The Navier-Stokes equations and weak Herz spaces*

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Abstract

In this paper, we discuss the Cauchy problem for Navier-Stokes equations in homogeneous weak Herz spaces $W\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$. More precisely, we construct the solution in the class $L^\infty(0,T;W\dot{K}^{\alpha}_{p,q})$ with the initial data in $W\dot{K}^{\alpha}_{p,q}$. Further, we consider the blow-up phenomena of time-local solutions and the uniqueness of global solutions with large initial data in $W\dot{K}^{\alpha}_{p,q}$. Also, we give several embeddings of weak Herz spaces into homogeneous Besov spaces $\dot{B}^{-\alpha}_{p,\infty}(\mathbb{R}^n)$, $(\alpha > 0)$, or $bmo^{-1}(\mathbb{R}^n)$.

1 Introduction

In this paper, we consider the Cauchy problem for the incompressible homogeneous Navier-Stokes equations on whole space $\mathbb{R}^n$,

$$\text{(N-S)} \quad \begin{cases} \partial_t u - \mathit{\Delta} u + (u\cdot\nabla)u + \nabla p = 0, \\ \text{div } u = 0, \\ u(0) = u_0 \end{cases}$$

with the initial data u_0 in homogeneous weak Herz space $W\dot{K}^{\alpha}_{p,q}$. Here $u = (u^1, \cdots, u^n)$ is the unknown velocity vector field, p is the unknown pressure scalar field and $u_0 = (u_0^1, \cdots, u_0^n)$ is the given initial velocity with div $u_0 = \nabla\cdot u_0 = 0$. Solving the Cauchy problem (N-S) can be reduced to finding a divergence free solution u of the integral equation

$$\begin{aligned} \text{(I.E.)} \quad u(t) &= e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}(u\cdot\nabla)u(s)ds \\ &= e^{t\Delta}u_0 - \int_0^t \nabla e^{(t-s)\Delta}\mathbb{P}(u\otimes u)(s)ds \\ &=: e^{t\Delta}u_0 - B(u,u)(t), \end{aligned}$$

where $u\otimes v := (u^i v^j)_{1\le i,j\le n}$ is a matrix valued function whose (i,j) component is $u^i v^j$, $e^{t\Delta}$ is the heat semigroup and $\mathbb{P} = \{\mathbb{P}_{i,j} := \delta_{i,j} + R_iR_j\}_{1\le i,j\le n}$ denotes the Leray-Hopf operator or the Weyl-Helmholtz projection which is the orthogonal projection on solenoidal vector field, where $R_j = (-\Delta)^{-1/2}\partial_j$ is the jth Riesz transform. Of course, the operator $e^{t\Delta}$ is defined by the convolution

$$e^{t\Delta}f(x) := f * G_{\sqrt{t}}(x),$$

*AMS Classifications: 35Q30,76D05.

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where G is the Gaussian $G(x) := \dfrac{1}{(4\pi)^{n/2}} e^{-|x|^2/4}$ and $G_t(x) := t^{-n}G(x/t)$. Formally, if u is a divergence free solution of (I.E.) and

$$
\begin{aligned}
p :&= (-\Delta)^{-1} \sum_{j,k=1}^{n} \partial_j \partial_k (u^j u^k) \\
&= \sum_{j,k=1}^{n} R_j R_k (u^j u^k)
\end{aligned}
$$

then (u, p) solves (N-S). We call the solution of (I.E.) the mild solution of (N-S). There are many papers which studied Navier-Stokes equations on several function spaces, for example, [2], [17], [9], [13], [14], [21], [23], [24], [27], [30], [33], [32], [38], [40], [44], [45] etc... In the present paper, we construct local and global mild solutions of (N-S) with the initial velocity u_0 belonging to the weak Herz space. We also discuss the uniqueness of our solutions and blow-up phenomena of our local solutions. Moreover, we investigate embeddings of weak Herz spaces into Besov spaces or bmo^{-1}.

Here we recall a remark of Cannone [7] to make clear advantages of using the dyadic decomposition of a function in x-space and ξ-space, respectively. In [7], he proved that if the initial data $u_0 \in L^3(\mathbb{R}^3)$ is small in the sense of $\dot{B}^{-1+3/p}_{p,\infty}$ where $3 < p \leq 6$, then there exists a global mild solution of (N-S) in $C([0,\infty), L^3)$. Also , he remarked that for any $f \in L^n(\mathbb{R}^n)$,

$$
\lim_{|k|\to\infty} \|\omega_k f \ ; \dot{B}^{-1+n/p}_{p,\infty}\| = 0
$$

where $\omega_k(x) := e^{ikx}$ and $n < p < \infty$. In other words, for the initial data $u_0 \in L^n$, if u_0 sufficiently oscillates, then there exists a global mild solution $u \in C((0,\infty); L^n)$ to (N-S) with u_0, even if $\|u_0 \ ; L^n\|$ is large. On the other hand, roughly speaking, Theorem 1.3 below allows us to deal with the following function f as the initial data;

$$
f(x) := \sum_{k\in\mathbb{Z}} \frac{1}{|x - x_k|} \chi_k(x)
$$

where $x_k := (\frac{3}{2}2^{k-1}, 0, \cdots, 0) \in \mathbb{R}^n$ and χ_k is the characteristic function of $\{2^{k-1} \leq |x| < 2^k\}$. It is not hard to see that $f \in W\dot{K}^0_{n,\infty}$ and $f \notin \cup_{n<p<\infty} \dot{B}^{-1+n/p}_{p,\infty}$. For the latter, see Theorem 1.7. In other words, the theorem says that even if the initial data u_0 has infinitely singular points, if $\|u_0 \ ; W\dot{K}^0_{n,\infty}\|$ is sufficiently small, then there exists a global mild solution $u \in C((0,\infty); W\dot{K}^0_{n,\infty})$ to (N-S) with u_0. It is well known that $L^{n,\infty} \hookrightarrow \dot{B}^{-1+n/p}_{p,\infty} \hookrightarrow BMO^{-1}$ when $n < p < \infty$ and the existence of global mild solution $u \in L^\infty(0,\infty; BMO^{-1})$ to (N-S) with u_0 when $\|u_0 \ ; BMO^{-1}\|$ is sufficiently small, see [24]. Since we disprove $\dot{B}^{-1+n/p}_{p,\infty} \hookrightarrow W\dot{K}^0_{n,\infty}$ and Prof. Akihiko Miyachi proved the inclusion $W\dot{K}^0_{n,\infty} \hookrightarrow BMO^{-1}$ recently, our result are independent from Cannone's result and we have;

$$
\begin{array}{c}
L^n \hookrightarrow L^{n,\infty} \hookrightarrow \dot{B}^{-1+n/p}_{p,\infty} \hookrightarrow BMO^{-1}. \\
\searrow \ W\dot{K}^0_{n,\infty} \nearrow.
\end{array}
$$

Then, our initial data in Theorem 1.3 below were dealt by Koch-Tataru in [24].

