g,z)$ be the Eisenstein series for the Borel subgroup B, where $g \in G_{\mathbb{A}}, z \in S$. For any character ψ_1 of $N_{\mathbb{A}}/N_k$, we consider the integral

$$\int_{N_{\mathbb{A}}/N_k} E_B(gn(u),z)\psi_1(n(u))du. \tag{1.1}$$

Moreover, for any character ψ_2 of $U_{\mathbb{A}}/U_k$, we consider the integral

$$\int_{U_{\mathbb{A}}/U_k} E_B\left(gn(u),z\right)\psi_2(n(u))du. \tag{1.2}$$

Then the function $E_B(g,z)$ can be expressed as follows:

$$\begin{aligned} E_B(g,z) =& \sum_{\psi_1} \sum_{\gamma_1 \in \Gamma_{\psi_1}} \int_{N_{\mathbb{A}}/N_k} E_B(g\gamma_1 n(u),z)\psi_1(n(u))du \\ &+ \sum_{\psi_2} \sum_{\gamma_2 \in \Gamma_{\psi_2}} \int_{U_{\mathbb{A}}/U_k} E_B(g\gamma_2 n(u),z)\psi_2(n(u))du, \end{aligned} \tag{1.3}$$

where $\Gamma_{\psi_1}, \Gamma_{\psi_2}$ are certain subsets of G_k. We call (1.3) the Fourier expansion of $E_B(g,z)$. The integrals (1.1), (1.2) can be expressed in terms of the global Whittaker functions, which have Euler products. We call the Euler factors the local Whittaker functions. The functions related to (1.1),(1.2) are called the global Whittaker functions for the Borel subgroup of G, the global generalized Whittaker functions for the Siegel parabolic subgroup of G respectively. The local generalized Whittaker functions for the Siegel parabolic subgroup of G are our main concern. The exact definition of p-adic generalized Whittaker functions will be given in Section 4.

We describe the outline of this paper. Let $\mathcal{O}$ be the integer ring of F, $\mathfrak{p}$ the maximal ideal of $\mathcal{O}$ and q the cardinality of the residue field $\mathcal{O}/\mathfrak{p}$. In Section 4, we will define the p-adic generalized Whittaker function $W(\alpha,z)$. This is a function of $\alpha = \left(\begin{smallmatrix} \alpha_2 & \alpha_3 \\ \alpha_3 & \alpha_4 \end{smallmatrix}\right) \in \mathrm{GL}(2)_F, z = (z_1,z_2,z_3,z_4) \in S$. Let $\overline{W}(\alpha,z)$ be the function defined by

$$W(\alpha,z) = \overline{W}(\alpha,z)(1+q^{-(z_1-z_3+1)})(1-q^{-(z_2-z_4+1)}).$$

The following four theorems are our main results.

Theorem 4.4. Unless $\alpha_2, 2\alpha_3, \alpha_4 \in \mathcal{O}$, $W(\alpha,z)=0$.

Theorem 4.5. Suppose that F is not dyadic. Then $W(\alpha,z)$ is computable.

Theorem 4.6. Suppose that F is not dyadic, $\alpha_2 \neq 0$, and $z \in S(0,0)$ (see Definition 4.1 for the definition). Then there exist constants $e_1, e_2 \geqq 0$ independent of F such that

$$\left|\overline{W}(\alpha,z)\right|_{\mathbb{C}} \leqq (\mathrm{ord}(\alpha_2)+1)^{e_1}(\mathrm{ord}(D_\alpha)+1)^{e_2}.$$

Theorem 8.3. Let $r_1, r_2, \varepsilon > 0$. Then there exists a constant $C(r_1,r_2,\varepsilon)$ such that if $\alpha_{2,\mathrm{f}} \in \mathbb{A}_{\mathrm{f}}^\times$ (the finite part of the idele group) and $z \in S(r_1,r_2)$, then

$$|\overline{W}_{\mathrm{f}}(\alpha_{\mathrm{f}},z)|_{\mathbb{C}} \leqq C(r_1,r_2,\varepsilon)\mathrm{N}(\alpha_2)^\varepsilon \mathrm{N}(D_\alpha)^\varepsilon,$$

where $\mathrm{N}(\alpha_2), \mathrm{N}(D_\alpha)$ are the ideal norms of α_2, D_α respectively.

We prove in Theorems 4.4, 4.5 that $W(\alpha, z)$ is computable. It does not mean that the value of $W(\alpha, z)$ cannot be actually obtained though it is possible to calculate theoretically. The proof of Theorem 4.5 shows the method of obtaining the value of $W(\alpha, z)$. However, since there are a lot of cases and we have not found an intrinsic interpretation of our result, we do not determine the value of $W(\alpha, z)$ for all cases. Next, we prove in Theorem 4.6 that $W(\alpha, z)$ is bounded by a power of divisor sums. Since we would like to use the bound to estimate the global generalized Whittaker function, it is desired that the bound does not depend on places. We prove in Theorem 8.3 that the product of these divisor sums in Theorem 4.6 is estimated by the ideal norm. Therefore, we expect that we can obtain a good estimate for the global generalized Whittaker function. We shall consider an application of the global generalized Whittaker function to the Eisenstein series for the Borel subgroup B in the near future.

The proofs of Theorems 4.4, 4.5 will be carried out in Sections 5, 6. First, we show that $W(\alpha, z) = 0$ unless $\alpha_2, 2\alpha_3, \alpha_4$ belong to $\mathcal{O}$ in Section 5. Second, we show that $W(\alpha, z)$ is computable for general α when F is not dyadic in Section 6. In the case $G = \mathrm{GL}(n)$, the orthogonality of characters was an essential tool. But in the case $G = \mathrm{GSp}(4)$, we must consider not only linear characters but also quadratic characters. To treat these quadratic characters without using Gauss sums, we use Hensel's lemma and try to reduce the consideration to that of linear characters. The proof of Theorem 4.6 will be carried out in Section 7. By the global consideration, it turns out to be enough to consider the case $\alpha_2 \in \mathbb{A}^\times$. The proof of Theorem 8.3 will be carried out in Section 8 by using Theorem 4.6. Since the number of the dyadic places is finite, we can use a trivial estimate of the p-adic Whittaker function at dyadic places. Notation regarding the global situation will also be introduced in this section. Finally, we provide some examples of the values of the p-adic generalized Whittaker function in Appendix A.

In this paper, we did not consider Whittaker functions at Archimedean places. However, they are also important. Some results on Whittaker functions on $\mathrm{Sp}(4, \mathbb{R})$ have been proved by Ishii [6], Miyazaki [9], Miyazaki-Oda [10], Moriyama [11], [12] and Oda [13].

2. Notation

In this section, we establish our basic notational conventions. More specialized notation will be introduced in the section where it is required.

The standard symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ will denote respectively the set of rational integers, rational numbers, real numbers and complex numbers. The set of positive integers is denoted by $\mathbb{N}$. If $a \in \mathbb{R}$, then the largest integer z such that $z \le a$ is denoted by $\lfloor a \rfloor$. The set of positive real numbers is denoted by $\mathbb{R}_+$. If R is any ring, then $R^\times$ is the set of invertible elements of R.

Let F be a finite extension of $\mathbb{Q}_p$. Then $\mathcal{O}$ denotes the ring of integers of F, $| \ |$ the normalized absolute value on F, π a uniformizer in $\mathcal{O}$, $\mathfrak{p}$ the maximal ideal of $\mathcal{O}$ and q the cardinality of $\mathcal{O}/\mathfrak{p}$. If $a \in F$ and $(a) = \mathfrak{p}^i$, then we write $\mathrm{ord}(a) = i$. Let $\langle \ \rangle$ be a character which is trivial on $\mathcal{O}$ and non-trivial on $\pi^{-1}\mathcal{O}$. We choose a Haar measure dx on F so that $\int_{\mathcal{O}} dx = 1$. For $z \in \mathbb{C}$, we denote Re (resp. Im) real (resp. imaginary) part of z. Let $|\ \ |_{\mathbb{C}}$ be the usual absolute value, i.e. $|z|_{\mathbb{C}} = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$ (not $\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2$!) for $z \in \mathbb{C}$.

3. Iwasawa decomposition of $\mathrm{GSp}(4)_F$

In this section, we review the Iwasawa decomposition of $\mathrm{GSp}(4)_F$ whose maximal compact subgroup is $\mathrm{GSp}(4)_{\mathcal{O}}$.

If R is any ring and $n \in \mathbb{N}$, then $M(n)_R$ is the ring of $n \times n$ matrices and $\mathrm{GL}(n)_R$ is the group of invertible elements of $M(n)_R$.

First, we establish basic notations regarding the group $\mathrm{GSp}(4)_F$.

Definition 3.1. Let I_n be the unit matrix of $\mathrm{GL}(n)_R$, O_n the zero matrix of $\mathrm{GL}(n)_R$, and $J = \begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix}$. We define

$$\mathrm{GSp}(4)_R = \{X \in M(4)_R \,|\, \exists \beta \in \mathrm{GL}(1)_R, {}^tXJX = \beta J\}.$$

This group is called the general symplectic group of degree 4.

Next, we define subgroups of $\mathrm{GSp}(4)_F$ which are needed to describe an Iwasawa decomposition of $\mathrm{GSp}(4)_F$.

Definition 3.2. We define

$$T = \left\{ \begin{pmatrix} t_3t_1 & & & \\ & t_3t_2 & & \\ & & t_1^{-1} & \\ & & & t_2^{-1} \end{pmatrix} \middle| \, t_i \in F^\times \right\}, N_1 = \left\{ \begin{pmatrix} 1 & & & \\ & 1 & & \\ u_2 & u_3 & 1 & \\ u_3 & u_4 & & 1 \end{pmatrix} \middle| \, u_i \in F \right\},$$

$$N_2 = \left\{ \begin{pmatrix} 1 & & & \\ u_1 & 1 & & \\ & & 1 & -u_1 \\ & & & 1 \end{pmatrix} \middle| \, u_1 \in F \right\}, \ N = N_1 \cdot N_2, K = \mathrm{GSp}(4)_{\mathcal{O}}.$$

The subgroup T is a maximal torus, $B = TN$ is a Borel subgroup, and N is the unipotent radical of B.

The following theorem is well-known and describes the Iwasawa decomposition and the Cartan decomposition of $\mathrm{GSp}(4)_F$.

Theorem 3.3. *The subgroup K is a maximal compact subgroup of* $\mathrm{GSp}(4)_F$*. The group* $\mathrm{GSp}(4)_F$ *has the following decompositions*

$$\mathrm{GSp}(4)_F = KTK = KTN.$$

Proof. The proof of this theorem is almost the same as in [15, Sec. 9]. Note that the only difference between [15] and our case is the choice of a skew-symmetric matrix J and a Borel subgroup B. In [15], $J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$ and B is the set of upper triangular matrices. □

4. The p-adic generalized Whittaker function

In this section, we define the p-adic generalized Whittaker function and state our main theorems. The proofs of these theorems will be carried out in the next three sections.

By Theorem 3.3, any element $g \in \mathrm{GSp}(4)_F$ can be written as $h(g)t(g)n(u(g))$ where $h(g) \in K, t(g) \in T$, and $n(u(g)) \in N$. Since we only consider elements whose diagonal part of the Iwasawa decomposition belongs to $T \cap \mathrm{Sp}(4)_F$, we use the following notation.

Definition 4.1. We use the notation

$$t = \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_1^{-1} & \\ & & & t_2^{-1} \end{pmatrix} \quad \text{for } t_1, t_2 \in F^\times.$$

Let $t^z = |t_1|^{z_1 - z_3}|t_2|^{z_2 - z_4}$ for $t \in T \cap \mathrm{Sp}(4), z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4$. We consider the sets

$$S = \{z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \,|\, z_1 + z_3 = z_2 + z_4 = 0\},$$
$$S(r_1, r_2) = \{z = (z_1, z_2, z_3, z_4) \in S \,|\, \mathrm{Re}(z_1 - z_3) > r_1, \mathrm{Re}(z_2 - z_4) > r_2\}.$$

Let ρ be half the sum of the weights of N with respect to conjugations by elements of T. We can regard ρ as an element of S. By an easy computation, $t^\rho = |t_1|^2|t_2|$. For $u = \left(\begin{smallmatrix} u_2 & u_3 \\ u_3 & u_4 \end{smallmatrix}\right) \in M(2)_F$, let $n(u), n'(u), n''(u)$ be the following matrices

$$n(u) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ u_2 & u_3 & 1 & \\ u_3 & u_4 & & 1 \end{pmatrix}, n'(u) = \begin{pmatrix} 1 & & & u_3 \\ & 1 & u_3 & u_2 \\ & & 1 & \\ & & & 1 \end{pmatrix}, n''(u) = \begin{pmatrix} 1 & & & \\ & 1 & & u_2 \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

and let

$$\tau = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}.$$

Next, we define the p-adic generalized Whittaker function.

Definition 4.2. For $\alpha = \left(\begin{smallmatrix} \alpha_2 & \alpha_3 \\ \alpha_3 & \alpha_4 \end{smallmatrix}\right) \in GL(2)_F, z = (z_1, z_2, z_3, z_4) \in S$, we define

$$W(\alpha, z) = \int_{F \times F \times F} t(n(u)\tau)^{z+\rho} \langle \mathrm{tr}(\alpha u) \rangle du_2 du_3 du_4. \tag{4.3}$$

We call this function the p-adic generalized Whittaker function for the Siegel parabolic subgroup of GSp(4).

We define a function $\overline{W}(\alpha, z)$ of α, z by

$$W(\alpha, z) = \overline{W}(\alpha, z)(1 + q^{-(z_1 - z_3 + 1)})(1 - q^{-(z_2 - z_4 + 1)}).$$

The following theorems are our main results.

Theorem 4.4. *Unless* $\alpha_2, 2\alpha_3, \alpha_4 \in \mathcal{O}$, $W(\alpha, z) = 0$.

Theorem 4.5. *Suppose that F is not dyadic. Then $W(\alpha, z)$ is computable.*

Theorem 4.6. *Suppose that F is not dyadic, $\alpha_2 \neq 0$, and $z \in S(0,0)$. Then there exist constants $e_1, e_2 \geqq 0$ independent of F such that*

$$\left|\overline{W}(\alpha, z)\right|_{\mathbb{C}} \leqq (\mathrm{ord}(\alpha_2)+1)^{e_1}(\mathrm{ord}(D_\alpha)+1)^{e_2}.$$

For the rest of this section, we will prove an integral formula for $W(\alpha, z)$ in Lemma 4.9. The remainder of the proofs will be carried out in Sections 5 to 7. To state Lemma 4.9, we use the Iwasawa decomposition of the element $\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$ in $\mathrm{GL}(2)_{\mathrm{F}}$. We need the following definition for that purpose.

Definition 4.7. For $u \in F$, we define

$$c(u) = \begin{cases} 1 & \text{if } u \in \mathcal{O}, \\ u^{-1} & \text{if } u \notin \mathcal{O}, \end{cases} \qquad s(u) = \begin{cases} 0 & \text{if } u \in \mathcal{O}, \\ 1 & \text{if } u \notin \mathcal{O}. \end{cases}$$

Then it is easy to see the following equation:

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-s(u) & c(u)u \\ -s(u) & c(u) \end{pmatrix} \begin{pmatrix} c(u) & 0 \\ 0 & c(u)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c(u)s(u) & 1 \end{pmatrix}, \tag{4.8}$$

which is the Iwasawa decomposition of $\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$.