In [26], they introduced the Fourier-Besov spaces $F\dot{B}^{\alpha}_{p,q}$ which are defined by

$$
\|f \ ; F\dot{B}^{\alpha}_{p,q}\| := \|\hat{f} \ ; \dot{K}^{\alpha}_{p,q}\|,
$$

where $\hat{f}$ is the Fourier transform of f, see Definition 1.1 below for the definition of the norm $\|\cdot \ ; \dot{K}^{\alpha}_{p,q}\|$.

In the present paper, we use the symbol $\dot{K}^{\alpha}_{p,q}$, instead of $\dot{K}^{\alpha,p}_{q}$, because the parameter p was adopted as the one describing local regularity of functions in many papers studying Navier-Stokes equations. See the below for the precise definition of $\dot{K}^{\alpha}_{p,q}$.

We explain about the Herz spaces. Nonhomogeneous Herz spaces $K^{n(1-1/p)}_{p,1}$ with $1<p<\infty$ are now called Beurling algebras which were introduced by Beurling [4]. Feichtinger [16] gave the different norms of Beurling algebras, which is equivalent to that in [4]. The space $\dot{K}^{\alpha}_{p,q}$ for all p,q and α was introduced by Herz [20] but his definition differs from our definition. In [8], the Hardy spaces associated to the Beurling algebras were introduced, and then these spaces are generalized, are now called Herz-type Hardy spaces. The theories of the Herz space and the Herz-type Hardy space were developed by, for example, [10], [11], [34], [35], [36], [37], [41] etc.... These spaces are useful in the analysis of mapping properties of important operators. For example, Baernstein II and Sawyer [1] showed some multiplier theorems on Hardy space $H^p(\mathbb{R}^n)$ by using a norm of the Herz space. Also, see [48] and the references there for applications of Herz spaces.

Firstly we recall the definitions of Herz spaces and weak Herz spaces.

Definition 1.1. *Let $0<p,q\le\infty$ and $\alpha\in\mathbb{R}$. One defines the homogeneous Herz space $\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$ as*

$$\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n):=\{f\in L^p_{loc}(\mathbb{R}^n\backslash\{0\})\ ;\ \|f\ ;\dot{K}^{\alpha}_{p,q}\|<\infty\}$$

where

$$\|f\ ;\dot{K}^{\alpha}_{p,q}\|:=\Big(\sum_{k\in\mathbb{Z}}2^{k\alpha q}\|f\ ;L^p(A_k)\|^q\Big)^{1/q},$$

with the usual modification in the case $q=\infty$ and $A_k:=\{x\in\mathbb{R}^n;2^{k-1}\le|x|<2^k\}$.

Definition 1.2. *With same exponents as above, one defines the weak Herz spaces $W\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$ by the space of measurable functions f such that*

$$\|f\ ;W\dot{K}^{\alpha}_{p,q}\|:=\Big(\sum_{k\in\mathbb{Z}}2^{k\alpha q}\|f\ ;L^{p,\infty}(A_k)\|^q\Big)^{1/q}<\infty,$$

with the usual modification in the case $q=\infty$, where

$$\|f\ ;L^{p,\infty}(A_k)\|:=\sup_{\lambda>0}\lambda|\{x\in A_k;|f(x)|>\lambda\}|^{1/p}$$

for $p<\infty$ and $L^{\infty,\infty}(A_k)=L^\infty(A_k)$.

A typecal example of $W\dot{K}^{\alpha}_{p,\infty}$ is

$$\sum_{k\in\mathbb{Z}}\frac{2^{-k\alpha}}{|x-x_k|^{n/p}}\chi_k(x),$$

where $x_k:=(\frac{3}{2}2^{k-1},0,\cdots,0)\in\mathbb{R}^n$ and χ_k is the characteristic function of A_k. These functions are used in the proof of Theorem 1.7 below.

Here we remark that there is another type of the weak Herz spaces, [25], which is defined by

$$\|f\ ;\mathscr{K}^{\alpha}_{(p,\infty),q}\|:=\sup_{\lambda>0}\lambda\Big(\sum_{k\in\mathbb{Z}}2^{k\alpha q}|\{x\in A_k;|f(x)|>\lambda\}|^{p/q}\Big)^{1/q}.$$

It is obvious $\|f\ ;\mathscr{K}^{\alpha}_{(p,\infty).q}\|\le\|f\ ;W\dot{K}^{\alpha}_{p,q}\|$, i.e. $W\dot{K}^{\alpha}_{p,q}\hookrightarrow\mathscr{K}^{\alpha}_{(p,\infty),q}$.

Moreover, to state our theorems, we introduce the "smooth" weak Herz spaces;

$$W\dot{\mathcal{K}}^{\alpha}_{p,\infty}:=\{f\in W\dot{K}^{\alpha}_{p,\infty};\ e^{t\Delta}f\ \to\ f\ in\ W\dot{K}^{\alpha}_{p,\infty}\ as\ t\ \searrow\ 0\}.$$

Now we state our theorems. The first one concerns the local existence of mild solution for large data;

Theorem 1.1 (local existence for large data)**.** *Let $n \geq 2,\ n < p < \infty$ and $0 \leq \alpha < 1 - n/p$. Then for every $u_0 \in W\dot{K}^\alpha_{p,\infty}$ with div $u_0 = 0$, there exist a positive constant $T > 0$ and a solution $u \in X_T \cap C((0,T); W\dot{K}^\alpha_{p,\infty})$ of (I.E.)with u_0 so that*

$$\sup_{0<t\leq T} t^{(\alpha+n/p)/2} \|u(t)\ ; L^\infty\| < \infty, \tag{1}$$

$$u(t) \in W\dot{\mathcal{K}}^\alpha_{p,\infty} \ \textit{for}\ t > 0, \tag{2}$$

$$u(t) \to u_0 \ \textit{in the weak}\ast\ \textit{topology as}\ t \searrow 0, \tag{3}$$

$$u(\cdot) - e^{\cdot\Delta} u_0 \in L^\infty(0,T; \dot{K}^\alpha_{p,\infty}), \tag{4}$$

and

$$u(\cdot) - e^{\cdot\Delta} u_0 \ \textit{is right continuous in}\ \dot{\mathcal{K}}^\alpha_{p,\infty}\ \textit{on}\ (0,T), \tag{5}$$

where $X_T = \{u \in L^\infty(0,T; W\dot{K}^\alpha_{p,\infty}); \|u\ ; X_T\| < \infty,\ \mathrm{div}\ u = 0\}$,

$$\begin{aligned}\|u\ ; X_T\| : &= \sup_{0<t\leq T} \|u(t)\ ; W\dot{K}^\alpha_{p,\infty}\| + \sup_{0<t\leq T} t^{1/2} \|\nabla u(t)\ ; W\dot{K}^\alpha_{p,\infty}\| \\ &=: \|u\ ; X_{T,1}\| + \|u\ ; X_{T,2}\|.\end{aligned}$$

Moreover, if $u_0 \in W\dot{\mathcal{K}}^\alpha_{p,\infty}$, then we can prove $u(t) \to u_0$ in $W\dot{K}^\alpha_{p,\infty}$ as $t \searrow 0$. As for uniqueness, u is the only solution of (I.E.) in the class $L^\infty(0,T; W\dot{K}^\alpha_{p,\infty})$.