Lemma 4.9. *The function $W(\alpha, z)$ has the following integral formula:*

$$\begin{aligned} W(\alpha, z) = \int_{F\times F\times F} & |c(u_2)|^{z_2-z_4+1}|c(u_3)|^{z_1+z_2-z_3-z_4+1}|c(u_4)|^{z_1-z_3+1} \\ & \times \langle \alpha_2 c(u_3)^{-2}u_2 + \alpha_2 s(u_4)c(u_4)^{-1}u_3^2 + 2\alpha_3 c(u_4)^{-1}u_3 + \alpha_4 u_4\rangle \\ & \times du_2du_3du_4. \end{aligned} \tag{4.10}$$

Proof. It is easy to see that $t(gt) = t(g)t$ for $g \in \mathrm{GSp}(4)_F, t \in T$. Note that $\tau \in K$. By multiplying τ from the left, we can bring variables u_2, u_3, u_4 to the upper right side as follows:

$$n(u)\tau = \tau\tau n(u)\tau = \tau \begin{pmatrix} 1 & & u_4 & u_3 \\ & 1 & u_3 & u_2 \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

We consider the Iwasawa decomposition of this element. By (4.8),

$$\begin{aligned} \begin{pmatrix} 1 & & u_4 & u_3 \\ & 1 & u_3 & u_2 \\ & & 1 & \\ & & & 1 \end{pmatrix} &= \begin{pmatrix} 1 & & u_4 & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & u_3 \\ & 1 & u_3 & u_2 \\ & & 1 & \\ & & & 1 \end{pmatrix} \\ &= h_1 \begin{pmatrix} c(u_4) & & & \\ & 1 & & \\ & & c(u_4)^{-1} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ c(u_4)s(u_4) & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & u_3 \\ & 1 & u_3 & u_2 \\ & & 1 & \\ & & & 1 \end{pmatrix}, \end{aligned}$$

where $h_1 \in K$. By easy computations,

$$
\begin{aligned}
&\begin{pmatrix} 1 & & & \\ & 1 & & \\ c(u_4)s(u_4) & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & u_3 \\ & 1 & u_3 & u_2 \\ & & 1 & \\ & & & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & & & u_3 \\ & 1 & u_3 & u_2 \\ c(u_4)s(u_4) & & 1 & c(u_4)s(u_4)u_3 \\ & & & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & & & u_3 \\ & 1 & u_3 & u_2 \\ c(u_4)s(u_4) & & 1 & c(u_4)s(u_4)u_3 \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ c(u_4)s(u_4)u_3 & 1 & & \\ & & 1 & -c(u_4)s(u_4)u_3 \\ & & & 1 \end{pmatrix} n_1 \\
&= \begin{pmatrix} 1 & & & u_3 \\ c(u_4)s(u_4)u_3 & 1 & u_3 & u_2 - c(u_4)s(u_4)u_3^2 \\ c(u_4)s(u_4) & & 1 & \\ & & & 1 \end{pmatrix} n_1 \\
&= \begin{pmatrix} 1 & & & u_3 \\ c(u_4)s(u_4)u_3 & 1 & u_3 & u_2 - c(u_4)s(u_4)u_3^2 \\ c(u_4)s(u_4) & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ -c(u_4)s(u_4) & & 1 & \\ & & & 1 \end{pmatrix} n_2 \\
&= \begin{pmatrix} 1 & & & u_3 \\ & 1 & u_3 & u_2 - c(u_4)s(u_4)u_3^2 \\ & & 1 & \\ & & & 1 \end{pmatrix} n_2,
\end{aligned}
$$

where $n_1, n_2 \in N$. Therefore

$$
\begin{aligned}
n(u)\tau &= h_2 \begin{pmatrix} c(u_4) & & & \\ & 1 & & \\ & & c(u_4)^{-1} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & u_3 \\ & 1 & u_3 & u_2 - c(u_4)s(u_4)u_3^2 \\ & & 1 & \\ & & & 1 \end{pmatrix} n_2 \\
&= h_2 \begin{pmatrix} 1 & & & c(u_4)u_3 \\ & 1 & c(u_4)u_3 & u_2 - c(u_4)s(u_4)u_3^2 \\ & & 1 & \\ & & & 1 \end{pmatrix} n_3 \begin{pmatrix} c(u_4) & & & \\ & 1 & & \\ & & c(u_4)^{-1} & \\ & & & 1 \end{pmatrix},
\end{aligned}
$$

where $h_2 \in K, n_3 \in N$. Let $u_2' = u_2 - c(u_4)s(u_4)u_3^2, u_3' = c(u_4)u_3$. Then $du_2du_3du_4 = |c(u_4)|^{-1}du_2'du_3'du_4$. By this change of variables, (4.3) is equal to

$$
\begin{aligned}
&\int_{F\times F\times F} |c(u_4)|^{z_1-z_3+1} t(n'(u))^{z+\rho} \\
&\times \langle \alpha_2 u_2 + \alpha_2 s(u_4)c(u_4)^{-1}u_3^2 + 2\alpha_3 c(u_4)^{-1}u_3 + \alpha_4 u_4 \rangle du_2du_3du_4.
\end{aligned}
\tag{4.11}
$$

Next we consider the Iwasawa decomposition of $n'(u)$. If $u_3 \notin \mathcal{O}$, then $|u_3^{-1}| \leqq 1$. So

$$\begin{aligned}
\begin{pmatrix} 1 & & & u_3 \\ & 1 & u_3 & u_2 \\ & & 1 & \\ & & & 1 \end{pmatrix} &= h_3 \begin{pmatrix} u_3^{-1} & & & -1 \\ & u_3^{-1} & -1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \begin{pmatrix} 1 & & & u_3 \\ & 1 & u_3 & u_2 \\ & & 1 & \\ & & & 1 \end{pmatrix} \\
&= h_3 \begin{pmatrix} u_3^{-1} & & & \\ & u_3^{-1} & & u_2u_3^{-1} \\ & 1 & u_3 & u_2 \\ 1 & & & u_3 \end{pmatrix} \\
&= h_3 \begin{pmatrix} u_3^{-1} & & & \\ & u_3^{-1} & & \\ & & u_3 & \\ & & & u_3 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & u_2 \\ & u_3^{-1} & 1 & u_2u_3^{-1} \\ u_3^{-1} & & & 1 \end{pmatrix},
\end{aligned}$$

where $h_3 \in K$. By easy computations,

$$\begin{aligned}
\begin{pmatrix} 1 & & & \\ & 1 & & u_2 \\ & u_3^{-1} & 1 & u_2u_3^{-1} \\ u_3^{-1} & & & 1 \end{pmatrix} &= \begin{pmatrix} 1 & & & \\ & 1 & & u_2 \\ & u_3^{-1} & 1 & u_2u_3^{-1} \\ u_3^{-1} & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ u_2u_3^{-1} & 1 & & \\ & & 1 & -u_2u_3^{-1} \\ & & & 1 \end{pmatrix} n_4 \\
&= \begin{pmatrix} 1 & & & \\ u_2u_3^{-1} & 1 & & u_2 \\ u_2u_3^{-2} & u_3^{-1} & 1 & \\ u_3^{-1} & & & 1 \end{pmatrix} n_4 \\
&= \begin{pmatrix} 1 & & & \\ u_2u_3^{-1} & 1 & & u_2 \\ u_2u_3^{-2} & u_3^{-1} & 1 & \\ u_3^{-1} & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ -u_2u_3^{-2} & -u_3^{-1} & 1 & \\ -u_3^{-1} & & & 1 \end{pmatrix} n_5 \\
&= \begin{pmatrix} 1 & & & \\ & 1 & & u_2 \\ & & 1 & \\ & & & 1 \end{pmatrix} n_5,
\end{aligned}$$

where $n_4, n_5 \in N$. Therefore

$$\begin{pmatrix} 1 & & & u_3 \\ & 1 & u_3 & u_2 \\ & & 1 & \\ & & & 1 \end{pmatrix} = h_3 \begin{pmatrix} u_3^{-1} & & & \\ & u_3^{-1} & & \\ & & u_3 & \\ & & & u_3 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & u_2 \\ & & 1 & \\ & & & 1 \end{pmatrix} n_5. \tag{4.12}$$

If $u_3 \in \mathcal{O}$, then

$$\begin{pmatrix} 1 & & & u_3 \\ & 1 & u_3 & u_2 \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & u_3 \\ & 1 & u_3 & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & u_2 \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \tag{4.13}$$

where

$$\begin{pmatrix} 1 & & & u_3 \\ & 1 & u_3 & \\ & & 1 & \\ & & & 1 \end{pmatrix} \in K.$$

By (4.12), (4.13), we can express $n'(u)$ as follows:

$$\begin{aligned}\begin{pmatrix} 1 & & & u_3 \\ & 1 & u_3 & u_2 \\ & & 1 & \\ & & & 1 \end{pmatrix} &= h_4 \begin{pmatrix} c(u_3) & & & \\ & c(u_3) & & \\ & & c(u_3)^{-1} & \\ & & & c(u_3)^{-1} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & u_2 \\ & & 1 & \\ & & & 1 \end{pmatrix} n_6 \\ &= h_4 \begin{pmatrix} 1 & & & \\ & 1 & & c(u_3)^2 u_2 \\ & & 1 & \\ & & & 1 \end{pmatrix} n_7 \begin{pmatrix} c(u_3) & & & \\ & c(u_3) & & \\ & & c(u_3)^{-1} & \\ & & & c(u_3)^{-1} \end{pmatrix},\end{aligned}$$

where $h_4 \in K, n_6, n_7 \in N$. By replacing $c(u_3)^2 u_2$ by u_2, (4.11) is equal to

$$\begin{aligned}&\int_{F\times F\times F} |c(u_4)|^{z_1-z_3+1} |c(u_3)|^{z_1+z_2-z_3-z_4+1} t(n''(u))^{z+\rho} \\ &\times \langle \alpha_2 c(u_3)^{-2} u_2 + \alpha_2 s(u_4) c(u_4)^{-1} u_3^2 + 2\alpha_3 c(u_4)^{-1} u_3 + \alpha_4 u_4 \rangle du_2 du_3 du_4.\end{aligned} \tag{4.14}$$

By (4.8),

$$\begin{aligned}\begin{pmatrix} 1 & & & \\ & 1 & & u_2 \\ & & 1 & \\ & & & 1 \end{pmatrix} &= h_5 \begin{pmatrix} 1 & & & \\ & c(u_2) & & \\ & & 1 & \\ & & & c(u_2)^{-1} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & c(u_2)s(u_2) & & 1 \end{pmatrix} \\ &= h_5 n_8 \begin{pmatrix} 1 & & & \\ & c(u_2) & & \\ & & 1 & \\ & & & c(u_2)^{-1} \end{pmatrix},\end{aligned}$$

where $h_5 \in K, n_8 \in N$. So $t(n''(u))^{z+\rho} = |c(u_2)|^{z_2-z_4+1}$ and (4.14) is equal to (4.10). □

5. Proof of Theorem 4.4

In this section, we prove Theorem 4.4. Throughout this section, we assume that either $\alpha_2, 2\alpha_3$ or α_4 does not belong to $\mathcal{O}$. We shall show that $W(\alpha, z) = 0$ under this assumption. Roughly speaking, the proof of Theorem 4.4 is based on the fact that integrals of non-trivial characters on compact groups are 0. The following lemma is well-known.

Lemma 5.1.

$$\int_{\pi^i\mathcal{O}} \langle \pi^j u\rangle du = \begin{cases} 0 & \textit{if } i+j<0, \\ q^{-i} & \textit{if } i+j \geqq 0, \end{cases} \tag{5.2}$$

$$\int_{\pi^i\mathcal{O}^\times} \langle \pi^j u\rangle du = \begin{cases} 0 & \textit{if } i+j+1<0, \\ -q^{-(i+1)} & \textit{if } i+j+1=0, \\ q^{-i}(1-q^{-1}) & \textit{if } i+j+1>0. \end{cases} \tag{5.3}$$

Proof. See [18, Lem. (2.3.3), p. 42]. □

By considering the integral with respect to u_2 in (4.10), the p-adic Whittaker function on GL(2) appears in $W(\alpha, z)$. So we recall known facts on the p-adic Whittaker function on GL(2).

Definition 5.4. For $\alpha \in F^\times, z \in \mathbb{C}$, we define

$$W_2(\alpha, z) = \int_F |c(u)|^{z+1} \langle \alpha u\rangle du.$$

This function is called the p-adic Whittaker function on GL(2).

Lemma 5.5. *Let $\alpha \in \pi^c\mathcal{O}^\times, z \in \mathbb{C}$. Then*

$$W_2(\alpha, z) = \begin{cases} 0 & \textit{if } c<0, \\ \dfrac{1-q^{-(z+1)}}{1-q^{-z}}(1-q^{-(c+1)z}) & \textit{if } c \geqq 0. \end{cases} \tag{5.6}$$

Proof. See [18, Prop. (2.3.4), p. 42]. □

First we consider the integral with respect to u_2 in (4.10).

Lemma 5.7. *If $\alpha_2 \notin \mathcal{O}$, then $W(\alpha, z) = 0$.*

Proof. Since $c(u_3) \in \mathcal{O}$, $|c(u_3)^{-2}| \geqq 1$. So this lemma follows from Lemma 5.5. □

So we assume that $2\alpha_3 \notin \mathcal{O}$ or $\alpha_4 \notin \mathcal{O}$. We divide the limit of the integral in (4.10) as follows:

$$\begin{aligned} W(\alpha, z) = &\int_{F\times F\times\mathcal{O}} |c(u_2)|^{z_2-z_4+1} |c(u_3)|^{z_1+z_2-z_3-z_4+1} \\ &\times \langle \alpha_2 c(u_3)^{-2} u_2 + 2\alpha_3 u_3 + \alpha_4 u_4\rangle du_2 du_3 du_4 \\ &+ \sum_{n=1}^{\infty} q^{-n(z_1-z_3+1)} \int_{F\times\mathcal{O}\times\pi^{-n}\mathcal{O}^\times} |c(u_2)|^{z_2-z_4+1} \\ &\qquad \times \langle \alpha_2 u_2 + (\alpha_2 u_3^2 + 2\alpha_3 u_3 + \alpha_4) u_4\rangle du_2 du_3 du_4 \\ &+ \sum_{m,n=1}^{\infty} q^{-m(z_1+z_2-z_3-z_4+1)-n(z_1-z_3+1)} \int_{F\times\pi^{-m}\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times} |c(u_2)|^{z_2-z_4+1} \\ &\qquad \times \langle \alpha_2 u_3^2 u_2 + (\alpha_2 u_3^2 + 2\alpha_3 u_3 + \alpha_4) u_4\rangle du_2 du_3 du_4. \end{aligned} \tag{5.8}$$

To prove Theorem 4.4, we show that each term of (5.8) is equal to 0.

Lemma 5.9. *If $2\alpha_3 \notin \mathcal{O}$ or $\alpha_4 \notin \mathcal{O}$, then the first term of (5.8) is equal to 0.*

Proof. If $\alpha_4 \notin \mathcal{O}$, then $\int_{\mathcal{O}} \langle \alpha_4 u_4 \rangle du_4 = 0$ by Lemma 5.1. By Lemma 5.7,

$$\begin{aligned}
&\int_{F\times\pi^{-m}\mathcal{O}^\times} |c(u_2)|^{z_2-z_4+1}|c(u_3)|^{z_1+z_2-z_3-z_4+1}\langle \alpha_2 c(u_3)^{-2}u_2 + 2\alpha_3 u_3\rangle du_2 du_3 \\
&= \int_F |c(u_2)|^{z_2-z_4+1}\langle \alpha_2\pi^{-2m}u_2\rangle du_2 \int_{\pi^{-m}\mathcal{O}^\times} |c(u_3)|^{z_1+z_2-z_3-z_4+1}\langle 2\alpha_3 u_3\rangle du_3
\end{aligned}$$

for $m \geqq 1$ and

$$\begin{aligned}
&\int_{F\times\mathcal{O}} |c(u_2)|^{z_2-z_4+1}|c(u_3)|^{z_1+z_2-z_3-z_4+1}\langle \alpha_2 c(u_3)^{-2}u_2 + 2\alpha_3 u_3\rangle du_2 du_3 \\
&= \int_F |c(u_2)|^{z_2-z_4+1}\langle \alpha_2 u_2\rangle du_2 \int_{\mathcal{O}} \langle 2\alpha_3 u_3\rangle du_3.
\end{aligned}$$

If $2\alpha_3 \notin \mathcal{O}$, these values are 0 by applying Lemma 5.1 to the integral with respect to u_3. Therefore the lemma follows. □

If we try to compute $W(\alpha, z)$, we have to consider integrals of the form

$$\int_F \langle au^2 + bu + c\rangle du.$$

We prove the following lemma for that purpose.

Lemma 5.10. *Suppose that* $|a| < |b|, |c| \leqq |b|$. *Then the map*

$$\varphi : \mathcal{O} \ni v \to \frac{1}{b}\left(av^2 + bv + c\right) \in \mathcal{O}$$

is bijective. If $|c| < |b|$, $\varphi|_{\mathcal{O}^\times} : \mathcal{O}^\times \to \mathcal{O}^\times$ *is bijective, too.*

Proof. Since $|v| \leqq 1$, φ is well-defined. If $\varphi(v_1) = \varphi(v_2)$, then $(v_1 - v_2)\left(a(v_1+v_2)+b\right) = 0$. Since $|a(v_1+v_2)| < |b|$, $v_1 = v_2$ follows. So φ is injective.