Remark 1.1. *1. The local solution above is in $\cap_{0<T_1<T_2<T} L^\infty(\mathbb{R}^n \times (T_1,T_2))$. Hence, thanks to the proof of Proposition 15.1 in [31], we obtain the smoothness $u \in C^\infty(\mathbb{R}^n \times (0,T))$.*

2. The life span T of the solution u in Theorem 1.1 is characterized by

$$T = \frac{C}{\|u_0\ ; W\dot{K}^\alpha_{p,\infty}\|^{\frac{2}{1-n/p-\alpha}}},$$

with a constant C depends on the exponents p, α and n only.

3. Because we show the inclusion

$$W\dot{K}^\alpha_{p,\infty} \hookrightarrow bmo^{-1}$$

for $n < p \leq \infty$ and $0 \leq \alpha < 1 - n/p$, see Theorem 1.6, our initial data u_0 was dealt by Koch and Tataru [24], where $bmo^{-1} = F^{-1}_{\infty,2}$ is defined by

$$\|f\ ; bmo^{-1}\| := \sup_{|Q|\leq 1} \left(\fint_Q \int_0^{l(Q)^2} |e^{t\Delta} f(x)|^2 dt dx \right)^{1/2},$$

where the supremum is taken over all cubes Q with the volume $|Q| \leq 1$, $l(Q)$ is the side length of Q and the slashed integral $\fint_Q f dx$ denotes the average $\frac{1}{|Q|} \int_Q f dx$. But Theorem 1.1 is not included in [24], because they used the condition

$$\sup_{0<t\leq T} t^{1/2} \|u(t)\ ; L^\infty\| < \infty$$

to prove the uniqueness of local solutions. We show the uniqueness of our local solutions without such a condition.

4. In the case $u_0 \in \dot{K}^\alpha_{p,\infty}$, we can show Theorem 1.1 in which $W\dot{K}^\alpha_{p,\infty}$ is replaced with $\dot{K}^\alpha_{p,\infty}$. However we omit the details, this fact says that the assumption of Theorem 1.2 below is natural.

Also, we are interested in whether the local solution in Theorem 1.1 blows up at $t = T$ or can be continued beyond $t = T$. Such a problem was considered by Beale, Kato and Majda [3], Giga [12], Kozono and Taniuchi [29], Kozono, Ogawa and Taniuchi [27] and Kozono and Shimada [28]. We can establish an extension criterion on our local-in-time mild solutions by using the new function space $\dot{K}^0_{BMO,\infty}$ which is strictly larger than BMO by applying our previous work [47].

Definition 1.3. *We define the space* $\dot{K}^0_{BMO,\infty}$ *as the space of locally integrable functions* f *on* $\mathbb{R}^n \backslash \{0\}$ *satisfying*

$$\begin{aligned} \|f\ ; \dot{K}^0_{BMO,\infty}\| &:= \sup_{k\in\mathbb{Z}} \|f\ ; BMO(Q_k^*)\| \\ &:= \sup_{k\in\mathbb{Z}} \sup_{Q\subset Q_k^*} \inf_{c\in\mathbb{C}} ⨍_Q |f - c|dy < \infty, \end{aligned}$$

where $Q_k^* := Q_{k-1} \cup Q_k \cup Q_{k+1},\ Q_k := (-2^k, 2^k)^n \backslash (-2^{k-1}, 2^{k-1})^n$ *and the supremum* $\sup\limits_{Q\subset Q_k^*}$ *is taken over all cubes* Q *contained in* Q_k^**. Then,* $\dot{K}^0_{BMO,\infty}/\mathbb{C}$ *is a Banach space under the above norm* $\|\cdot\ ; \dot{K}^0_{BMO,\infty}\|$.

Our extension criterion theorem is the following. Theorems 1.1 and 1.2 are proved in Section 3.

Theorem 1.2 (continuation principle for local mild solutions)**.** *Let* $n \geq 2,\ n < p < \infty,\ 0 \leq \alpha < 1 - n/p$ *and* $0 < T^* < \infty$*. Let* $u_0 \in \dot{K}^\alpha_{p,\infty}$ *with div* $u_0 = 0$ *and* u *be a solution of* (I.E.) *in the class* $L^\infty(0, T; \dot{K}^\alpha_{p,\infty})$ *for all* $T \in (0, T^*)$*. Suppose that* u *satisfies the condition*

$$\int_0^{T^*} \|\nabla u(t)\ ; \dot{K}^0_{BMO,\infty}\|dt < \infty,$$

then there exists $\widetilde{T} > T^*$ *so that* u *can be extended to a solution of* (I.E.) *in the class* $L^\infty(0, \widetilde{T}; \dot{K}^\alpha_{p,\infty})$.