For any $w \in \mathcal{O}$, let

$$f(X) = b^{-1}(aX^2 + bX + c) - w \in \mathcal{O}[X], \quad \beta = -\frac{c}{b} + w \in \mathcal{O}.$$

Since

$$f(\beta) = b^{-1}a\beta^2 \in \mathfrak{p}, \quad f'(\beta) = b^{-1}(2a\beta + b) \in \mathcal{O}^\times,$$

there exists $v \in \mathcal{O}$ such that $f(v) = 0$, i.e. $\varphi(v) = w$ by Hensel's lemma. So φ is surjective. We can show that $\varphi|_{\mathcal{O}^\times}$ is bijective by an argument similar to the case $|c| < |b|$. □

Lemma 5.11. *If* $2\alpha_3 \notin \mathcal{O}$ *or* $\alpha_4 \notin \mathcal{O}$, *then*

$$\int_{\mathcal{O}\times\pi^{-n}\mathcal{O}^\times} \langle (\alpha_2 u_3^2 + 2\alpha_3 u_3 + \alpha_4)u_4\rangle du_3 du_4 = 0$$

for $n \geqq 1$. *Therefore the second term of (5.8) is equal to* 0.

Proof. If $\mathrm{ord}(\alpha_4) < \mathrm{ord}(2\alpha_3)$, then $|\alpha_2 u_3^2 + 2\alpha_3 u_3 + \alpha_4| = |\alpha_4| > 1$ for $u_3 \in \mathcal{O}$. By Lemma 5.1,

$$\int_{\mathcal{O}\times\pi^{-n}\mathcal{O}^\times} \langle (\alpha_2 u_3^2 + 2\alpha_3 u_3 + \alpha_4)u_4\rangle du_3 du_4 = \int_{\mathcal{O}} du_3 \int_{\pi^{-n}\mathcal{O}^\times} \langle \alpha_4 u_4\rangle du_4 = 0.$$

We assume that $\mathrm{ord}(\alpha_4) \geqq \mathrm{ord}(2\alpha_3)$. Then $2\alpha_3 \notin \mathcal{O}$ by assumption. By Lemmas 5.1 and 5.10,

$$\int_{\mathcal{O}\times\pi^{-n}\mathcal{O}^\times} \langle(\alpha_2 u_3^2 + 2\alpha_3 u_3 + \alpha_4)u_4\rangle du_3 du_4 = \int_{\mathcal{O}\times\pi^{-n}\mathcal{O}^\times} \langle 2\alpha_3 u_3 u_4\rangle du_3 du_4 = 0.$$

□

Next we consider the third term of (5.8). It is equal to

$$\sum_{m,n=1}^{\infty} q^{-m(z_1+z_2-z_3-z_4+1)-n(z_1-z_3+1)} \int_F |c(u_2)|^{z_2-z_4+1} \langle \alpha_2 \pi^{-2m} u_2\rangle du_2$$
$$\times \int_{\pi^{-m}\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times} \langle(\alpha_2 u_3^2 + 2\alpha_3 u_3 + \alpha_4)u_4\rangle du_3 du_4.$$

If $\alpha_2\pi^{-2m} \notin \mathcal{O}$, then the third term of (5.8) is equal to 0 by considering the integral with respect to u_2. So we can assume that $\alpha_2\pi^{-2m} \in \mathcal{O}$.

Lemma 5.12. *Suppose that $\alpha_2\pi^{-2m} \in \mathcal{O}$. If $2\alpha_3 \notin \mathcal{O}$ or $\alpha_4 \notin \mathcal{O}$, then the integral*

$$\int_{\pi^{-m}\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times} \langle(\alpha_2 u_3^2 + 2\alpha_3 u_3 + \alpha_4)u_4\rangle du_3 du_4 \tag{5.13}$$

is equal to 0 *for $m, n \geqq 1$. Therefore the third term of (5.8) is equal to* 0.

Proof. Note that $\alpha_4 \notin \mathcal{O}$ or $2\alpha_3 u_3 \notin \mathcal{O}$. If the orders of $\alpha_2\pi^{-2m}, 2\alpha_3\pi^{-m}, \alpha_4$ are distinct, (5.13) is equal to 0 by considering the integral with respect to u_4. So we only consider the following cases:

(1) $|\alpha_2\pi^{-2m}| = |2\alpha_3\pi^{-m}|, m > 0$,
(2) $|\alpha_2\pi^{-2m}| = |\alpha_4|, m > 0$,
(3) $|2\alpha_3\pi^{-m}| = |\alpha_4|, m > 0$.

First we consider the case (1). By assumption of the case (1), $\alpha_2\pi^{-2m}, 2\alpha_3\pi^{-m} \in \mathcal{O}, \alpha_4 \notin \mathcal{O}$. So

$$\int_{\pi^{-m}\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times} \langle(\alpha_2 u_3^2 + 2\alpha_3 u_3 + \alpha_4)u_4\rangle du_3 du_4 = \int_{\pi^{-m}\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times} \langle \alpha_4 u_4\rangle du_3 du_4 = 0$$

by Lemma 5.1.

Next we consider the case (2). By assumption of the case (2), $\alpha_2\pi^{-2m}, \alpha_4 \in \mathcal{O}, 2\alpha_3\pi^{-m} \notin \mathcal{O}$. So

$$\int_{\pi^{-m}\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times} \langle(\alpha_2 u_3^2 + 2\alpha_3 u_3 + \alpha_4)u_4\rangle du_3 du_4 = \int_{\pi^{-m}\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times} \langle 2\alpha_3 u_3 u_4\rangle du_3 du_4 = 0$$

by Lemmas 5.1 and 5.10.

Finally, we consider the case (3). Since $\mathrm{ord}(\alpha_4) = \mathrm{ord}(2\alpha_3\pi^{-m}) = \mathrm{ord}(2\alpha_3 u_3)$, $\alpha_4 \notin \mathcal{O}$ in this case. We choose $x_i \in \mathcal{O}^\times (i = 1, \ldots, q-1)$ so that

$$\mathcal{O}^\times = \coprod_i (x_i + \mathfrak{p}) \quad (x_i \in \mathcal{O}^\times). \tag{5.14}$$

By (5.14) and Lemma 5.10,

$$\begin{aligned}
&\int_{\pi^{-m}\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times}\langle(\alpha_2u_3^2+2\alpha_3u_3+\alpha_4)u_4\rangle du_3du_4\\
&=q^m\int_{\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times}\langle(\alpha_2\pi^{-2m}u_3^2+2\alpha_3\pi^{-m}u_3+\alpha_4)u_4\rangle du_3du_4\\
&=q^{m-1}\sum_i\int_{\mathcal{O}\times\pi^{-n}\mathcal{O}^\times}\langle(\alpha_2\pi^{-2m}(x_i+\pi u_3)^2+2\alpha_3\pi^{-m}(x_i+\pi u_3)+\alpha_4)u_4\rangle du_3du_4\\
&=q^{m-1}\sum_i\int_{\mathcal{O}\times\pi^{-n}\mathcal{O}^\times}\langle\left(\alpha_2x_i^2\pi^{-2m}+2\alpha_3x_i\pi^{-m}+\alpha_4\right)u_4\\
&\qquad+\pi\left(\alpha_2\pi^{-2m+1}u_3^2+2\left(\alpha_2x_i\pi^{-2m}+\alpha_3\pi^{-m}\right)u_3\right)u_4\rangle du_3du_4\\
&=q^{m-1}\sum_i\int_{\mathcal{O}\times\pi^{-n}\mathcal{O}^\times}\langle\left(\alpha_2x_i^2\pi^{-2m}+2\alpha_3x_i\pi^{-m}+\alpha_4\right)u_4\\
&\qquad+2\pi\left(\alpha_2x_i\pi^{-2m}+\alpha_3\pi^{-m}\right)\tilde{u_3}u_4\rangle d\tilde{u_3}du_4,
\end{aligned}$$

where we applied Lemma 5.10 for the case $c=0$. For any $u_4\in\pi^{-n}\mathcal{O}^\times$,

$$\int_{\mathcal{O}}\langle 2\pi\left(\alpha_2x_i\pi^{-2m}+\alpha_3\pi^{-m}\right)\tilde{u_3}u_4\rangle d\tilde{u_3}=\int_{\mathcal{O}}\langle 2\alpha_3\pi^{-m+1}\tilde{u_3}u_4\rangle d\tilde{u_3}=0$$

by Lemma 5.1. So the lemma follows. □

Proof of Theorem 4.4. If $\alpha_2\notin\mathcal{O}$, then $W(\alpha,z)=0$ by Lemma 5.7. Otherwise, $2\alpha_3\notin\mathcal{O}$ or $\alpha_4\notin\mathcal{O}$. Then each term of (5.8) is equal to 0 by Lemmas 5.9, 5.11 and 5.12. This completes the proof of Theorem 4.4. □

6. Proof of Theorem 4.5

In this section, we prove that $W(\alpha,z)$ is computable. In Sections 6, 7, we assume that F is not dyadic and $\alpha_i\in\mathcal{O}$ for $i=2,3,4$. We define $D_\alpha=\alpha_3^2-\alpha_2\alpha_4$. For convenience, we use the following notation:

$$a=q^{-(z_1-z_3)},b=q^{-(z_2-z_4)},T=q^{-1},c_i=\mathrm{ord}(\alpha_i),c_\alpha=\mathrm{ord}(D_\alpha). \tag{6.1}$$

First, we show that $W(\alpha,z)$ is invariant by the action of lower triangular matrices of $\mathrm{GL}(2)_{\mathcal{O}}$ whose diagonal parts are 1.

Lemma 6.2. *Let $g=\left(\begin{smallmatrix}1&0\\x&1\end{smallmatrix}\right)\in GL(2)_{\mathcal{O}}$ and $\beta=g\alpha{}^tg$. Then $W(\alpha,z)=W(\beta,z)$.*

Proof. Let $\nu=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, $v={}^tg^{-1}ug^{-1}$. By easy computations,

$$\begin{aligned}
&\mathrm{tr}(\alpha u)=\mathrm{tr}\left(g^{-1}\beta\,{}^tg^{-1}u\right)=\mathrm{tr}(\beta v),\\
&\begin{pmatrix}1&\\u&1\end{pmatrix}\begin{pmatrix}&\nu\\\nu&\end{pmatrix}=\begin{pmatrix}g^{-1}&\\&{}^tg\end{pmatrix}\begin{pmatrix}1&\\v&1\end{pmatrix}\begin{pmatrix}&\nu\\\nu&\end{pmatrix}\begin{pmatrix}\nu\,{}^tg^{-1}\nu&\\&\nu g\nu\end{pmatrix},\\
&\begin{pmatrix}g^{-1}&\\&{}^tg\end{pmatrix}\in K,\qquad\begin{pmatrix}\nu\,{}^tg^{-1}\nu&\\&\nu g\nu\end{pmatrix}\in N,\\
&du_2du_3du_4=dv_2dv_3dv_4.
\end{aligned}$$

So

$$\begin{aligned} W(\alpha, z) &= \int_{F\times F\times F} t\left(\begin{pmatrix} 1 & \\ u & 1 \end{pmatrix}\begin{pmatrix} & \nu \\ \nu & \end{pmatrix}\right)^{z+\rho} \langle \mathrm{tr}(\alpha u)\rangle du_2 du_3 du_4 \\ &= \int_{F\times F\times F} t\left(\begin{pmatrix} 1 & \\ v & 1 \end{pmatrix}\begin{pmatrix} & \nu \\ \nu & \end{pmatrix}\right)^{z+\rho} \langle \mathrm{tr}(\beta v)\rangle dv_2 dv_3 dv_4 \\ &= W(\beta, z). \end{aligned}$$

□

It is easy to see that $\beta_2 = \alpha_2, \beta_3 = \alpha_2 x + \alpha_3, \beta_4 = \alpha_2 x^2 + 2\alpha_3 x + \alpha_4$.

If $|\alpha_3| \leqq |\alpha_2|$, then $\beta_3 = 0$ for $x = -\alpha_2^{-1}\alpha_3$.

If $|\alpha_3| > |\alpha_2|, |\alpha_3| \geqq |\alpha_4|$, then there exists $x \in \mathcal{O}$ such that $\beta_4 = 0$ by Lemma 5.10. For this x, $|\beta_3| = |\alpha_2 x + \alpha_3| = |\alpha_3| > |\alpha_2| = |\beta_2|$.

Otherwise, $|\alpha_4| > |\alpha_3| > |\alpha_2|$.

By Lemma 6.2 and the above consideration, we only consider α in the following five cases:

(1) $\alpha_3 = 0$,
(2) $\alpha_4 = 0, |\alpha_3| > |\alpha_2| \neq 0$,
(3) $\alpha_2 = \alpha_4 = 0$,
(4) $|\alpha_4| > |\alpha_3|, \alpha_2 = 0$,
(5) $|\alpha_4| > |\alpha_3| > |\alpha_2| \neq 0$.

We divide the limit of the integral in (4.10) as follows:

$$\begin{aligned} W(\alpha, z) = &\int_{F\times F\times \mathcal{O}} |c(u_2)|^{z_2-z_4+1}|c(u_3)|^{z_1+z_2-z_3-z_4+1} \\ &\times \langle \alpha_2 c(u_3)^{-2}u_2 + 2\alpha_3 u_3 + \alpha_4 u_4\rangle du_2 du_3 du_4 \\ &+ \sum_{n=1}^{\infty}\sum_{m=-\infty}^{0} q^{-n(z_1-z_3+1)} \int_{F\times \pi^{-m}\mathcal{O}^\times \times \pi^{-n}\mathcal{O}^\times} |c(u_2)|^{z_2-z_4+1} \\ &\qquad \times \langle \alpha_2 u_2 + (\alpha_2 u_3^2 + 2\alpha_3 u_3 + \alpha_4)u_4\rangle du_2 du_3 du_4 \\ &+ \sum_{m,n=1}^{\infty} q^{-m(z_1+z_2-z_3-z_4+1)-n(z_1-z_3+1)} \int_{F\times \pi^{-m}\mathcal{O}^\times \times \pi^{-n}\mathcal{O}^\times} |c(u_2)|^{z_2-z_4+1} \\ &\qquad \times \langle \alpha_2 u_3^2 u_2 + (\alpha_2 u_3^2 + 2\alpha_3 u_3 + \alpha_4)u_4\rangle du_2 du_3 du_4 \end{aligned} \tag{6.3}$$

and show that each term of (6.3) is computable. The computation of the first term of (6.3) is straightforward, and will be given in Proposition 6.4 and Corollary 6.8. We shall compute the second and third terms of (6.3) in Proposition 6.13.

Now we consider the first term of (6.3).