Remark 1.2. *Kozono and Shimada [28] showed that if*

$$\int_0^{T^*} \|u(t)\ ; \dot{F}^{-\alpha}_{\infty,\infty}\|^\gamma dt < \infty$$

for some $0 < \alpha < 1$ *with* $\gamma = 2(1 - \alpha)$*, then the similar continuation principle of strong solutions holds. Here* $\dot{F}^{-\alpha}_{\infty,\infty}$ *is the homogeneous Triebel-Lizorkin space.*

Although the Navier-Stokes equations are invariant with respect to the scaling

$$(u(x, t), p(x, t)) \to (\lambda u(\lambda x, \lambda^2 t), \lambda^2 p(\lambda x, \lambda^2 t)),\ (\lambda > 0)$$

the weak Herz spaces appeared in Theorem 1.1 are not invariant with respect to such a scaling. Roughly speaking, it is well-known that if a function space X is invariant with respect to the scaling, i.e., $\|f(\lambda\cdot)\ ; X\| \approx \lambda^{-1}\|f\ ; X\|$, then the global existence of solutions for small data to (I.E.) is expected. It is not hard to show that $W\dot{K}^{1-n/p}_{p,q}$ and $\dot{K}^{1-n/p}_{p,q}$ are so, see Lemma 2.5 in Section 2. Then the global existence of mild solutions for small data is expected on these spaces. Under the restriction $n \leq p \leq \infty$, since the largest space is $W\dot{K}^0_{n,\infty}$ among $\dot{K}^{1-n/p}_{p,q}$ and $W\dot{K}^{1-n/p}_{p,q}$, we state our global existence theorem for $W\dot{K}^0_{n,\infty}$ only. More precisely, we have the following theorem

Theorem 1.3 (global existence for small data)**.** *Let $n \geq 2$ and $n < p < \infty$. Then, there exists $\delta > 0$ such that for all $u_0 \in W\dot{K}^0_{n,\infty}$ with $\|u_0 \,; W\dot{K}^0_{n,\infty}\| \leq \delta$ and div $u_0 = 0$ there exists a solution $u \in X \cap C((0,\infty); W\dot{K}^0_{n,\infty})$ of* (I.E.) *such that*

$$u(t) \in W\dot{\mathcal{K}}^0_{n,\infty} \ for\ t > 0, \tag{6}$$

$$u(t) \to u_0 \ in\ weak* \ topology\ as\ t \searrow 0, \tag{7}$$

$$u(\cdot) - e^{\cdot\Delta}u_0 \in L^\infty(0,\infty; \dot{K}^0_{n,\infty}), \tag{8}$$

and

$$u(\cdot) - e^{\cdot\Delta}u_0 \ is\ right\ continuous\ in\ \dot{K}^0_{n,\infty}\ on\ (0,\infty), \tag{9}$$

where $X = \{u \in L^\infty(0,\infty; W\dot{K}^0_{n,\infty}); \|u \,; X\| < \infty,\ \mathrm{div}\ u = 0\}$,

$$\begin{aligned}\|u \,; X\| :&= \sup_{t>0} \|u(t) \,; W\dot{K}^0_{n,\infty}\| + \sup_{t>0}\ t^{(1-n/p)/2}\|u(t) \,; \dot{K}^0_{p,\infty}\| \\ &+ \sup_{t>0} t^{1/2}\|u(t) \,; L^\infty\| + \sup_{t>0}\ t^{1/2}\|\nabla u(t) \,; W\dot{K}^0_{n,\infty}\| \\ &=: \|u \,; X_1\| + \|u \,; X_2\| + \|u \,; X_3\| + \|u \,; X_4\|.\end{aligned}$$

Moreover, if $u_0 \in W\dot{\mathcal{K}}^0_{n,\infty}$, then we can show that

$$\begin{aligned}&u(t) \to u_0 \ in\ W\dot{K}^0_{n,\infty}\ as\ t \searrow 0,\ and \\ &\lim_{t\searrow 0} t^{(1-n/p)/2}\|u(t) \,; \dot{K}^0_{p,\infty}\| = \lim_{t\searrow 0} t^{1/2}\|u(t) \,; L^\infty\| = 0.\end{aligned}$$

Remark 1.3. *1. The solution above is in $\bigcap_{0<T_1<T_2<T} L^\infty(\mathbb{R}^n \times (T_1,T_2))$ for every $T \in (0,\infty)$. Hence, thanks to the proof of Proposition 15.1 in [31], we obtain the smoothness $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$.*

2. We know the following inclusion relations from Theorem 1.6; for $2 \leq n < p < \sigma < \infty$

$$W\dot{K}^{1-n/p}_{p,\infty} \hookrightarrow \dot{B}^{-1+n/\sigma}_{\sigma,\infty} \hookrightarrow BMO^{-1},$$

and for $q < \infty$

$$W\dot{K}^0_{n,q} \hookrightarrow \dot{B}^{-1+n/\sigma}_{\sigma,\infty} \hookrightarrow BMO^{-1},$$

where $BMO^{-1} = \dot{F}^{-1}_{\infty,2}$ is defined by

$$\|f \,; BMO^{-1}\| := \sup_Q \Big(\fint_Q \int_0^{l(Q)^2} |e^{t\Delta}f(x)|^2 dtdx \Big)^{1/2},$$

where the supremum is taken over all cubes Q. Therefore if we replace $W\dot{K}^0_{n,\infty}$ with $W\dot{K}^{1-n/p}_{p,\infty}$ or $W\dot{K}^0_{n,q}$ in Theorem 1.3 where $p > n$ and $q < \infty$, then the initial data was dealt in [24]. On the other hand, $W\dot{K}^0_{n,\infty}$ is not included in $\cup_{0<p<\infty} \dot{B}^{-1+n/p}_{p,\infty}$, see Theorem 1.7.

As for uniqueness of global-in-time mild solution, in the case $u_0 \in L^3(\mathbb{R}^3)$, it is well-known that the solution is unique in the class $C([0,T); L^3)$ even if initial data is large, see [18], [19] and [39]. Nevertheless, the existence of global solutions for large data in $L^3(\mathbb{R}^3)$ is open problem. In this connection, Cannone [7] showed the existence of global solution $u \in C([0,\infty); L^3(\mathbb{R}^3))$ when an initial data $u_0 \in L^3(\mathbb{R}^3)$ is sufficiently small in the scale of $\dot{B}^{-1+3/p}_{p,\infty}$ norm, where $3 < p \leq 6$. We can prove an analogous result of the uniqueness for large data in the weak Herz space setting.

Theorem 1.4 (uniqueness of global solution for large data)**.** *Let $n \geq 3$. Let $u_0 \in W\dot{\mathcal{K}}^0_{n,\infty}$ with div $u_0 = 0$. If u and v are divergence free mild solutions of (I.E.) with initial data u_0 in the class $C([0,\infty); W\dot{K}^0_{n,\infty})$ satisfying $u(t),\ v(t) \to u_0$ in $W\dot{K}^0_{n,\infty}$ as $t \searrow 0$, then $u = v$ on $[0,\infty)$.*

Also, as a corollary of the boundedness of B, see Remark 4.1, we can show a stability result of global solution for small data, see [22].