Proposition 6.4. *Suppose that $\alpha_2, \alpha_3 \neq 0$. If $\lfloor \mathrm{ord}(\alpha_2)/2\rfloor \leqq \mathrm{ord}(\alpha_3)$, then the first term of (6.3) is equal to*

$$\frac{(1-bT)(1-b^{\mathrm{ord}(\alpha_2)+1})}{1-b} + \frac{ab(1-bT)(1-T)(1-(ab)^{\lfloor \mathrm{ord}(\alpha_2)/2\rfloor})}{(1-b)(1-ab)}$$
$$- \frac{ab^{\mathrm{ord}(\alpha_2)}(1-bT)(1-T)(1-(a/b)^{\lfloor \mathrm{ord}(\alpha_2)/2\rfloor})}{(1-b)(1-(a/b))}.$$

If $\mathrm{ord}(\alpha_3)+1 \leqq \lfloor \mathrm{ord}(\alpha_2)/2 \rfloor$, *then the first term of (6.3) is equal to*

$$\begin{aligned}
&\frac{(1-bT)(1-b^{\mathrm{ord}(\alpha_2)+1})}{1-b} + \frac{ab(1-bT)(1-(ab)^{\mathrm{ord}(\alpha_3)})}{(1-b)(1-ab)} \\
&- \frac{ab^{\mathrm{ord}(\alpha_2)}(1-bT)(1-(a/b)^{\mathrm{ord}(\alpha_3)})}{(1-b)(1-(a/b))} - \frac{abT(1-bT)(1-(ab)^{\mathrm{ord}(\alpha_3)+1})}{(1-b)(1-ab)} \\
&+ \frac{ab^{\mathrm{ord}(\alpha_2)}T(1-bT)(1-(a/b)^{\mathrm{ord}(\alpha_3)+1})}{(1-b)(1-(a/b))}.
\end{aligned}$$

Proof. By an easy computation, the first term of (6.3) is equal to

$$\begin{aligned}
&\int_{F^2\times\mathcal{O}} |c(u_2)|^{z_2-z_4+1}|c(u_3)|^{z_1+z_2-z_3-z_4+1} \\
&\quad\times \langle \alpha_2 c(u_3)^{-2}u_2 + 2\alpha_3u_3 + \alpha_4u_4\rangle du_2du_3du_4 \\
&= \int_{F^2} |c(u_2)|^{z_2-z_4+1}|c(u_3)|^{z_1+z_2-z_3-z_4+1} \\
&\quad\times \langle \alpha_2 c(u_3)^{-2}u_2 + 2\alpha_3u_3\rangle du_2du_3 \int_{\mathcal{O}} \langle \alpha_4u_4\rangle du_4 \\
&= \int_{F^2} |c(u_2)|^{z_2-z_4+1}|c(u_3)|^{z_1+z_2-z_3-z_4+1}\langle \alpha_2 c(u_3)^{-2}u_2 + 2\alpha_3u_3\rangle du_2du_3 \\
&= \int_{F\times\mathcal{O}} |c(u_2)|^{z_2-z_4+1}\langle \alpha_2u_2 + 2\alpha_3u_3\rangle du_2du_3 \\
&\quad+ \sum_{m=1}^{\infty}(abT)^m \int_{F\times\pi^{-m}\mathcal{O}^\times} |c(u_2)|^{z_2-z_4+1}\langle \alpha_2u_2u_3^2 + 2\alpha_3u_3\rangle du_2du_3.
\end{aligned} \tag{6.5}$$

Then the first term of (6.5) is equal to

$$\int_F |c(u_2)|^{z_2-z_4+1}\langle \alpha_2u_2\rangle du_2 \int_{\mathcal{O}} \langle 2\alpha_3u_3\rangle du_3 = (1-bT)\sum_{k_1=0}^{c_2} b^{k_1} \tag{6.6}$$

and the second term of (6.5) is equal to

$$\begin{aligned}
&\sum_{m=1}^{\infty}(abT)^m \int_{\pi^{-m}\mathcal{O}^\times} \langle 2\alpha_3u_3\rangle \left(\int_F |c(u_2)|^{z_2-z_4+1}\langle \alpha_2u_2u_3^2\rangle du_2\right) du_3 \\
&= \sum_{\substack{1\leqq m\leqq\infty \\ c_2-2m\geqq 0}} (abT)^m(1-bT)\sum_{k_1=0}^{c_2-2m} b^{k_1} \int_{\pi^{-m}\mathcal{O}^\times} \langle 2\alpha_3u_3\rangle du_3 \\
&= (1-bT)\sum_{\substack{1\leqq m\leqq\infty \\ c_2-2m\geqq 0 \\ c_3-m\geqq 0}} (abT)^m \sum_{k_1=0}^{c_2-2m} b^{k_1}\cdot T^{-m}
\end{aligned} \tag{6.7}$$

$$+ (1-bT) \sum_{\substack{1 \leqq m \leqq \infty \\ c_2-2m \geqq 0 \\ c_3+1-m \geqq 0}} (abT)^m \sum_{k_1=0}^{c_2-2m} b^{k_1} \cdot (-T^{-(m-1)})$$

$$= (1-bT) \sum_{\substack{1 \leqq m \leqq \infty \\ c_2-2m \geqq 0 \\ c_3-m \geqq 0}} (ab)^m \sum_{k_1=0}^{c_2-2m} b^{k_1} - T(1-bT) \sum_{\substack{1 \leqq m \leqq \infty \\ c_2-2m \geqq 0 \\ c_3+1-m \geqq 0}} (ab)^m \sum_{k_1=0}^{c_2-2m} b^{k_1}.$$

So this proposition follows easily by (6.6), (6.7). □

Corollary 6.8. *If $\alpha_2 = \alpha_3 = 0$, then the first term of (6.3) is equal to*

$$\frac{(1-bT)(1-abT)}{(1-b)(1-ab)}.$$

If $\alpha_2 = 0, \alpha_3 \neq 0$, then the first term of (6.3) is equal to

$$\frac{(1-bT)(1-abT)(1-(ab)^{\mathrm{ord}(\alpha_3)+1})}{(1-b)(1-ab)}.$$

If $\alpha_2 \neq 0, \alpha_3 = 0$, then the first term of (6.3) is equal to

$$\frac{(1-bT)(1-b^{\mathrm{ord}(\alpha_2)+1})}{1-b} + \frac{ab(1-T)(1-bT)(1-(ab)^{\lfloor \mathrm{ord}(\alpha_2)/2 \rfloor})}{(1-b)(1-ab)}$$
$$- \frac{ab^{\mathrm{ord}(\alpha_2)}(1-T)(1-bT)(1-(a/b)^{\mathrm{ord}(\alpha_2)})}{(1-b)(1-(a/b))}.$$

Proof. We regard c_i as ∞. □

Then we consider the second and third terms of (6.3). The second term of (6.3) is equal to

$$\begin{aligned} &\frac{(1-q^{-(z_2-z_4+1)})(1-q^{-(c_2+1)(z_2-z_4)})}{1-q^{-(z_2-z_4)}} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{0} q^{-n(z_1-z_3+1)} \\ &\quad \times \int_{\pi^{-m}\mathcal{O}^\times \times \pi^{-n}\mathcal{O}^\times} \langle (\alpha_2 u_3^2 + 2\alpha_3 u_3 + \alpha_4) u_4 \rangle du_3 du_4 \\ &= (1-bT) \sum_{k_1=0}^{c_2} b^{k_1} \sum_{-\infty \leqq m \leqq 0} \sum_{1 \leqq n \leqq \infty} (aT)^n \\ &\quad \times \int_{\pi^{-m}\mathcal{O}^\times \times \pi^{-n}\mathcal{O}^\times} \langle (\alpha_2 u_3^2 + 2\alpha_3 u_3 + \alpha_4) u_4 \rangle du_3 du_4 \end{aligned} \tag{6.9}$$

and the third term of (6.3) is equal to

$$\begin{aligned} &\frac{1-q^{-(z_2-z_4+1)}}{1-q^{-(z_2-z_4)}} \sum_{m,n=1}^{\infty} q^{-m(z_1+z_2-z_3-z_4+1)-n(z_1-z_3+1)}(1-q^{-(c_2-2m+1)(z_2-z_4)}) \\ &\times \int_{\pi^{-m}\mathcal{O}^\times \times \pi^{-n}\mathcal{O}^\times} \langle (\alpha_2 u_3^2 + 2\alpha_3 u_3 + \alpha_4) u_4 \rangle du_3 du_4 \end{aligned} \tag{6.10}$$

$$= (1-bT)\sum_{\substack{1\leqq m\leqq\infty\\ c_2-2m\geqq 0}}\sum_{1\leqq n\leqq\infty}(abT)^m(aT)^n\sum_{k_1=0}^{c_2-2m}b^{k_1}$$
$$\times\int_{\pi^{-m}\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times}\langle(\alpha_2u_3^2+2\alpha_3u_3+\alpha_4)u_4\rangle du_3du_4$$

by Lemma 5.5. Therefore, we can reduce the problem of computing $W(\alpha,z)$ to the problem of computing

$$\int_{\pi^{-m}\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times}\langle(\alpha_2u_3^2+2\alpha_3u_3+\alpha_4)u_4\rangle du_3du_4 \tag{6.11}$$

for $n\geqq 1, m\in\mathbb{Z}$. The following definitions are used to describe the condition of Proposition 6.13.

Definition 6.12. Let F^2 be the set of square elements of F, i.e.

$$F^2=\{a\in F\,|\,\exists x\in F, a=x^2\}.$$

For $a,b\in\mathbb{Z}$, we denote $a\sim b$ (resp. $a\not\sim b$) if $a-b\in 2\mathbb{Z}$ (resp. $a-b\notin 2\mathbb{Z}$).

Proposition 6.13. *Let $D=D(\alpha,m)=D_\alpha/\alpha_2^2\pi^{-2m}$. Then (6.11) is equal to the following:*

(1) *If $c_4<c_2-2m, c_4<c_3-m$, then*

$$\begin{cases}0 & \text{if } c_4+1<n,\\ -q^{m+n-1}(1-q^{-1}) & \text{if } c_4+1=n,\\ q^{m+n}(1-q^{-1})^2 & \text{if } c_4+1>n.\end{cases}$$

(2) *If $c_3-m<c_2-2m, c_3-m<c_4$, then*

$$\begin{cases}0 & \text{if } c_3-m+1<n,\\ -q^{m+n-1}(1-q^{-1}) & \text{if } c_3-m+1=n,\\ q^{m+n}(1-q^{-1})^2 & \text{if } c_3-m+1>n.\end{cases}$$

(3) *If $c_2-2m<c_3-m, c_2-2m<c_4$, then*

$$\begin{cases}0 & \text{if } c_2-2m+1<n,\\ -q^{m+n-1}(1-q^{-1}) & \text{if } c_2-2m+1=n,\\ q^{m+n}(1-q^{-1})^2 & \text{if } c_2-2m+1>n.\end{cases}$$

(4) *If $c_2-2m=c_3-m<c_4$, then*

$$\begin{cases}0 & \text{if } c_3-m+1<n,\\ q^{m+n-2} & \text{if } c_3-m+1=n,\\ q^{m+n}(1-q^{-1})^2 & \text{if } c_3-m+1>n.\end{cases}$$

(5) *If $c_2-2m>c_3-m=c_4$, then*

$$\begin{cases}0 & \text{if } c_3-m+1<n,\\ q^{m+n-2} & \text{if } c_3-m+1=n,\\ q^{m+n}(1-q^{-1})^2 & \text{if } c_3-m+1>n.\end{cases}$$

(6) *If* $c_2 - 2m = c_4 < c_3 - m$*, then*

$$\begin{cases} 0 & \textit{if } c_4 + 1 < n, \\ q^{m+n-1}(1+q^{-1}) & \textit{if } c_4 + 1 = n, D \in F^2, \\ -q^{m+n-1}(1-q^{-1}) & \textit{if } c_4 + 1 = n, D \notin F^2, \\ q^{m+n}(1-q^{-1})^2 & \textit{if } c_4 + 1 > n. \end{cases}$$

(7) *If* $c_2 - 2m = c_3 - m = c_4$*, then (6.11) is equal to the following:*
If $\mathrm{ord}(D) = 0$*, then*

$$\begin{cases} 0 & \textit{if } c_4 + 1 < n \\ q^{m+n-1}(1+q^{-1}) & \textit{if } c_4 + 1 = n, D \in F^2, \\ -q^{m+n-1}(1-q^{-1}) & \textit{if } c_4 + 1 = n, D \notin F^2, \\ q^{m+n}(1-q^{-1})^2 & \textit{if } c_4 + 1 > n. \end{cases}$$

If $\mathrm{ord}(D) = 2j+1$ $(j \geqq 0)$*, then*

$$\begin{cases} 0 & \textit{if } c_4 + 2j + 2 < n, \\ -q^{-j+m+n-2} & \textit{if } c_4 + 2j + 2 = n, \\ q^{(c_4+n)/2+m}(1-q^{-1}) & \textit{if } c_4 + 3 < n \leqq c_4 + 2j + 1, c_4 \sim n, \\ 0 & \textit{if } c_4 + 3 \leqq n \leqq c_4 + 2j + 1, c_4 \not\sim n, \\ q^{m+n-1}(1-q^{-1}) & \textit{if } c_4 + 2 = n, j \geqq 1, \\ q^{m+n-2} & \textit{if } c_4 + 1 = n, \\ q^{m+n}(1-q^{-1})^2 & \textit{if } c_4 + 1 > n. \end{cases}$$

If $\mathrm{ord}(D) = 2j+2$ $(j \geqq 0)$*, then*

$$\begin{cases} 0 & \textit{if } c_4 + 2j + 3 < n, \\ q^{-j+m+n-2} & \textit{if } c_4 + 2j + 3 = n, D \in F^2, \\ -q^{-j+m+n-2} & \textit{if } c_4 + 2j + 3 = n, D \notin F^2, \\ q^{-j+m+n-1}(1-q^{-1}) & \textit{if } c_4 + 2j + 2 = n, \\ q^{(c_4+n)/2+m}(1-q^{-1}) & \textit{if } c_4 + 3 < n \leqq c_4 + 2j + 1, c_4 \sim n, \\ 0 & \textit{if } c_4 + 3 \leqq n \leqq c_4 + 2j + 1, c_4 \not\sim n, \\ q^{m+n-1}(1-q^{-1}) & \textit{if } c_4 + 2 = n, j \geqq 1, \\ q^{m+n-2} & \textit{if } c_4 + 1 = n, \\ q^{m+n}(1-q^{-1})^2 & \textit{if } c_4 + 1 > n. \end{cases}$$

Proof. In case (1), (6.11) is equal to $\int_{\pi^{-m}\mathcal{O}^\times \times \pi^{-n}\mathcal{O}^\times} \langle \alpha_4 u_4 \rangle du_3 du_4$. So case (1) follows by Lemma 5.1.

The argument is similar in cases (2), (3).

In case (4), note that by making the change of variables and using the decomposition (5.14), (6.11) is equal to

$$\begin{aligned} & q^{m-1}\sum_{i=1}^{q-1}\int_{\mathcal{O}\times\pi^{-n}\mathcal{O}^\times}\langle\{\alpha_2\pi^{-2m+2}u_3^2+2(\alpha_2\pi^{-2m+1}x_i+\alpha_3\pi^{-m+1})u_3 \\ & +(\alpha_2\pi^{-2m}x_i^2+2\alpha_3\pi^{-m}x_i+\alpha_4)\}u_4\rangle du_3du_4. \end{aligned} \tag{6.14}$$

So we consider the integral

$$\begin{aligned} & \int_{\mathcal{O}\times\pi^{-n}\mathcal{O}^\times}\langle\{\alpha_2\pi^{-2m+2}u_3^2+2(\alpha_2\pi^{-2m+1}x_i+\alpha_3\pi^{-m+1})u_3 \\ & +(\alpha_2\pi^{-2m}x_i^2+2\alpha_3\pi^{-m}x_i+\alpha_4)\}u_4\rangle du_3du_4. \end{aligned} \tag{6.15}$$

There exists unique i such that $\mathrm{ord}(\alpha_2\pi^{-2m}x_i^2+2\alpha_3\pi^{-m}x_i)\geqq c_3-m+1$. Then $\mathrm{ord}(\alpha_2\pi^{-2m+1}x_i+\alpha_3\pi^{-m+1})=\mathrm{ord}(\pi(\alpha_2\pi^{-2m}x_i+2\alpha_3\pi^{-m})-\alpha_3\pi^{-m+1})=c_3-m+1$. For this i, (6.15) is equal to

$$\int_{\mathcal{O}\times\pi^{-n}\mathcal{O}^\times}\langle\pi^{c_3-m+1}u_3u_4\rangle du_3du_4=\begin{cases}0 & \text{if } c_3-m+1<n,\\ q^n(1-q^{-1}) & \text{if } c_3-m+1\geqq n\end{cases}$$

by Lemmas 5.1 and 5.10. For other i's, $\mathrm{ord}(\alpha_2\pi^{-2m}x_i^2+2\alpha_3\pi^{-m}x_i)=c_3-m$. Therefore (6.15) is equal to

$$\int_{\mathcal{O}\times\pi^{-n}\mathcal{O}^\times}\langle\pi^{c_3-m}u_4\rangle du_3du_4=\begin{cases}0 & \text{if } c_3-m+1<n,\\ -q^{n-1} & \text{if } c_3-m+1=n,\\ q^n(1-q^{-1}) & \text{if } c_3-m+1>n\end{cases}$$

by Lemma 5.1. So case (4) follows by an easy computation.

In case (5), by Lemma 5.10, (6.11) is equal to

$$\begin{aligned} & q^m\int_{\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times}\langle\left((\alpha_2\pi^{-2m}u_3^2+2\alpha_3\pi^{-m}u_3)+\alpha_4\right)u_4\rangle du_3du_4 \\ & =q^m\int_{\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times}\langle(\pi^{c_3-m}u_3+\alpha_4)u_4\rangle du_3du_4 \\ & =q^m\int_{\pi^{-n}\mathcal{O}^\times}\langle\pi^{c_4}u_4\rangle\left(\int_{\mathcal{O}^\times}\langle\pi^{c_3-m}u_3u_4\rangle du_3\right)du_4 \\ & =q^m\int_{\mathcal{O}^\times}\langle\pi^{c_3-m-n}u_3\rangle du_3\int_{\pi^{-n}\mathcal{O}^\times}\langle\pi^{c_4}u_4\rangle du_4, \end{aligned}$$

where we applied Lemma 5.10 for the case $c=0$ in the first step. So case (5) follows by Lemma 5.1.