Theorem 1.5 (stability of global solution for small data)**.** *Let $n \geq 3$ and $u_0, v_0 \in W\dot{K}^0_{n,\infty}$. There exists $\delta > 0$ such that if $\|u_0\ ; W\dot{K}^0_{n,\infty}\|,\ \|v_0\ ; W\dot{K}^0_{n,\infty}\| < \delta$, and*

$$\lim_{t\nearrow\infty} \|e^{t\Delta}(u_0 - v_0)\ ; W\dot{K}^0_{n,\infty}\| = 0,$$

then for the solutions u, v constructed in Theorem 1.3 with initial data u_0, v_0, respectively, we have

$$\lim_{t\nearrow\infty} \|u(t) - v(t)\ ; W\dot{K}^0_{n,\infty}\| = 0.$$

Theorems 1.3, 1.4 and 1.5 are proved in Section 4.

Furthermore, we discuss about embeddings of weak Herz spaces into Besov spaces or bmo^{-1}. To prove negative result, we use the characterization of homogeneous Besov space $\dot{B}^{-\alpha}_{p,\infty}$, $(\alpha > 0)$, in terms of the heat semigroup $e^{t\Delta}$. The characterization we use is the following; for $1 \leq \sigma \leq \infty$, $\alpha > 0$ and $f \in \mathcal{S}'$;

$$\sup_{t>0} t^{\alpha/2} \|e^{t\Delta} f\ ; L^\sigma\| \approx \|f\ ; \dot{B}^{-\alpha}_{\sigma,\infty}\|. \tag{10}$$

See [7] or [46], for the details. Theorems 1.6 and 1.7 are proved in Section 5.

Theorem 1.6. (I) (The case $\sigma < \infty$.)
(i): *For $1 < p < \sigma < \infty$ and $0 < \alpha < n(1 - 1/p)$, we have*

$$W\dot{K}^\alpha_{p,\infty} \hookrightarrow \dot{B}^{-(\alpha + n(1/p - 1/\sigma))}_{\sigma,\infty} \tag{11}$$

(ii): *For $1 < p < \infty$ and $0 < \alpha < n(1 - 1/p)$, we have*

$$\dot{K}^\alpha_{p,\infty} \hookrightarrow \dot{B}^{-\alpha}_{p,\infty} \tag{12}$$

(iii): *For $1 < p < \sigma < \infty$, one has*

$$W\dot{K}^0_{p,\sigma} \hookrightarrow \dot{B}^{-n(1/p - 1/\sigma)}_{\sigma,\infty}. \tag{13}$$

(II) (The case $\sigma = \infty$.)
(iv): *For $1 < p < \infty$ and $0 \leq \alpha < n(1 - 1/p)$, we have*

$$W\dot{K}^\alpha_{p,\infty} \hookrightarrow \dot{B}^{-(\alpha + n/p)}_{\infty,\infty} \tag{14}$$

(v): *For $0 \leq \alpha < n$, we have*

$$\dot{K}^\alpha_{\infty,\infty} \hookrightarrow \dot{B}^{-\alpha}_{\infty,\infty} \tag{15}$$

(vi): *For $1 < p \leq \infty$ and $0 \leq \alpha \leq n(1 - 1/p)$, one has*

$$W\dot{K}^\alpha_{p,1} \hookrightarrow \dot{B}^{-(\alpha + n/p)}_{\infty,\infty} \tag{16}$$

(vii): *We have*

$$L^1 = \dot{K}^0_{1,1} \hookrightarrow \dot{B}^{-n}_{\infty,\infty}. \tag{17}$$

(III)
(viii): *For $n < p \leq \infty$ and $0 \leq \alpha < 1 - n/p$, one obtains*

$$W\dot{K}^\alpha_{p,\infty} \hookrightarrow bmo^{-1}. \tag{18}$$

In (11) and (14), lower bounds of α are sharp in the following sense.

Theorem 1.7. (i): (The case $\sigma < \infty$) *For* $1 < p < \sigma < \infty$ *and* $-n(1/p - 1/\sigma) < \alpha \le 0$, *an embedding*

$$W\dot{K}^{\alpha}_{p,\infty} \hookrightarrow \dot{B}^{-(\alpha+n(1/p-1/\sigma))}_{\sigma,\infty} \tag{19}$$

dose not hold.

(ii): (The case $\sigma = \infty$) *For* $1 < p < \infty$ *and* $-n/p < \alpha < 0$, *an embedding*

$$W\dot{K}^{\alpha}_{p,\infty} \hookrightarrow \dot{B}^{-(\alpha+n/p)}_{\infty,\infty} \tag{20}$$

dose not hold.

Remark 1.4. *1. We give some examples of* bmo^{-1}. *For* $n < p < \infty$, $0 \le \alpha < 1 - n/p$ *and* $0 \le \beta < 1$,

$$\sum_{k\in\mathbb{Z}} \frac{2^{-k\alpha}}{|x - x_k|^{n/p}}\chi_k(x),\ \frac{1}{|x|^{\beta}} \in \bigcup_{\substack{n<p\le\infty \\ 0\le\alpha<1-n/p}} W\dot{K}^{\alpha}_{p,\infty},$$

where $x_k := \frac{3}{2}2^{k-1}e_1$ *and* $e_1 := (1, 0, \cdots, 0)$. *Then from (18), these functions belong to* bmo^{-1}.

2. A special case of Theorem 1.7 is

$$W\dot{K}^{1-n/p}_{p,\infty} \not\hookrightarrow \dot{B}^{-1}_{\infty,\infty},$$

where $0 < p < n$. *We remark that Bourgain and Pavlović [5] proved the ill-posedness of the Navier-Stokes equations in* $\dot{B}^{-1}_{\infty,\infty}$. *Recently, in [49] he showed the ill-posedness in* $\dot{B}^{-1}_{\infty,q}$ *for* $q \in (2, \infty)$.

Many of our results above are consequences of the boundedness of convolution operators with some good function on weak Herz spaces, see Propositions 2.1 and 2.2 in Section 2.

This paper is organized as follows. In Section 2 we recall the fundamental facts for Herz spaces and weak Herz spaces, and establish several estimates on weak Herz spaces, that is, the $W\dot{K}^{\alpha}_{p,q} - W\dot{K}^{\beta}_{\sigma,\delta}$ estimates of the heat semigroup $e^{t\Delta}$, the boundedness of a class of operators which contains the fractional integral operator I_α, and the critical estimates of the bilinear form B. In Section 3, we construct the unique local-in-time solution in weak Herz spaces and we prove the extension criterion Theorem 1.2. In Section 4, we prove the existence of global-in-time solution on a scaling invariant space $W\dot{K}^{0}_{n,\infty}$ for small data and the uniqueness of the solution in the class $C([0,\infty); W\dot{K}^{0}_{n,\infty})$ for smooth large data. Finally, in Section 5, we give the proof of Theorems 1.6 and 1.7.