In case (6), we consider the integral (6.15). First, we note that

$$\mathrm{ord}(2(\alpha_2\pi^{-2m+1}x_i+\alpha_3\pi^{-m+1}))=c_2-2m+1=c_4+1.$$

If $\mathrm{ord}(\alpha_2\pi^{-2m}x_i^2+\alpha_4)=c_4$, then (6.15) is equal to

$$\int_{\mathcal{O}\times\pi^{-n}\mathcal{O}^\times}\langle\pi^{c_4}u_4\rangle du_3du_4=\begin{cases}0 & \text{if } c_4+1<n,\\ -q^{n-1} & \text{if } c_4+1=n,\\ q^n(1-q^{-1}) & \text{if } c_4+1>n\end{cases}$$

by Lemma 5.1. Otherwise, $\mathrm{ord}(\alpha_2\pi^{-2m}x_i^2+\alpha_4) \geqq c_4+1$. The number of such i's is 0 or 2. For such i, (6.15) is equal to

$$\int_{\mathcal{O}\times\pi^{-n}\mathcal{O}^\times} \langle \pi^{c_4+1}u_3u_4\rangle du_3du_4 = \begin{cases} 0 & \text{if } c_4+1<n, \\ q^n(1-q^{-1}) & \text{if } c_4+1 \geqq n \end{cases}$$

by Lemmas 5.1 and 5.10. By Hensel's lemma,

$$\mathrm{ord}(\alpha_2\pi^{-2m}x_i^2+\alpha_4) > c_4 \text{ for some } x_i \iff -\alpha_4/\alpha_2\pi^{-2m} \in F^2$$
$$\iff -\alpha_4\alpha_2 \in F^2 \iff D \in F^2.$$

So case (6) follows by an easy computation.

In case (7), let

$$f(X) = X^2 + 2\frac{\alpha_3\pi^{-m}}{\alpha_2\pi^{-2m}}X + \frac{\alpha_4}{\alpha_2\pi^{-2m}} \in \mathcal{O}[X].$$

We consider the decomposition (5.14). There are three cases as follows:

(i) $f(x_i) \in \mathcal{O}^\times$ for all i,

(ii) There are two i's such that $f(x_i) \in \mathfrak{p}, f'(x_i) \in \mathcal{O}^\times$, and $f(x_j) \in \mathcal{O}^\times$ for other j's,

(iii) There is unique i such that $f(x_i) \in \mathfrak{p}$, $f'(x_i) \in \mathfrak{p}$, and $f(x_j) \in \mathcal{O}^\times$ for other j's. Note that (ii) (resp. (iii)) occurs if and only if $D \in (\mathcal{O}^\times)^2$ (resp. $D \in \mathfrak{p}$).

If $f(x_i) \in \mathcal{O}^\times$, then (6.15) is equal to

$$\int_{\mathcal{O}\times\pi^{-n}\mathcal{O}^\times} \langle \pi^{c_4}u_4\rangle du_3du_4 = \begin{cases} 0 & \text{if } c_4+1<n, \\ -q^{n-1} & \text{if } c_4+1=n, \\ q^n(1-q^{-1}) & \text{if } c_4+1>n \end{cases} \tag{6.16}$$

by Lemma 5.1.

If $f(x_i) \in \mathfrak{p}, f'(x_i) \in \mathcal{O}^\times$, then (6.15) is equal to

$$\int_{\mathcal{O}\times\pi^{-n}\mathcal{O}^\times} \langle \pi^{c_4+1}u_3u_4\rangle du_3du_4 = \begin{cases} 0 & \text{if } c_4+1<n, \\ q^n(1-q^{-1}) & \text{if } c_4+1 \geqq n \end{cases} \tag{6.17}$$

by Lemmas 5.1 and 5.10.

If $f(x_i), f'(x_i) \in \mathfrak{p}$, then we choose $x_i = -(\alpha_3\pi^{-m})/(\alpha_2\pi^{-2m})$. Then (6.15) is equal to

$$\int_{\mathcal{O}\times\pi^{-n}\mathcal{O}^\times} \langle \alpha_2\pi^{-2m}(\pi^2u_3^2-D)u_4\rangle du_3du_4. \tag{6.18}$$

In this case, $D \in \mathfrak{p}$. If $\mathrm{ord}(D) = 2j+1 (j \geqq 0)$, then by the decomposition

$$\mathcal{O} = \mathcal{O}^\times \sqcup \cdots \sqcup \pi^{j-1}\mathcal{O}^\times \sqcup \pi^j\mathcal{O},$$

(6.18) is equal to

$$\sum_{i=0}^{j-1}\int_{\pi^i\mathcal{O}^\times\times\pi^{-n}o^\times} \langle \pi^{c_4+2}u_3^2u_4\rangle du_3du_4 + \int_{\pi^j\mathcal{O}\times\pi^{-n}\mathcal{O}^\times} \langle \pi^{c_4+2j+1}u_4\rangle du_3du_4. \tag{6.19}$$

Lemma 6.20. *The first term of (6.19) is equal to*

$$\begin{cases} 0 & \text{if } c_4+2j+1<n, \\ q^n(1-q^{-1})(q^{(c_4-n+2)/2}-q^{-j}) & \text{if } c_4+3<n\leqq c_4+2j+1, c_4\sim n, \\ -q^{n-j}(1-q^{-1}) & \text{if } c_4+3\leqq n<c_4+2j, c_4\not\sim n, \\ q^n(1-q^{-1})(1-q^{-j}) & \text{if } c_4+3>n. \end{cases}$$

Proof. By Lemma 5.1,

$$\int_{\pi^i\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times}\langle\pi^{c_4+2}u_3^2u_4\rangle du_3du_4=\begin{cases} 0 & \text{if } c_4+2i+3<n, \\ -q^{n-i-1}(1-q^{-1}) & \text{if } c_4+2i+3=n, \\ q^{n-i}(1-q^{-1})^2 & \text{if } c_4+2i+3>n. \end{cases}$$

So if $c_4+3>n$, then

$$\sum_{i=0}^{j-1}\int_{\pi^i\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times}\langle\pi^{c_4+2}u_3^2u_4\rangle du_3du_4=\sum_{i=0}^{j-1}q^{n-i}(1-q^{-1})^2=q^n(1-q^{-1})(1-q^{-j}).$$

If $c_4+3\leqq n\leqq c_4+2j+1, c_4\not\sim n$, then

$$\begin{aligned}
&\sum_{i=0}^{j-1}\int_{\pi^i\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times}\langle\pi^{c_4+2}u_3^2u_4\rangle du_3du_4 \\
&=\sum_{i=-(c_4-n+3)/2}^{j-1}\int_{\pi^i\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times}\langle\pi^{c_4+2}u_3^2u_4\rangle du_3du_4 \\
&=-q^{n+(c_4-n+3)/2-1}(1-q^{-1})+\sum_{i=-(c_4-n+1)/2}^{j-1}q^{n-i}(1-q^{-1})^2 \\
&=-q^{n-j}(1-q^{-1}).
\end{aligned}$$

If $c_4+2<n\leqq c_4+2j+1, c_4\sim n$, then

$$\begin{aligned}
&\sum_{i=0}^{j-1}\int_{\pi^i\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times}\langle\pi^{c_4+2}u_3^2u_4\rangle du_3du_4 \\
&=\sum_{i=-(c_4-n+2)/2}^{j-1}\int_{\pi^i\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times}\langle\pi^{c_4+2}u_3^2u_4\rangle du_3du_4 \\
&=\sum_{i=-(c_4-n+2)/2}^{j-1}q^{n-i}(1-q^{-1})^2 \\
&=q^n(1-q^{-1})(q^{(c_4-n+2)/2}-q^{-j}).
\end{aligned}$$

If $c_4+2j+3<n$, then

$$\sum_{i=0}^{j-1}\int_{\pi^i\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times}\langle\pi^{c_4+2}u_3^2u_4\rangle du_3du_4=0.$$

□

If $j = 0$, then we regard the first term of (6.19) as 0. We continue the proof of Proposition 6.13. The second term of (6.19) is equal to

$$\begin{cases} 0 & \text{if } c_4 + 2j + 2 < n, \\ -q^{n-j-1} & \text{if } c_4 + 2j + 2 = n, \\ q^{n-j}(1 - q^{-1}) & \text{if } c_4 + 2j + 2 > n. \end{cases}$$

So (6.19) is equal to

$$\begin{cases} 0 & \text{if } c_4 + 2j + 2 < n, \\ -q^{-j+n-1} & \text{if } c_4 + 2j + 2 = n, \\ q^{(c_4+n)+1}(1 - q^{-1}) & \text{if } c_4 + 3 < n \leqq c_4 + 2j + 1, c_4 \sim n, \\ 0 & \text{if } c_4 + 3 \leqq n \leqq c_4 + 2j + 1, c_4 \not\sim n, \\ q^n(1 - q^{-1}) & \text{if } c_4 + 2j + 2 - \max(0, 2j - 1) > n. \end{cases} \tag{6.21}$$

If $\mathrm{ord}(D) = 2j + 2$, then by the decomposition

$$\mathcal{O} = \mathcal{O}^\times \sqcup \cdots \sqcup \pi^{j-1}\mathcal{O}^\times \sqcup \pi^j\mathcal{O}^\times \sqcup \pi^{j+1}\mathcal{O},$$

(6.18) is equal to

$$\begin{aligned} &\sum_{i=0}^{j-1} \int_{\pi^i\mathcal{O}^\times \times \pi^{-n}\mathcal{O}^\times} \langle \pi^{c_4+2} u_3^2 u_4 \rangle du_3 du_4 \\ &+ \int_{\pi^j\mathcal{O}^\times \times \pi^{-n}\mathcal{O}^\times} \langle \pi^{c_4}(\pi^2 u_3^2 - D/4) u_4 \rangle du_3 du_4 \\ &+ \int_{\pi^{j+1}\mathcal{O} \times \pi^{-n}\mathcal{O}^\times} \langle \pi^{c_4+2j+2} u_4 \rangle du_3 du_4, \end{aligned} \tag{6.22}$$

where the first term does not appear if $j = 0$. The first term of (6.22) is computed in Lemma 6.20 and the third term of (6.22) is equal to

$$\begin{cases} 0 & \text{if } c_4 + 2j + 3 < n, \\ -q^{n-j-2} & \text{if } c_4 + 2j + 3 = n, \\ q^{n-j-1}(1 - q^{-1}) & \text{if } c_4 + 2j + 3 > n \end{cases}$$

by Lemma 5.1. We consider the second term of (6.22). Then by the decomposition (5.14), the second term is equal to

$$\begin{aligned} &q^{-j} \int_{\mathcal{O}^\times \times \pi^{-n}\mathcal{O}^\times} \langle \pi^{c_4+2j+2}(u_3^2 - \pi^{-2j-2}D) u_4 \rangle du_3 du_4 \\ &= q^{-j-1} \sum_{i=1}^{q-1} \int_{\mathcal{O} \times \pi^{-n}\mathcal{O}^\times} \langle \pi^{c_4+2j+2}(\pi^2 u_3^2 + 2\pi x_i u_3 + x_i^2 - \pi^{-2j-2}D) u_4 \rangle du_3 du_4. \end{aligned} \tag{6.23}$$

If there exists x_i such that $x_i^2 - \pi^{-2j-2}D \in \mathfrak{p}$, (i.e. $D \in F^2$), then (6.23) is equal to

$$\begin{aligned} & q^{-j-1} \cdot (q-3) \int_{\mathcal{O}\times\pi^{-n}\mathcal{O}^\times} \langle \pi^{c_4+2j+2} u_4 \rangle du_3 du_4 \\ & + q^{-j-1} \cdot 2 \int_{\mathcal{O}\times\pi^{-n}\mathcal{O}^\times} \langle \pi^{c_4+2j+3} u_3 u_4 \rangle du_3 du_4 \\ & = \begin{cases} 0 & \text{if } c_4+2j+3<n, \\ q^{n-j-1}(1+q^{-1}) & \text{if } c_4+2j+3=n, \\ q^{n-j}(1-q^{-1})^2 & \text{if } c_4+2j+3>n \end{cases} \end{aligned}$$

by Lemmas 5.1 and 5.10.

If $D \notin F^2$, then (6.23) is equal to

$$q^{-j} \cdot (q-1) \int_{\mathcal{O}\times\pi^{-l}\mathcal{O}^\times} \langle \pi^{c_4+2j} u_4 \rangle du_3 du_4 = \begin{cases} 0 & \text{if } c_4+2j+3<n, \\ -q^{n-j-1}(1-q^{-1}) & \text{if } c_4+2j+3=n, \\ q^{n-j}(1-q^{-1})^2 & \text{if } c_4+2j+3>n. \end{cases}$$

So (6.22) is equal to

$$\begin{cases} 0 & \text{if } c_4+2j+3<n, \\ q^{n-j-1} & \text{if } c_4+2j+3=n, D\in F^2, \\ -q^{n-j-1} & \text{if } c_4+2j+3=n, D\notin F^2, \\ q^{n-j}(1-q^{-1}) & \text{if } c_4+2j+2=n, \\ q^{(c_4+n+2)/2}(1-q^{-1}) & \text{if } c_4+3<n\leqq c_4+2j+1, c_4\sim n, \\ 0 & \text{if } c_4+3<n\leqq c_4+2j+1, c_4\not\sim n, \\ q^{n}(1-q^{-1}) & \text{if } c_4+2j+3-\max(1,2j)>n. \end{cases} \tag{6.24}$$

□

By Proposition 6.13, we can compute sums (6.9), (6.10). So we can compute each term of (6.3). It completes the proof of Theorem 4.5. □

7. Proof of Theorem 4.6

In this section, we prove Theorem 4.6. Throughout this section, we assume that F is not dyadic, $\alpha_2 \in \mathcal{O}^\times, \alpha_3, \alpha_4 \in \mathcal{O}$. We use the notation (6.1) again. In addition, let $\beta_i = \alpha_i$ (resp. 1) if $\alpha_i \neq 0$ (resp. $\alpha_i = 0$).

Lemma 7.1. *There exists a constant $C>0$ independent of F such that*

$$\begin{aligned} |W(\alpha,z)|_{\mathbb{C}} \leqq & C(\mathrm{ord}(\alpha_2)+1)(\mathrm{ord}(\beta_3)+1)(\mathrm{ord}(\beta_4)+1) \\ & \times (\mathrm{ord}(D_\alpha)+1)|1-q^{-(z_2-z_4+1)}|_{\mathbb{C}}. \end{aligned}$$

Proof. To prove the lemma, we show that each term of (6.3) is bounded by

$$C(\mathrm{ord}(\alpha_2)+1)(\mathrm{ord}(\beta_3)+1)(\mathrm{ord}(\beta_4)+1)(\mathrm{ord}(D_\alpha)+1)|1-q^{-(z_2-z_4+1)}|_{\mathbb{C}}$$

in the next three lemmas. □

Lemma 7.2. *The first term of (6.3) is bounded by*

$$3(\mathrm{ord}(\alpha_2)+1)(\mathrm{ord}(\beta_3)+1)(\mathrm{ord}(\beta_4)+1)(\mathrm{ord}(D_\alpha)+1)|1-q^{-(z_2-z_4+1)}|_{\mathbb{C}}.$$

Proof. Since $|a|_{\mathbb{C}}, |b|_{\mathbb{C}}, |T|_{\mathbb{C}} \leqq 1$, it follows from the last steps of (6.6), (6.7). □

Lemma 7.3. *The second term of (6.3) is bounded by*

$$19(\mathrm{ord}(\alpha_2)+1)(\mathrm{ord}(\beta_3)+1)(\mathrm{ord}(\beta_4)+1)(\mathrm{ord}(D_\alpha)+1)|(1-q^{-(z_2-z_4+1)})|_{\mathbb{C}}.$$

Proof. We divide the sum (6.9) into at most seven terms by Proposition 6.13 and show that each term is bounded by

$$7(\mathrm{ord}(\alpha_2)+1)(\mathrm{ord}(\beta_3)+1)(\mathrm{ord}(\beta_4)+1)(\mathrm{ord}(D_\alpha)+1)|(1-q^{-(z_2-z_4+1)})|_{\mathbb{C}}.$$

We divide the sum as follows:

$$\begin{aligned}\sum_{-\infty\leqq m\leqq 0} &= \sum_{\substack{-\infty\leqq m\leqq 0\\ c_4<c_2-2m\\ c_4<c_3-m}} + \sum_{\substack{-\infty\leqq m\leqq 0\\ c_3-m<c_2-2m\\ c_3-m<c_4}} + \sum_{\substack{-\infty\leqq m\leqq 0\\ c_2-2m<c_3-m\\ c_2-2m<c_4}} + \sum_{\substack{-\infty\leqq m\leqq 0\\ c_2-2m=c_3-m<c_4}}\\ &+ \sum_{\substack{-\infty\leqq m\leqq 0\\ c_3-m=c_4<c_2-2m}} + \sum_{\substack{-\infty\leqq m\leqq 0\\ c_2-2m=c_4<c_3-m}} + \sum_{\substack{-\infty\leqq m\leqq 0\\ c_2-2m=c_3-m=c_4}}.\end{aligned} \tag{7.4}$$

If there exists no such m, then we regard the corresponding term of (6.9) as 0.