2 Preliminaries

Throughout this paper we use the following notations. $\mathcal{S}$ and $\mathcal{S}'$ denote the Schwartz spaces of rapidly decreasing smooth functions and tempered ditributions, respectively. $A \lesssim B$ means $A \le cB$ with a positive constant c. Also, $A \approx B$ means $c_1 B \le A \le c_2 B$ with positive constants c_1 and c_2. In what follows, c denotes a constant that is independent of the functions involved, which may differ from line to line.

In this section, we recall the fundamental facts on Herz and weak Herz spaces and also establish some propositions, for example, the estimates of heat semigroup and the boundedness of some operator on the weak Herz spaces. We begin with discussions for the relations between the Herz spaces or the weak Herz spaces and other function spaces.

Lebesgue spaces with power weight are special cases of Herz spaces; in the case $p < \infty$

$$\dot{K}^{\alpha}_{p,p} = L^p(|x|^{\alpha p}dx),$$

and

$$\|f\ ;\dot{K}^{\alpha}_{\infty,\infty}\| \approx \||x|^{\alpha}f\ ;L^{\infty}\|.$$

Further, Herz spaces $\dot{K}^{\alpha}_{p,\infty}$ include Morrey spaces $\mathcal{M}^{p}_{q}(\mathbb{R}^n)$ and Lorentz spaces $L^{p,q}(\mathbb{R}^n)$; for $0 < q < p < \infty$,

$$L^{p,\infty} \hookrightarrow \mathcal{M}^{p}_{q} \hookrightarrow \dot{K}^{n(1/p-1/q)}_{q,\infty}.$$

More precisely, we obtain the following;

Lemma 2.1. *Let $0 < q \le p \le \infty$. Then one has the embedding*

$$\mathcal{M}^{p}_{q} \hookrightarrow \dot{K}^{n(1/p-1/q)}_{q,\infty}.$$

Here,

$$\|f\ ;\mathcal{M}^{p}_{q}\| := \sup_{Q} |Q|^{1/p-1/q}\|f\ ;L^{q}(Q)\|,$$

where the supremum is taken over all cubes Q, and

$$\|f\ ;L^{p,q}\| := \Big(\int_0^{\infty} t^{q/p}f^{*}(t)^{q}\frac{dt}{t}\Big)^{1/q},$$

where f^{*} is the decreasing rearrangement of f. The usual modifications in the definitions above are made when $q=\infty$. The scaling invariant spaces $\dot{K}^{0}_{n,\infty}$ are not included in $L^{n,\infty}$. In fact, let $h(x)=1$ if $2^k \le x_i < 2^k+1$ for some $k \in \mathbb{N}$ and all i, $h(x)=0$ if else. Then, it is easy to see that $h \notin L^{n,\infty}$ and $h \in \dot{K}^{0}_{n,\infty}$.

On the other hand, $W\dot{K}^{0}_{n,\infty}$ is scaling invariant, also includes $L^{n,\infty}$. A typical member of $W\dot{K}^{0}_{n,\infty}$ is

$$f(x) := \sum_{k\in\mathbb{Z}} \frac{1}{|x-x_k|}\chi_{A_k}(x).$$

Then, we see that f does not belong to $\cup_{0<p<\infty}\dot{B}^{-1+n/p}_{p,\infty}$, $\cup_{n<p\le\infty}W\dot{K}^{1-n/p}_{p,\infty}$, $\dot{K}^{0}_{n,\infty}$ and $L^{n,\infty}$. Remark that $L^{n,\infty} \hookrightarrow \cup_{n<p<\infty}\dot{B}^{-1+n/p}_{p,\infty}$. See Theorem 1.7 for the proof of $f \notin \dot{B}^{-1+n/p}_{p,\infty}$. But the author knows no inclusion relations between Herz space and Morrey spaces, or weak Herz spaces and weak Morrey spaces which are scaling invariant, see [40] for the definition of weak Morrey spaces.

The following four lemmas are fundamental facts of homogeneous Herz spaces and weak Herz spaces. Because it is not hard to prove them, we omit the proofs.

Lemma 2.2. *Let $1 \le p,q \le \infty$ and $\alpha \in \mathbb{R}$.*
(i): $\mathcal{S} \hookrightarrow \dot{K}^{\alpha}_{p,q} \hookrightarrow W\dot{K}^{\alpha}_{p,q}$, *provided that $q<\infty$ and $-n/p<\alpha<\infty$ or $q=\infty$ and $-n/p \le \alpha < \infty$.*
(ii): $\dot{K}^{\alpha}_{p,q} \hookrightarrow W\dot{K}^{\alpha}_{p,q} \hookrightarrow \mathcal{S}'$, *provided that $q=1$ and $-\infty<\alpha\le n(1-1/p)$ or $1<q\le\infty$ and $-\infty<\alpha<n(1-1/p)$.*

Lemma 2.3 ([41], Lemma 2.3)**.** *Let $0<p,q\le\infty$, $\alpha\in\mathbb{R}$ and $f \in W\dot{K}^{\alpha}_{p,q}$. Then, $f \in L^{r}_{loc}(\mathbb{R}^n)$ for every r satisfying* $\max(\alpha,0)/n + 1/p < 1/r < \infty$.

Lemma 2.4. *Let $0<p_i,q_i\le\infty$ and $\alpha_i\in\mathbb{R}$, $(i=1,2)$. Then, if*

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2},\ \frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}\ \text{and}\ \alpha=\alpha_1+\alpha_2$$

then

$$\|fg\ ;W\dot{K}^{\alpha}_{p,q}\| \le c\|f\ ;W\dot{K}^{\alpha_1}_{p_1,q_1}\|\ \|g\ ;W\dot{K}^{\alpha_2}_{p_2,q_2}\|.$$

To construct the global solutions of (N-S), we need to know the parameters (p,q,α) for which $\dot{K}^{\alpha}_{p,q}$ and $W\dot{K}^{\alpha}_{p,q}$ are scaling invariant. From lemma below, we see that the spaces $\dot{K}^{1-n/p}_{p,q}$ and $W\dot{K}^{1-n/p}_{p,q}$ are so.

Lemma 2.5 ([41], Lemma 2.1)**.** *For $0 < p,q \le \infty,\ \alpha \in \mathbb{R}$ and $0 < \lambda < \infty$,*

$$\|f(\lambda\cdot)\ ;\dot{K}^{\alpha}_{p,q}\| \approx \lambda^{-(\alpha+n/p)}\|f\ ;\dot{K}^{\alpha}_{p,q}\|$$

and

$$\|f(\lambda\cdot)\ ;W\dot{K}^{\alpha}_{p,q}\| \approx \lambda^{-(\alpha+n/p)}\|f\ ;W\dot{K}^{\alpha}_{p,q}\|.$$

To establish the $W\dot{K}^{\alpha}_{p,q} - \dot{K}^{\beta}_{\sigma,q}$ or $W\dot{K}^{\alpha}_{p,q} - W\dot{K}^{\beta}_{\sigma,q}$ estimates of heat semigroup $e^{t\Delta}$, we apply the following lemma to the Gaussian G, see Corollary 2.1 below.

Lemma 2.6. *Let $1 < p \le \sigma \le \infty,\ k,j \in \mathbb{Z},\ 1+1/\sigma = 1/p+1/r$ and $\phi \in L^{r,1}\cap L^{\infty}$ with $|\phi(x)| \le C_*|x|^{-m}$ for some $m \ge 0$ and $x \ne 0$.*
(i): *In the case $p < \sigma \le \infty$, we have*

$$\|f_j * \phi\ ;L^{\sigma}(A_k)\| \le c\|f_j\ ;L^{p,\infty}\|$$

$$\times \begin{cases} 2^{kn/\sigma}2^{jn(1-1/p)}\min\,(2^{-km},1) =: R_1, & \text{if } j \le k-2,\\ \min\,(2^{kn/\sigma}2^{jn(1-1/p)},1) =: R_2, & \text{if } k-1 \le j \le k+1,\\ 2^{kn/\sigma}2^{jn(1-1/p)}\min\,(2^{-jm},1) =: R_3, & \text{if } k+2 \le j, \end{cases}$$

where $f_j := f\chi_{A_j}$ and a constant c depends on $n,\ p,\ \sigma,\ \phi$ and C_* only.
(ii): *In the case $p = \sigma \le \infty$, we have the same estimate for $\|f_j * \phi\ ;L^{\sigma,\infty}(A_k)\|$ as above* (i).

Proof. (i): In the case $j \le k-2$, for $x \in A_k$ and $y \in A_j$, one has $|x-y| \gtrsim 2^k$. Then it is easy to see that

$$\|f_j * \phi\ ;L^{\sigma}(A_k)\| \lesssim |A_k|^{1/\sigma}\min(2^{-km},1)\|f_j\ ;L^1\|$$

Since supp $f_j \subset A_j$, the inequality $\|f_j\ ;L^1\| \le p/(p-1)|A_j|^{1-1/p}\|f_j\ ;L^{p,\infty}\|$ holds. Therefore we obtain the desired inequality in this case.

In the case $k-1 \le j \le k+1$, from the same argument as above, we have $\|f_j * \phi\ ;L^{\sigma}(A_k)\| \lesssim 2^{kn/\sigma}2^{jn(1-1/p)}\|f_j\ ;L^{p,\infty}\|$. On the other hand, in the case $\sigma < \infty$, by using the stronger Young's inequality, see pp. 63 in [15], we have

$$\begin{aligned}\|f_j * \phi\ ;L^{\sigma}(A_k)\| &\le \|f_j * \phi\ ;L^{\sigma}(\mathbb{R}^n)\|\\ &\lesssim \|\phi\ ;L^{r}\|\|f_j\ ;L^{p,\infty}\|.\end{aligned}$$

In the case $\sigma = \infty$, from the inequality of Hardy and Littlewood for rearrangements, [1], and the shift-invariance of $L^{r,1}$ norm, we have

$$\|f_j * \phi\ ;L^{\infty}(A_k)\| \lesssim \|\phi\ ;L^{r,1}\|\|f_j\ ;L^{p,\infty}\|.$$

The desired inequality in the case $k+2 \le j$ can be showed by the same argument as the first case.

(ii): In this case, since the stronger Young's inequality is not available, the left hand side is replaced by $L^{\sigma,\infty}$. □

The following proposition is a consequence of Lemma 2.6 and yields the $W\dot{K}^{\alpha}_{p,q} - \dot{K}^{\alpha}_{p,q}$ or $W\dot{K}^{\alpha}_{p,q} - W\dot{K}^{\beta}_{\sigma,q}$ estimates of the heat semigroup $e^{t\Delta}$ and the operator $\nabla e^{t\Delta}\mathbb{P}$, see the below.

Proposition 2.1. *Let $1 < p \le \sigma \le \infty$, $0 < q \le \infty$ and $m \ge 0$. Suppose that $\phi \in L^{r,1} \cap L^\infty$, with $1 + 1/\sigma = 1/p + 1/r$, satisfies $|\phi(x)| \le C_*|x|^{-m}$ for all $x \ne 0$. Then we obtain the following estimates.*
(i): *In the case $p < \sigma$*

$$\|f * \phi\ ; \dot{K}^{\beta}_{\sigma,q}\| \le c\|f\ ; W\dot{K}^{\alpha}_{p,q}\|, \text{ and}$$

(ii): *in the case $p = \sigma$*

$$\|f * \phi\ ; W\dot{K}^{\beta}_{p,q}\| \le c\|f\ ; W\dot{K}^{\alpha}_{p,q}\|$$

provided that one of the following cases holds;
(1) $0 < q \le 1$, $-n/\sigma < \beta \le \alpha \le n(1 - 1/p)$, $n(1 - 1/p) - \alpha + \beta + n/\sigma \le m$ *and* $\beta + n/\sigma < m$,
(2) $1 < q < \infty$, $-n/\sigma < \beta \le \alpha < n(1 - 1/p)$ *and* $n(1 - 1/p) - \alpha + \beta + n/\sigma \le m$,
(3) $q = \infty$, $-n/\sigma \le \beta \le \alpha < n(1 - 1/p)$, $n(1 - 1/p) - \alpha + \beta + n/\sigma \le m$ *and* $n(1 - 1/p) - \alpha < m$.

Proof. We prove the case (i) only. By usig Lemma 2.6, we decompose

$$\|f * \phi\ ; \dot{K}^{\beta}_{\sigma,q}\| \le c(\mathrm{I} + \mathrm{II} + \mathrm{III}),$$

where

$$\mathrm{I} = (\sum_{k\in\mathbb{Z}} 2^{k\beta q}(\sum_{j=-\infty}^{k-2} \|f\ ; L^{p,\infty}(A_j)\|R_1)^q)^{1/q},$$
$$\mathrm{II} = (\sum_{k\in\mathbb{Z}} 2^{k\beta q}(\sum_{j=k-1}^{k+1} \|f\ ; L^{p,\infty}(A_j)\|R_2)^q)^{1/q},$$
$$\mathrm{III} = (\sum_{k\in\mathbb{Z}} 2^{k\beta q}(\sum_{j=k+2}^{\infty} \|f\ ; L^{p,\infty}(A_j)\|R_3)^q)^{1/q}.$$