(1) The partial sum of (6.9) corresponding to the first term of (7.4) is equal to

$$\begin{aligned}&(1-bT)\sum_{k_1=0}^{c_2} b^{k_1} \sum_{\substack{-\infty\leqq m\leqq 0\\ c_4<c_2-2m\\ c_4<c_3-m}} \sum_{n=1}^{\infty}(aT)^n \int_{\pi^{-m}\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times} \langle(\alpha_2 u_3^2+2\alpha_3 u_3+\alpha_4)u_4\rangle du_3 du_4\\ &= (1-bT)(1-T)\sum_{k_1=0}^{c_2} b^{k_1}\sum_m T^{-m}\left(\sum_{n=1}^{c_4} a^n - T\sum_{n=1}^{c_4+1} a^n\right).\end{aligned}$$

It is bounded by

$$\begin{aligned}&|1-bT|_{\mathbb{C}}\cdot|1-T|_{\mathbb{C}}\cdot(c_2+1)\cdot|1-T|_{\mathbb{C}}^{-1}\cdot(2(c_4+1))\\ &= 2(c_2+1)(c_4+1)|1-bT|_{\mathbb{C}}.\end{aligned}$$

(2) Let $m' = \min(0, c_2-c_3-1)$. Then the partial sum of (6.9) corresponding to the second term of (7.4) is equal to

$$\begin{aligned}&\sum_{n=1}^{\infty}(aT)^n(1-bT)\sum_{k_1=0}^{c_2} b^{k_1}\int_{\pi^{-m'}\mathcal{O}\times\pi^{-n}\mathcal{O}^\times}\langle(\alpha_2 u_3^2+2\alpha_3 u_3+\alpha_4)u_4\rangle du\\ &= (1-bT)\sum_{k_1=0}^{c_2} b^{k_1}T^{-m'}\sum_{n=1}^{c_3-m'}(aT)^n T^{-n}(1-T)\\ &= T^{-m'}(1-bT)(1-T)\sum_{k_1=0}^{c_2} b^{k_1}\sum_{n=1}^{c_3-m'} a^n.\end{aligned}$$

Since

$$\begin{cases} c_3-m' = c_3 \leqq c_3+1 & \text{if } m'=0,\\ c_3-m' = c_3-(c_2-c_3-1) \leqq 2(c_3+1) & \text{if } m' = c_2-c_3-1,\end{cases}$$

it is bounded by

$$2(c_2+1)(c_3+1)|1-bT|_{\mathbb{C}}.$$

(3) The partial sum of (6.9) corresponding to the third term of (7.4) is equal to

$$\begin{aligned}
&(1-bT)\sum_{k_1=0}^{c_2} b^{k_1} \sum_{\substack{-\infty \leqq m \leqq 0\\ c_2-2m<c_3-m\\ c_2-m<c_4}} \sum_{n=1}^{\infty}(aT)^n \int_{\pi^{-m}\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times} \langle(\alpha_2 u_3^2+2\alpha_3 u_3+\alpha_4)u_4\rangle du_3 du_4\\
&= (1-bT)(1-T)\sum_{k_1=0}^{c_2} b^{k_2} \sum_{m}\left(T^{-m}\left(\sum_{n=1}^{c_2-2m} a^n - T\sum_{n=1}^{c_2-2m+1} a^n\right)\right).
\end{aligned}$$

Since

$$\begin{cases} c_2-2m+1 < c_2-2(c_2-c_3)+1 < 2(c_3+1) & \text{if } \alpha_3 \neq 0,\\ c_2-2m+1 < c_4+1 & \text{if } \alpha_4 \neq 0,\end{cases}$$

it is bounded by

$$\begin{cases} 4(c_2+1)(c_3+1)|1-bT|_{\mathbb{C}} & \text{if } \alpha_3 \neq 0,\\ 2(c_2+1)(c_4+1)|1-bT|_{\mathbb{C}} & \text{if } \alpha_4 \neq 0.\end{cases}$$

(4) The partial sum of (6.9) corresponding to the fourth term of (7.4) is equal to

$$\begin{aligned}
&(1-bT)\sum_{k_1=0}^{c_2} b^{k_1} \sum_{\substack{-\infty \leqq m \leqq 0\\ c_2-2m=c_3-m<c_4}} \sum_{n=1}^{\infty}(aT)^n\\
&\quad\times \int_{\pi^{-m}\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times} \langle(\alpha_2 u_3^2+2\alpha_3 u_3+\alpha_4)u_4\rangle du_3 du_4\\
&= T^{-m}(1-bT)\sum_{k_1=0}^{c_2} b^{k_1}\left((1-2T)\sum_{n=1}^{c_3-m} a^n + T^2\sum_{n=1}^{c_3-m+1} a^n\right).
\end{aligned}$$

Since

$$\begin{cases} c_3-m+1 = c_3-(c_2-c_3)+1 < 2(c_3+1) & \text{if } \alpha_3 \neq 0,\\ c_3-m+1 < c_4+1 & \text{if } \alpha_4 \neq 0,\end{cases}$$

it is bounded by

$$\begin{cases} 4(c_2+1)(c_3+1)|1-bT|_{\mathbb{C}} & \text{if } \alpha_3 \neq 0,\\ 2(c_2+1)(c_4+1)|1-bT|_{\mathbb{C}} & \text{if } \alpha_4 \neq 0.\end{cases}$$

(5) The partial sum of (6.9) corresponding to the fifth term of (7.4) is equal to

$$(1-bT)\sum_{k_1=0}^{c_2} b^{k_1} \sum_{\substack{-\infty \leqq m \leqq 0 \\ c_3-m=c_4<c_2-2m}} \sum_{n=1}^{\infty} (aT)^n$$
$$\times \int_{\pi^{-m}\mathcal{O}^\times \times \pi^{-n}\mathcal{O}^\times} \langle (\alpha_2 u_3^2 + 2\alpha_3 u_3 + \alpha_4) u_4 \rangle du_3 du_4$$
$$= T^{-m}(1-bT)\sum_{k_1=0}^{c_2} b^{k_1} \left((1-2T) \sum_{n=1}^{c_3-m} a^n + T^2 \sum_{n=1}^{c_3-m+1} \right).$$

It is bounded by

$$2(c_2+1)(c_4+1)|1-bT|_{\mathbb{C}}.$$

(6) Since

$$\sum_{n=1}^{c_4} (aT)^n T^{-(m+n)} (1-T)^2 - (aT)^{c_4+1} T^{-(m+(c_4+1)-1)} (1-T)$$
$$= T^{-m}(1-T) \left(\sum_{n=1}^{c_4} a^n - T \sum_{n=1}^{c_4+1} a^n \right)$$

and

$$\sum_{n=1}^{c_4} (aT)^n T^{-(m+n)} (1-T)^2 + (aT)^{c_4+1} T^{-(m+(c_4+1)-1)} (1+T)$$
$$= T^{-m} \left((1-3T) \sum_{n=1}^{c_4} a^n + T(1+T) \sum_{n=1}^{c_4+1} a^n \right),$$

the partial sum of (6.9) corresponding to the sixth term of (7.4) is equal to

$$(1-bT)\sum_{k_1=0}^{c_2} b^{k_1} \sum_{\substack{-\infty \leqq m \leqq 0 \\ c_2-2m=c_4<c_3-m}} \sum_{n=1}^{\infty} (aT)^n$$
$$\times \int_{\pi^{-m}\mathcal{O}^\times \times \pi^{-n}\mathcal{O}^\times} \langle (\alpha_2 u_3^2 + 2\alpha_3 u_3 + \alpha_4) u_4 \rangle du_3 du_4$$
$$= \begin{cases} T^{-m}(1-bT)\sum_{k_1=0}^{c_2} b^{k_1} \left((1-3T) \sum_{n=1}^{c_4} a^n + T(1+T) \sum_{n=1}^{c_4+1} a^n \right) & \text{if } D \in F^2, \\ T^{-m}(1-bT)(1-T)\sum_{k_1=0}^{c_2} b^{k_1} \left(\sum_{n=1}^{c_4} a^n - T \sum_{n=1}^{c_4+1} a^n \right) & \text{if } D \notin F^2. \end{cases}$$

It is bounded by

$$2(c_2+1)(c_4+1)|1-bT|_{\mathbb{C}}.$$

(7) The partial sum of (6.9) corresponding to the seventh term of (7.4) is equal to the following:

If $\operatorname{ord}(D)=0$, then

$$(1-bT)\sum_{k_1=0}^{c_2} b^{k_1} \sum_{\substack{-\infty\leqq m\leqq 0\\ c_2-2m=c_3-m=c_4}} \sum_{n=1}^{\infty}(aT)^n$$
$$\times \int_{\pi^{-m}\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times} \langle(\alpha_2u_3^2+2\alpha_3u_3+\alpha_4)u_4\rangle du_3du_4$$
$$=\begin{cases} T^{-m}(1-bT)\displaystyle\sum_{k_1=0}^{c_2} b^{k_1}\left((1-3T)\sum_{n=1}^{c_4}a^n+T(1+T)\sum_{n=1}^{c_4+1}a^n\right) & \text{if } D\in F^2,\\ T^{-m}(1-bT)(1-T)\displaystyle\sum_{k_1=0}^{c_2} b^{k_1}\left(\sum_{n=1}^{c_4}a^n-T\sum_{n=1}^{c_4+1}a^n\right) & \text{if } D\notin F^2.\end{cases}$$

If $\operatorname{ord}(D)=2j+1$, then

$$(1-bT)\sum_{k_1=0}^{c_2} b^{k_1} \sum_{\substack{-\infty\leqq m\leqq 0\\ c_2-2m=c_3-m=c_4}} \sum_{n=1}^{\infty}(aT)^n$$
$$\times \int_{\pi^{-m}\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times} \langle(\alpha_2u_3^2+2\alpha_3u_3+\alpha_4)u_4\rangle du_3du_4$$
$$=T^{-m}(1-bT)\sum_{k_1=0}^{c_2} b^{k_1}\Bigg(\sum_{n=1}^{c_4}(aT)^nT^{-n}(1-T)^2+(aT)^{c_4+1}T^{-((c_4+1)-2)}$$
$$+(aT)^{c_4+2}T^{-((c_4+2)-1)}(1-T)+\sum_{\substack{c_4+4\leqq n\leqq c_4+2j\\ c_4-n:\text{even}}}(aT)^nT^{-(c_4+n)/2}(1-T)$$
$$-(aT)^{c_4+2j+2}T^{-(-j+(c_4+2j+2)-2)}\Bigg)$$
$$=T^{-m}(1-bT)\sum_{k_1=0}^{c_2} b^{k_1}\Bigg((1-2T)\sum_{n=1}^{c_4}a^n-T(1-2T)\sum_{n=1}^{c_4+1}a^n+T(1-T)\sum_{n=1}^{c_4+2}a^n$$
$$+T^{2+j}\sum_{n=1}^{c_4+2j+1}a^n-T^{2+j}\sum_{n=1}^{c_4+2j+2}a^n+a^{c_4+4}T^2(1-T)\sum_{k_2=0}^{j-2}(a^2T)^{k_2}\Bigg).$$

Since

$$c_4+2j+2=c_\alpha+1-c_2\leqq c_\alpha+1,$$

it is bounded by

$$6(c_2+1)(c_\alpha+1)|1-bT|_{\mathbb{C}}.$$

If $\mathrm{ord}(D)=2j+2$, then

$$
\begin{aligned}
&(1-bT)\sum_{k_1=0}^{c_2} b^{k_1} \sum_{\substack{-\infty\leqq m\leqq 0\\ c_2-2m=c_3-m=c_4}} \sum_{n=1}^{\infty}(aT)^n \\
&\times \int_{\pi^{-m}\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times} \langle(\alpha_2u_3^2+2\alpha_3u_3+\alpha_4)u_4\rangle du_3du_4 \\
&=T^{-m}(1-bT)\sum_{k_1=0}^{c_2} b^{k_1}\Bigg(\sum_{n=1}^{c_4}(aT)^nT^{-n}(1-T)^2+(aT)^{c_4+1}T^{-((c_4+1)-2)} \\
&+(aT)^{c_4+2}T^{-((c_4+2)-1)}(1-T)+\sum_{\substack{c_4+4\leqq n\leqq c_4+2j\\ c_4-n:\text{even}}}(aT)^nT^{-(c_4+n)/2}(1-T) \\
&+(aT)^{c_4+2j+2}T^{-(-j+(c_4+2j+2)-1)}(1-T)\pm(aT)^{c_4+2j+3}T^{-(-j+(c_4+2j+3)-2)}\Bigg) \\
&=T^{-m}(1-bT)\sum_{k_1=0}^{c_2} b^{k_1}\Bigg((1-2T)\sum_{n=1}^{c_4}a^n-T(1-2T)\sum_{n=1}^{c_4+1}a^n+T(1-T)\sum_{n=1}^{c_4+2}a^n \\
&-T^{1+j}(1-T)\sum_{n=1}^{c_4+2j+1}a^n+T^{1+j}(1-T)\sum_{n=1}^{c_4+2j+2}a^n\pm T^{j+2}a^{c_4+2j+3} \\
&+a^{c_4+4}T^2(1-T)\sum_{k_2=0}^{j-2}(a^2T)^{k_2}\Bigg),
\end{aligned}
$$

where the $+$ (resp. $-$) sign correspond to $D\in F^2$ (resp. $D\notin F^2$). Since $c_4+2j+3\leqq c_\alpha+1$, it is bounded by

$$7(c_2+1)(c_\alpha+1)|1-bT|_{\mathbb{C}}.$$

Therefore, the second term of (6.3) is bounded by

$$
\begin{cases}
19(c_2+1)(c_4+1)(c_\alpha+1)|1-bT|_{\mathbb{C}} & \text{if } \alpha_4\neq 0,\\
10(c_2+1)(c_3+1)|1-bT|_{\mathbb{C}} & \text{if } \alpha_4=0.
\end{cases}
$$

□

Lemma 7.5. *The third term of (6.3) is bounded by*

$$20(\mathrm{ord}(\alpha_2)+1)(\mathrm{ord}(\beta_3)+1)(\mathrm{ord}(\beta_4)+1)(\mathrm{ord}(D_\alpha)+1)|1-bT|_{\mathbb{C}}.$$

Proof. We divide the sum (6.10) into at most seven terms by Proposition 6.13 and show that each term is bounded by

$$8(\mathrm{ord}(\alpha_2)+1)(\mathrm{ord}(\beta_3)+1)(\mathrm{ord}(\beta_4)+1)(\mathrm{ord}(D_\alpha)+1)|1-bT|_{\mathbb{C}}.$$

We divide the sum as follows:

$$\sum_{\substack{1\leqq m\leqq\infty\\ c_2-2m\geqq 0}} = \sum_{\substack{1\leqq m\leqq\infty\\ c_2-2m\geqq 0\\ c_4<c_2-2m\\ c_4<c_3-m}} + \sum_{\substack{1\leqq m\leqq\infty\\ c_2-2m\geqq 0\\ c_3-m<c_2-2m\\ c_3-m<c_4}} + \sum_{\substack{1\leqq m\leqq\infty\\ c_2-2m\geqq 0\\ c_2-2m<c_3-m\\ c_2-2m<c_4}} + \sum_{\substack{1\leqq m\leqq\infty\\ c_2-2m\geqq 0\\ c_2-2m=c_3-m<c_4}} \\ + \sum_{\substack{1\leqq m\leqq\infty\\ c_2-2m\geqq 0\\ c_3-m=c_4<c_2-2m}} + \sum_{\substack{1\leqq m\leqq\infty\\ c_2-2m\geqq 0\\ c_2-2m=c_4<c_3-m}} + \sum_{\substack{1\leqq m\leqq\infty\\ c_2-2m\geqq 0\\ c_2-2m=c_3-m=c_4}} . \tag{7.6}$$

If there exists no such m, then we regard the corresponding term of (6.10) as 0.

(1) The partial sum of (6.10) corresponding to the first term of (7.6) is equal to

$$(1-bT)\sum_{\substack{1\leqq m\leqq\infty\\ c_2-2m\geqq 0\\ c_4<c_2-2m\\ c_4<c_3-m}} \sum_{n=1}^{\infty}(abT)^m(aT)^n\sum_{k_1=0}^{c_2-2m} b^{k_1} \\ \times\int_{\pi^{-m}\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times}\langle(\alpha_2u_3^2+2\alpha_3u_3+\alpha_4)u_4\rangle du_3du_4 \\ =(1-bT)(1-T)\sum_m(ab)^m\sum_{k_1=0}^{c_2-2m}b^{k_1}\left(\sum_{n=1}^{c_4}a^n-T\sum_{n=1}^{c_4+1}a^n\right).$$

Since

$$\sum_m 1\leqq\begin{cases} c_3-c_4<c_3+1 & \text{if } \alpha_3\neq 0,\\ (c_2-c_4)/2<c_2+c_4+1=c_\alpha+1 & \text{if } \alpha_3=0,\end{cases}$$

$$\left|\sum_m(ab)^m\sum_{k_1=0}^{c_2-2m}b^{k_1}\right|_{\mathbb{C}}\leqq\left|\sum_m(c_2+1)\right|_{\mathbb{C}}\leqq\begin{cases}(c_2+1)(c_3+1) & \text{if } \alpha_3\neq 0,\\ (c_2+1)(c_\alpha+1) & \text{if } \alpha_3=0.\end{cases}$$

So it is bounded by

$$\begin{cases}2(c_2+1)(c_3+1)(c_4+1)|1-bT|_{\mathbb{C}} & \text{if } \alpha_3\neq 0,\\ 2(c_2+1)(c_4+1)(c_\alpha+1)|1-bT|_{\mathbb{C}} & \text{if } \alpha_3=0.\end{cases}$$

(2) The partial sum of (6.10) corresponding to the second term of (7.6) is equal to

$$(1-bT)\sum_{\substack{1\leqq m\leqq\infty\\ c_2-2m\geqq 0\\ c_3-m<c_2-2m\\ c_3-m<c_4}} \sum_{n=1}^{\infty}(abT)^m(aT)^n\sum_{k_1=0}^{c_2-2m} b^{k_1} \\ \times\int_{\pi^{-m}\mathcal{O}^\times\times\pi^{-n}\mathcal{O}^\times}\langle(\alpha_2u_3^2+2\alpha_3u_3+\alpha_4)u_4\rangle du_3du_4$$

$$= (1-bT)(1-T) \sum_{\substack{1\leqq m\leqq \infty \\ c_2-2m\geqq 0 \\ c_3-m<c_2-2m \\ c_3-m<c_4 \\ c_3-m\geqq 0}} (ab)^m \sum_{k_1=0}^{c_2-2m} b^{k_1} \left(\sum_{n=1}^{c_3-m} a^n - T \sum_{n=1}^{c_3-m+1} a^n \right).$$

Since

$$c_3 - m + 1 \leqq \begin{cases} c_4+1 & \text{if } \alpha_4 \neq 0, \\ 2c_3+1 = c_\alpha + 1 & \text{if } \alpha_4 = 0, \end{cases}$$

it is bounded by

$$\begin{cases} 2(c_2+1)(c_3+1)(c_4+1)|1-bT|_{\mathbb{C}} & \text{if } \alpha_4 \neq 0, \\ 2(c_2+1)(c_3+1)(c_\alpha+1)|1-bT|_{\mathbb{C}} & \text{if } \alpha_4 = 0. \end{cases}$$

(3) The partial sum of (6.10) corresponding to the third term of (7.6) is equal to

$$(1-bT) \sum_{\substack{1\leqq m\leqq \infty \\ c_2-2m\geqq 0 \\ c_2-2m<c_3-m \\ c_2-2m<c_4}} \sum_{n=1}^{\infty} (abT)^m (aT)^n \sum_{k_1=0}^{c_2-2m} b^{k_1}$$

$$\times \int_{\pi^{-m}\mathcal{O}^\times \times \pi^{-n}\mathcal{O}^\times} \langle (\alpha_2 u_3^2 + 2\alpha_3 u_3 + \alpha_4) u_4 \rangle du_3 du_4$$

$$= (1-bT)(1-T) \sum_m (ab)^m \sum_{k_1=0}^{c_2-2m} b^{k_1} \left(\sum_{n=1}^{c_2-2m} a^n - T \sum_{n=1}^{c_2-2m+1} a^n \right).$$

Since

$$\sum_m 1 \leqq \begin{cases} c_2/2 - (c_2-c_3) \leqq c_3+1 & \text{if } \alpha_3 \neq 0 \\ c_2/2 - (c_2-c_4)/2 \leqq c_4+1 & \text{if } \alpha_3 = 0 \end{cases}$$

and

$$c_2 - 2m + 1 \leqq \begin{cases} c_4+1 & \text{if } \alpha_3\alpha_4 \neq 0, \\ c_\alpha + 1 & \text{if } \alpha_3\alpha_4 = 0, \end{cases}$$

it is bounded by

$$\begin{cases} 2(c_2+1)(c_3+1)(c_4+1)|1-bT|_{\mathbb{C}} & \text{if } \alpha_3\alpha_4 \neq 0, \\ 2(c_2+1)(c_3+1)(c_\alpha+1)|1-bT|_{\mathbb{C}} & \text{if } \alpha_3 \neq 0, \alpha_4 = 0, \\ 2(c_2+1)(c_4+1)(c_\alpha+1)|1-bT|_{\mathbb{C}} & \text{if } \alpha_3 = 0, \alpha_4 \neq 0. \end{cases}$$

(4) The partial sum of (6.10) corresponding to the fourth term of (7.6) is equal to

$$(1-bT) \sum_{\substack{1\leqq m\leqq \infty \\ c_2-2m\geqq 0 \\ c_2-2m=c_3-m<c_4}} \sum_{n=1}^{\infty} (abT)^m (aT)^n \sum_{k_1=0}^{c_2-2m} b^{k_1}$$

$$\times \int_{\pi^{-m}\mathcal{O}^\times \times \pi^{-n}\mathcal{O}^\times} \langle (\alpha_2 u_3^2 + 2\alpha_3 u_3 + \alpha_4) u_4 \rangle du_3 du_4$$
$$= (1-bT)(ab)^m \sum_{k_1=0}^{c_2-2m} b^{k_1} \left((1-2T) \sum_{n=1}^{c_3-m} a^n + T^2 \sum_{n=1}^{c_3-m+1} a^n \right).$$

It is bounded by

$$2(c_2+1)(c_3+1)|1-bT|_{\mathbb{C}}.$$

(5) The partial sum of (6.10) corresponding to the fifth term of (7.6) is equal to

$$(1-bT) \sum_{\substack{1 \leqq m \leqq \infty \\ c_2-2m \geqq 0 \\ c_3-m=c_4<c_2-2m}} \sum_{n=1}^{\infty} (abT)^m (aT)^n \sum_{k_1=0}^{c_2-2m} b^{k_1}$$
$$\times \int_{\pi^{-m}\mathcal{O}^\times \times \pi^{-n}\mathcal{O}^\times} \langle (\alpha_2 u_3^2 + 2\alpha_3 u_3 + \alpha_4) u_4 \rangle du_3 du_4$$
$$= (1-bT)(ab)^m \sum_{k_1=0}^{c_2-2m} b^{k_1} \left((1-2T) \sum_{n=1}^{c_3-m} a^n + T^2 \sum_{n=1}^{c_3-m+1} a^n \right).$$

It is bounded by

$$2(c_2+1)(c_3+1)|1-bT|_{\mathbb{C}}.$$

(6) The partial sum of (6.10) corresponding to the sixth term of (7.6) is equal to

$$(1-bT) \sum_{\substack{1 \leqq m \leqq \infty \\ c_2-2m \geqq 0 \\ c_2-2m=c_4<c_3-m}} \sum_{n=1}^{\infty} (abT)^m (aT)^n \sum_{k_1=0}^{c_2-2m} b^{k_1}$$
$$\times \int_{\pi^{-m}\mathcal{O}^\times \times \pi^{-n}\mathcal{O}^\times} \langle (\alpha_2 u_3^2 + 2\alpha_3 u_3 + \alpha_4) u_4 \rangle du_3 du_4$$
$$= \begin{cases} (1-bT)(ab)^m \sum_{k_1=0}^{c_2-2m} b^{k_1} \left((1-3T) \sum_{n=1}^{c_4} a^n + T(1+T) \sum_{n=1}^{c_4+1} a^n \right) & \text{if } D \in F^2, \\ (1-bT)(1-T)(ab)^m \sum_{k_1=0}^{c_2-2m} b^{k_1} \left(\sum_{n=1}^{c_4} a^n - T \sum_{n=1}^{c_4+1} a^n \right) & \text{if } D \notin F^2. \end{cases}$$

It is bounded by

$$2(c_2+1)(c_4+1)|1-bT|_{\mathbb{C}}.$$

(7) The partial sum of (6.10) corresponding to the seventh term of (7.6) is equal to the following:
If $\mathrm{ord}(D) = 0$, then

$$(1-bT) \sum_{\substack{1 \leqq m \leqq \infty \\ c_2-2m \geqq 0 \\ c_2-2m=c_3-m=c_4}} \sum_{n=1}^{\infty} (abT)^m (aT)^n \sum_{k_1=0}^{c_2-2m} b^{k_1}$$

$$\times \int_{\pi^{-m}\mathcal{O}^\times \times \pi^{-n}\mathcal{O}^\times} \langle (\alpha_2 u_3^2 + 2\alpha_3 u_3 + \alpha_4) u_4 \rangle du_3 du_4$$

$$= \begin{cases} (1-bT)(ab)^m \sum_{k_1=0}^{c_2-2m} b^{k_1} \left((1-3T) \sum_{n=1}^{c_4} a^n + T(1+T) \sum_{n=1}^{c_4+1} a^n \right) & \text{if } D \in F^2, \\ (1-bT)(1-T)(ab)^m \sum_{k_1=0}^{c_2-2m} b^{k_1} \left(\sum_{n=1}^{c_4} a^n - T \sum_{n=1}^{c_4+1} a^n \right) & \text{if } D \notin F^2. \end{cases}$$

It is bounded by

$$2(c_2+1)(c_4+1)|1-bT|_{\mathbb{C}}.$$

If $\mathrm{ord}(D) = 2j+1$, then

$$(1-bT) \sum_{\substack{1 \leqq m \leqq \infty \\ c_2-2m \geqq 0 \\ c_2-2m=c_3-m=c_4}} \sum_{n=1}^{\infty} (abT)^m (aT)^n \sum_{k_1=0}^{c_2-2m} b^{k_1}$$

$$\times \int_{\pi^{-m}\mathcal{O}^\times \times \pi^{-n}\mathcal{O}^\times} \langle (\alpha_2 u_3^2 + 2\alpha_3 u_3 + \alpha_4) u_4 \rangle du_3 du_4$$

$$= (1-bT)(ab)^m \sum_{k_1=0}^{c_2-2m} b^{k_1} \Bigg((1-2T) \sum_{n=1}^{c_4} a^n - T(1-2T) \sum_{n=1}^{c_4+1} a^n + T(1-T) \sum_{n=1}^{c_4+2} a^n$$

$$+ T^{j+2} \sum_{n=1}^{c_4+2j+1} a^n - T^{j+2} \sum_{n=1}^{c_4+2j+2} a^n + a^{c_4+4} T^2 (1-T) \sum_{k_1=0}^{j-2} (a^2 T)^{k_1} \Bigg).$$

Since

$$c_4 + 2j + 2 = c_\alpha + 1 - c_2 \leqq c_\alpha + 1,$$

it is bounded by

$$6(c_2+1)(c_\alpha+1)|1-bT|_{\mathbb{C}}.$$

If $\mathrm{ord}(D) = 2j+2$, then

$$(1-bT) \sum_{\substack{1 \leqq m \leqq \infty \\ c_2-2m \geqq 0 \\ c_2-2m=c_3-m=c_4}} \sum_{n=1}^{\infty} (abT)^m (aT)^n \sum_{k_1=0}^{c_2-2m} b^{k_1}$$

$$\times \int_{\pi^{-m}\mathcal{O}^\times \times \pi^{-n}\mathcal{O}^\times} \langle (\alpha_2 u_3^2 + 2\alpha_3 u_3 + \alpha_4) u_4 \rangle du_3 du_4$$

$$= (1-bT)(ab)^m \sum_{k_1=0}^{c_2-2m} b^{k_1} \Bigg((1-2T) \sum_{n=1}^{c_4} a^n - T(1-2T) \sum_{n=1}^{c_4+1} a^n + T(1-T) \sum_{n=1}^{c_4+2} a^n$$

$$- T^{j+1}(1-T) \sum_{n=1}^{c_4+2j+1} a^n + T^{j+1}(1-T) \sum_{n=1}^{c_4+2j+2} a^n + a^{c_4+4} T^2 (1-T) \sum_{k_1=0}^{j-2} (a^2 T)^{k_1}$$

$$\pm T^{j+2} a^{c_4+2j+3} \Bigg),$$

where the + (resp. $-$) sign of the last term corresponds to $D \in F^2$ (resp. $D \notin F^2$). Since

$$c_4 + 2j + 3 = c_\alpha + 1 - c_2 \leqq c_\alpha + 1,$$

it is bounded by

$$8(c_2 + 1)(c_\alpha + 1)|1 - bT|_{\mathbb{C}}.$$

□

We continue the proof of Theorem 4.6. If α_2, D_α are units, then

$$|\overline{W}(\alpha, z)|_{\mathbb{C}} = \left| \frac{1 \pm q^{-(z_1 - z_3 + 1)}}{1 + q^{-(z_1 - z_3 + 1)}} \right|_{\mathbb{C}} \leqq 1$$

by easy computations. In this case, we take $e_1 = e_2 = 0$. Suppose that $\alpha_2 D_\alpha$ is not a unit. By Lemma 7.1, there exists a constant $C > 0$ such that

$$\begin{aligned} |\overline{W}(\alpha, z)|_{\mathbb{C}} &= \left| \frac{W(\alpha, z)}{(1 + q^{-(z_1 - z_3 + 1)})(1 - q^{-(z_2 - z_4 + 1)})} \right|_{\mathbb{C}} \\ &\leqq C(\mathrm{ord}(\alpha_2) + 1)(\mathrm{ord}(\beta_3) + 1)(\mathrm{ord}(\beta_4) + 1)(\mathrm{ord}(D_\alpha) + 1). \end{aligned}$$

If $|\alpha_4| > |\alpha_3| > |\alpha_2|$, then

$$C(\mathrm{ord}(\alpha_2)+1)(\mathrm{ord}(\beta_3)+1)(\mathrm{ord}(\beta_4)+1)(\mathrm{ord}(D_\alpha)+1) \leqq C(\mathrm{ord}(\alpha_2)+1)^3(\mathrm{ord}(D_\alpha)+1).$$

Otherwise, we can assume that α is one of the following two types:

$$\begin{pmatrix} \alpha_2 & \\ & -D_\alpha/\alpha_2 \end{pmatrix}, \begin{pmatrix} \alpha_2 & \sqrt{D_\alpha} \\ \sqrt{D_\alpha} & \end{pmatrix}$$

by the remark after Lemma 6.2. Since

$$\mathrm{ord}(D_\alpha) - \mathrm{ord}(\alpha_2) + 1 \leqq \mathrm{ord}(D_\alpha) + 1, \mathrm{ord}(D_\alpha)/2 + 1 \leqq \mathrm{ord}(D_\alpha) + 1,$$

$$\begin{aligned} &(\mathrm{ord}(\alpha_2) + 1)(\mathrm{ord}(\beta_3) + 1)(\mathrm{ord}(\beta_4) + 1)(\mathrm{ord}(D_\alpha) + 1) \\ \leqq &(\mathrm{ord}(\alpha_2) + 1)(\mathrm{ord}(D_\alpha) + 1)^2 \leqq (\mathrm{ord}(\alpha_2) + 1)^3(\mathrm{ord}(D_\alpha) + 1)^2. \end{aligned}$$

Let $e_1 = \log_2(C) + 3, e_2 = \log_2(C) + 2$. Since $\mathrm{ord}(\alpha_2) + 1 \geqq 2$ or $\mathrm{ord}(D_\alpha) + 1 \geqq 2$,

$$|\overline{W}(\alpha, z)|_{\mathbb{C}} \leqq (\mathrm{ord}(\alpha_2) + 1)^{e_1}(\mathrm{ord}(D_\alpha) + 1)^{e_2}.$$

This completes the proof of Theorem 4.6. □

8. Estimate of the Euler product

In this section, we prove an estimate of the finite part of the generalized Whittaker function. We first establish basic notations used in this section. Let k be a number field, o_k the integer ring of k and $\mathfrak{M}_{\mathrm{f}}$ the set of all finite places of k. We denote by k_v the completion of k at the place $v \in \mathfrak{M}_{\mathrm{f}}$. Let $\mathbb{A}_{\mathrm{f}}$ (resp. $\mathbb{A}_{\mathrm{f}}^\times$) be the restricted product of the k_v's (resp. $k_v^\times$'s) over $v \in \mathfrak{M}_{\mathrm{f}}$ and We express elements of $\mathbb{A}_{\mathrm{f}}$ as $x_{\mathrm{f}} = (x_v)_{v \in \mathfrak{M}_{\mathrm{f}}}$. Let $W_v(\alpha_v, z)$ be the p-adic generalized Whittaker function for $F = k_v$ defined in Definition 4.2.

Definition 8.1. We define

$$W_{\mathrm{f}}(\alpha_{\mathrm{f}}, z) = \prod_{v \in \mathfrak{M}_{\mathrm{f}}} W_v(\alpha_v, z) \text{ for } \alpha_{\mathrm{f}} = (\alpha_v)_{v \in \mathfrak{M}_{\mathrm{f}}} \in \mathrm{GL}(2)_{\mathbb{A}_{\mathrm{f}}^\times},$$

$$\overline{W}_{\mathrm{f}}(\alpha_{\mathrm{f}}, z) = \prod_{v \in \mathfrak{M}_{\mathrm{f}}} \overline{W}_v(\alpha_v, z) \text{ for } \alpha_{\mathrm{f}} = (\alpha_v)_{v \in \mathfrak{M}_{\mathrm{f}}} \in \mathrm{GL}(2)_{\mathbb{A}_{\mathrm{f}}^\times}.$$

Definition 8.2. For $\alpha \in \mathbb{A}_{\mathrm{f}}^\times$, we define

$$d(\alpha) = \prod_{v \in \mathfrak{M}_{\mathrm{f}}} (\mathrm{ord}_v(\alpha) + 1), \ \mathrm{N}(\alpha) = \prod_{v \in \mathfrak{M}_{\mathrm{f}}} |\alpha|_v^{-1} = |\alpha|_{\mathrm{f}}^{-1}.$$

The function $d(\alpha)$ is the divisor sum and $\mathrm{N}(\alpha)$ is the ideal norm.

Theorem 8.3. *Let $r_1, r_2, \varepsilon > 0$. Then there exists a constant $C(r_1, r_2, \varepsilon)$ such that if $\alpha_{2,\mathrm{f}} \in \mathbb{A}_{\mathrm{f}}^\times$ and $z \in S(r_1, r_2)$, then*

$$|\overline{W}_{\mathrm{f}}(\alpha_{\mathrm{f}}, z)|_{\mathbb{C}} \leqq C(r_1, r_2, \varepsilon) \mathrm{N}(\alpha_2)^\varepsilon \mathrm{N}(D_\alpha)^\varepsilon.$$

The estimate of the p-adic Whittaker function at non-dyadic places has already been obtained. So we estimate $\overline{W}(\alpha, z)$ at the dyadic places in the following lemma.

Lemma 8.4. *Suppose that v is any finite place. Then there exists a constant $C(r_1, r_2)$ such that*

$$|\overline{W}_v(\alpha_v, z)|_{\mathbb{C}} \leqq C(r_1, r_2)$$

for $z \in S(r_1, r_2)$.

Proof. By Lemma 4.9,

$$\begin{aligned} |W_v(\alpha_v, z)|_{\mathbb{C}} &\leqq \int_{F \times F \times F} \left| |c(u_2)|^{z_2 - z_4 + 1} |c(u_3)|^{z_1 + z_2 - z_3 - z_4 + 1} |c(u_4)|^{z_1 - z_3 + 1} \right|_{\mathbb{C}} du_2 du_3 du_4 \\ &\leqq \left| \frac{(1 - aT)(1 - bT)(1 - abT)}{(1 - a)(1 - b)(1 - ab)} \right| \\ &\leqq \frac{8}{(1 - q^{-r_1})(1 - q^{-r_1 - r_2})(1 - q^{-r_2})}. \end{aligned}$$

Therefore,

$$|\overline{W}_v(\alpha_v, z)|_{\mathbb{C}} = \left| \frac{W_v(\alpha_v, z)}{(1 + aT)(1 - bT)} \right|_{\mathbb{C}} \leqq \frac{32}{(1 - q^{-r_1})(1 - q^{-r_1 - r_2})(1 - q^{-r_2})}.$$

□

Proof of Theorem 8.3. Note that the number of the dyadic places is finite. Therefore, there exist constants $C(r_1, r_2), e_1, e_2$ such that

$$|\overline{W}_{\mathrm{f}}(\alpha_{\mathrm{f}}, z)|_{\mathbb{C}} \leqq C(r_1, r_2) d(\alpha_2)^{e_1} d(D_\alpha)^{e_2}$$

for $z \in S(r_1, r_2)$ by Theorem 4.6, Lemma 8.4.

Since for any $\varepsilon > 0$

$$\frac{d(\alpha)}{\mathrm{N}(\alpha)^\varepsilon} \to 0 \text{ as } |\alpha| \to \infty,$$

there exists a constant $C(r_1, r_2, \varepsilon)$ such that

$$|\overline{W}_{\mathrm{f}}(\alpha_{\mathrm{f}}, z)|_{\mathbb{C}} \leqq C(r_1, r_2, \varepsilon) \mathrm{N}(\alpha_2)^\varepsilon \mathrm{N}(D_\alpha)^\varepsilon$$

for $z \in S(r_1, r_2), \varepsilon > 0$. This finishes the proof of Theorem 8.3. □

Acknowledgment

The author would like to express his great gratitude to his advisor Professor Akihiko Yukie for his careful and constant advice for the study of this paper and suggesting the study of generalized Whittaker functions to the author. The author would like to express his hearty thanks to Professor Toyofumi Takahashi for guiding the author in the undergraduate seminar. The author also thanks to Professor Yasuo Morita for his many valuable comments.

Appendix A. Some examples

In this section, we provide examples of the values of $W(\alpha, z)$. Since $W(\alpha, z)$ depends only on the order of $\alpha_2, \alpha_3, \alpha_4, D_\alpha$ and whether or not D_α is a quadratic residue in F, we use the following notation to describe the cases of $W(\alpha, z)$:

$$\left[\begin{pmatrix} \beta_2 & \beta_3 \\ \beta_3 & \beta_4 \end{pmatrix}, \beta_5, \beta_6 \right],$$

where $\beta_i = \mathrm{ord}(\alpha_i)$ for $i = 2, 3, 4, \beta_5 = \mathrm{ord}(D_\alpha), \beta_6 = \begin{cases} + & \text{if } D_\alpha \in F^2, \\ - & \text{if } D_\alpha \notin F^2. \end{cases}$

If some α_i is equal to zero, then we leave the corresponding entry blank.

Example A.1. By the above notation and (6.1) , $W(\alpha, z)$ is as follows;

	$W(\alpha, z)$
$[\left(\begin{smallmatrix} 0 & \\ & 0 \end{smallmatrix}\right), 0, +]$	$(1+aT)(1-bT)$
$[\left(\begin{smallmatrix} 0 & \\ & 0 \end{smallmatrix}\right), 0, -]$	$(1-aT)(1-bT)$
$[\left(\begin{smallmatrix} 0 & \\ & 1 \end{smallmatrix}\right), 1, \pm]$	$(1+aT)(1-aT)(1-bT)$
$[\left(\begin{smallmatrix} 0 & \\ & 2 \end{smallmatrix}\right), 2, +]$	$(1+aT)(1-bT)(1-aT+a^2T)$
$[\left(\begin{smallmatrix} 0 & \\ & 2 \end{smallmatrix}\right), 2, -]$	$(1-aT)(1-bT)(1+aT+a^2T)$
$[\left(\begin{smallmatrix} 0 & \\ & 3 \end{smallmatrix}\right), 3, \pm]$	$(1+aT)(1-aT)(1-bT)(1+a^2T)$
$[\left(\begin{smallmatrix} 0 & \\ & 4 \end{smallmatrix}\right), 4, +]$	$(1+aT)(1-bT)(1-aT+a^2T-a^3T^2+a^4T^2)$
$[\left(\begin{smallmatrix} 0 & \\ & 4 \end{smallmatrix}\right), 4, -]$	$(1-aT)(1-bT)(1+aT+a^2T+a^3T^2+a^4T^2)$
$[\left(\begin{smallmatrix} 1 & \\ & 0 \end{smallmatrix}\right), 1, \pm]$	$(1+b)(1-aT)(1-bT)$
$[\left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right), 2, +]$	$(1+b)(1-bT)(1+a-aT+a^2T)$
$[\left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right), 2, -]$	$(1+a)(1+b)(1-aT)(1-bT)$
$[\left(\begin{smallmatrix} 1 & \\ & 2 \end{smallmatrix}\right), 3, \pm]$	$(1+b)(1-aT)(1-bT)(1+a+a^2T)$
$[\left(\begin{smallmatrix} 1 & \\ & 3 \end{smallmatrix}\right), 4, +]$	$(1+b)(1-bT)(1+a-aT+a^3T-a^3T^2+a^4T^2)$
$[\left(\begin{smallmatrix} 1 & \\ & 3 \end{smallmatrix}\right), 4, -]$	$(1+a)(1+b)(1-aT)(1-bT)(1+a^2T)$
$[\left(\begin{smallmatrix} 2 & \\ & 0 \end{smallmatrix}\right), 2, +]$	$(1-bT)(1+b+ab+b^2-aT-2abT+a^2bT-ab^2T+a^2bT^2)$
$[\left(\begin{smallmatrix} 2 & \\ & 0 \end{smallmatrix}\right), 2, -]$	$(1-aT)(1-bT)(1+b+ab+b^2-abT)$
$[\left(\begin{smallmatrix} 2 & \\ & 1 \end{smallmatrix}\right), 3, \pm]$	$(1-aT)(1-bT)(1+a+b+2ab+b^2+ab^2-abT)$
$[\left(\begin{smallmatrix} 2 & \\ & 2 \end{smallmatrix}\right), 4, +]$	$(1-bT)(1+a+b+a^2+2ab+b^2+a^2b+ab^2+a^2b^2-aT-a^2T-2abT+a^3T$ $-2a^2bT-ab^2T+a^3bT-a^2b^2T+a^3b^2T+a^2bT^2)$
$[\left(\begin{smallmatrix} 2 & \\ & 2 \end{smallmatrix}\right), 4, -]$	$(1-aT)(1-bT)(1+a+b+a^2+2ab+b^2+a^2b+ab^2+a^2b^2-abT)$
$[\left(\begin{smallmatrix} 3 & \\ & 0 \end{smallmatrix}\right), 3, \pm]$	$(1+b)(1-aT)(1-bT)(1+ab+b^2-abT)$
$[\left(\begin{smallmatrix} 3 & \\ & 1 \end{smallmatrix}\right), 4, +]$	$(1+b)(1-bT)(1+a+ab+b^2+a^2b+ab^2-aT-a^2T-abT-2a^2bT-ab^2T$ $+a^3bT-a^2b^2T+a^2bT^2+a^3bT^2)$
$[\left(\begin{smallmatrix} 3 & \\ & 1 \end{smallmatrix}\right), 4, -]$	$(1+a)(1+b)(1-aT)(1-bT)(1+ab+b^2-abT)$
$[\left(\begin{smallmatrix} 4 & \\ & 0 \end{smallmatrix}\right), 4, +]$	$(1-bT)(1+b+ab+b^2+ab^2+b^3+a^2b^2+ab^3+b^4-aT-2abT-a^2bT-2ab^2T$ $-2a^2b^2T-2ab^3T+a^3b^2T-a^2b^3T-ab^4T+a^2bT^2+a^2b^2T^2+a^3b^2T^2+a^2b^3T^2)$
$[\left(\begin{smallmatrix} 4 & \\ & 0 \end{smallmatrix}\right), 4, -]$	$(1-aT)(1-bT)(1+b+ab+b^2+ab^2+b^3+a^2b^2+ab^3+b^4-abT-ab^2T-ab^3T$ $-a^2b^2T)$
$[\left(\begin{smallmatrix} 1 & 0 \\ 0 & \end{smallmatrix}\right), 0, \pm]$	$(1+b)(1-bT)$
$[\left(\begin{smallmatrix} 2 & 0 \\ 0 & \end{smallmatrix}\right), 0, \pm]$	$(1-bT)(1+b+b^2-abT)$
$[\left(\begin{smallmatrix} 2 & 1 \\ 1 & \end{smallmatrix}\right), 2, \pm]$	$(1-bT)(1+a+b+2ab+b^2+ab^2-aT-2abT-ab^2T+a^2bT^2)$
$[\left(\begin{smallmatrix} 3 & 0 \\ 0 & \end{smallmatrix}\right), 0, \pm]$	$(1+b)(1-bT)(1+b^2-abT)$
$[\left(\begin{smallmatrix} 3 & 1 \\ 1 & \end{smallmatrix}\right), 2, \pm]$	$(1-bT)(1+a+b+2ab+b^2+2ab^2+b^3+ab^3-aT-2abT-a^2bT$ $-2ab^2T-a^2b^2T-ab^3T+a^2bT^2+a^2b^2T^2)$
$[\left(\begin{smallmatrix} 3 & 2 \\ 2 & \end{smallmatrix}\right), 4, \pm]$	$(1+b)(1-bT)(1+a+a^2+ab+b^2+a^2b+ab^2+a^2b^2-aT-a^2T-abT$ $-2a^2bT-ab^2T-a^2b^2T+a^2bT^2+a^3bT^2)$
$[\left(\begin{smallmatrix} 4 & 0 \\ 0 & \end{smallmatrix}\right), 0, \pm]$	$(1-bT)(1+b+b^2+b^3+b^4-abT-ab^2T-ab^3T)$
$[\left(\begin{smallmatrix} 4 & 1 \\ 1 & \end{smallmatrix}\right), 2, \pm]$	$(1-bT)(1+a+b+2ab+b^2+2ab^2+b^3+2ab^3+b^4+ab^4-aT-2abT-a^2bT$ $-2ab^2T-2a^2b^2T-2ab^3T-a^2b^3T-ab^4T+a^2bT^2+a^2b^2T^2+a^2b^3T^2)$
$[\left(\begin{smallmatrix} 4 & 2 \\ 2 & \end{smallmatrix}\right), 4, \pm]$	$(1-bT)(1+a+b+a^2+2ab+b^2+2a^2b+2ab^2+b^3+3a^2b^2+2ab^3+b^4+2a^2b^3$ $+ab^4+a^2b^4-aT-a^2T-2abT-3a^2bT-2ab^2T-a^3bT-4a^2b^2T-2ab^3T$ $-a^3b^2T-3a^2b^3T-ab^4T-a^3b^3T-a^2b^4T+a^2bT^2+a^3bT^2+a^2b^2T^2+2a^3b^2T^2$ $+a^2b^3T^2+a^3b^3T^2)$
$[\left(\begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix}\right), 2, +]$	$(1-bT)(1+b+ab+b^2-aT-2abT-ab^2T+a^3bT+a^2bT^2-a^3bT^2+a^4bT^2)$
$[\left(\begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix}\right), 2, -]$	$(1-aT)(1-bT)(1+b+b^2+ab-abT+a^2bT+a^3bT)$
$[\left(\begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix}\right), 3, \pm]$	$(1-aT)(1-bT)(1+b+b^2+ab-abT+a^2bT+a^3bT+a^4bT^2)$
$[\left(\begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix}\right), 4, +]$	$(1-bT)(1+b+ab+b^2-aT-2abT-ab^2T+a^3bT+a^2bT^2-a^3bT^2+a^5bT^2$ $-a^5bT^3+a^6bT^3)$
$[\left(\begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix}\right), 4, -]$	$(1-aT)(1-bT)(1+b+ab+b^2-abT+a^2bT+a^3bT+a^4bT^2+a^5bT^2)$
$[\left(\begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix}\right), 5, \pm]$	$(1-aT)(1-bT)(1+b+ab+b^2-abT+a^2bT+a^3bT+a^4bT^2+a^5bT^2+a^6bT^3)$
$[\left(\begin{smallmatrix} 3 & 1 \\ 1 & 0 \end{smallmatrix}\right), 2, \pm]$	$(1+b)(1-bT)(1+ab+b^2-aT-abT-ab^2T+a^2bT^2)$
$[\left(\begin{smallmatrix} 3 & 2 \\ 2 & 0 \end{smallmatrix}\right), 3, \pm]$	$(1+b)(1-aT)(1-bT)(1+ab+b^2-abT)$
$[\left(\begin{smallmatrix} 3 & 2 \\ 2 & 1 \end{smallmatrix}\right), 4, \pm]$	$(1+b)(1-aT)(1-bT)(1+a+ab+b^2+a^2b+ab^2-abT-a^2bT+a^3bT$ $+a^4bT+a^5bT^2)$

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