We shall show the case (1). By the assumption $n(1 - 1/p) - \alpha + \beta + n/\sigma \le m$ and $\beta \le \alpha$, II can be estimated as follows;

$$\begin{aligned}
\mathrm{II} &\le (\sum_{k\in\mathbb{Z}} 2^{k\beta q} \sum_{j=k-1}^{k+1} \|f\ ; L^{p,\infty}(A_j)\|^q R_2^q)^{1/q} \\
&\le c(\sum_{j\in\mathbb{Z}} 2^{j\alpha q}\|f\ ; L^{p,\infty}(A_j)\|^q \min\ (2^{j(n(1-1/p)-\alpha+\beta+n/\sigma)q},\ 2^{j(\beta-\alpha)q})^{1/q} \\
&\le c\|f\ ; W\dot{K}^{\alpha}_{p,q}\|.
\end{aligned}$$

To estimate I, we decompose again

$$\begin{aligned}
\mathrm{I} \le (&\sum_{k=-\infty}^{2} 2^{k\beta q} \sum_{j=-\infty}^{k-2} \|f\ ; L^{p,\infty}(A_j)\|^q R_1^q)^{1/q} \\
&+ (\sum_{k=3}^{\infty} 2^{k\beta q} \sum_{j=-\infty}^{0} \cdots)^{1/q} + (\sum_{k=3}^{\infty} 2^{k\beta q} \sum_{j=1}^{k-2} \cdots)^{1/q}.
\end{aligned}$$

Because $-n/\sigma < \beta$ and $\alpha \le n(1 - 1/p)$, we have

$$\begin{aligned}
\text{1st term} &\le (\sum_{j=-\infty}^{0} 2^{j\alpha q}\|f\ ; L^{p,\infty}(A_j)\|^q 2^{j(n(1-1/p)-\alpha)q} \sum_{k=-\infty}^{2} 2^{k(\beta+n/\sigma)q})) \\
&\le c(\sum_{j=-\infty}^{0} 2^{j\alpha q}\|f\ ; L^{p,\infty}(A_j)\|^q 2^{j(n(1-1/p)-\alpha)q})^{1/q} \\
&\le c\|f\ ; W\dot{K}^{\alpha}_{p,q}\|.
\end{aligned}$$

Also, since $\alpha \leq n(1-1/p)$ and $\beta + n/\sigma < m$, one has

$$\begin{aligned}
\text{2nd term} &\leq (\sum_{k=3}^{\infty} 2^{k(\beta+n/\sigma-m)q})^{1/q} \ (\sum_{j=-\infty}^{0} 2^{j\alpha q}\|f\ ; L^{p,\infty}(A_j)\|^q 2^{j(n(1-1/p)-\alpha)q})^{1/q} \\
&\leq c\|f\ ; W\dot{K}^{\alpha}_{p,q}\|.
\end{aligned}$$

Furthermore, since $n(1-1/p) - \alpha + \beta + n/\sigma \leq m$, we obtain

$$\begin{aligned}
\text{3rd term} &= (\sum_{j=1}^{\infty} 2^{j\alpha q}\|f\ ; L^{p,\infty}(A_j)\|^q 2^{j(n(1-1/p)-\alpha)q} \sum_{k=j+2}^{\infty} 2^{k(\beta+n/\sigma-m)q})^{1/q} \\
&\leq c\|f\ ; W\dot{K}^{\alpha}_{p,q}\|,
\end{aligned}$$

which imply $\mathrm{I} \leq c\|f\ ; W\dot{K}^{\alpha}_{p,q}\|$.

Going through a similar argument as above, one obtains

$$\mathrm{III} \leq c\|f\ ; W\dot{K}^{\alpha}_{p,q}\|,$$

which completes the proof of the case (i).

Next we consider the case (3) with the same decompositions as above. By using $n(1-1/p) - \alpha + \beta + n/\sigma \geq 0$ and $\beta - \alpha \leq 0$, we have

$$\begin{aligned}
\mathrm{II} &= \sup_{k\in\mathbb{Z}} 2^{k\beta} \sum_{j=k-1}^{k+1} \|f\ ; L^{p,\infty}(A_j)\| \min\,(2^{kn/\sigma}2^{jn(1-1/p)},\ 1) \\
&\leq c\|f\ ; W\dot{K}^{\alpha}_{p,\infty}\| \sup_{k\in\mathbb{Z}} \min\,(2^{k(n(1-1/p)-\alpha+\beta+n/\sigma)},\ 2^{k(\beta-\alpha)}) \\
&\leq c\|f\ ; W\dot{K}^{\alpha}_{p,\infty}\|.
\end{aligned}$$

I is dominated by $\|f\ ; W\dot{K}^{\alpha}_{p,\infty}\|$ in the following way;

$$\begin{aligned}
\text{1st term} &= \sup_{k\leq 2} 2^{k\beta} \sum_{j=-\infty}^{k-2} \|f\ ; L^{p,\infty}(A_j)\| 2^{kn/\sigma}2^{jn(1-1/p)} \\
&\leq \|f\ ; W\dot{K}^{\alpha}_{p,\infty}\| \sup_{k\leq 2} 2^{k(\beta+n/\sigma)} \sum_{j=-\infty}^{0} 2^{j(n(1-1/p)-\alpha)} \\
&\leq c\|f\ ; W\dot{K}^{\alpha}_{p,\infty}\|,
\end{aligned}$$

$$\begin{aligned}
\text{2nd term} &= \sup_{k\geq 3} 2^{k\beta} \sum_{j=-\infty}^{0} \|f\ ; L^{p,\infty}(A_j)\| 2^{k(n/\sigma-m)}2^{jn(1-1/p)} \\
&\leq \|f\ ; W\dot{K}^{\alpha}_{p,\infty}\| \sup_{k\geq 3} 2^{k(\beta+n/\sigma-m)} \sum_{j=-\infty}^{0} 2^{j(n(1-1/p)-\alpha)} \\
&\leq c\|f\ ; W\dot{K}^{\alpha}_{p,\infty}\|,
\end{aligned}$$

$$\begin{aligned}
\text{3rd term} &= \sup_{k\geq 3} 2^{k\beta} \sum_{j=1}^{k-2} \|f\ ; L^{p,\infty}(A_j)\| 2^{k(n/\sigma-m)}2^{jn(1-1/p)} \\
&\leq c\|f\ ; W\dot{K}^{\alpha}_{p,\infty}\| \sup_{k\geq 3} 2^{k(\beta+n/\sigma-m)}2^{k(n(1-1/p)-\alpha)} \\
&\leq c\|f\ ; W\dot{K}^{\alpha}_{p,\infty}\|.
\end{aligned}$$

Hence we have $\mathrm{I} \le c\|f\ ;W\dot{K}^{\alpha}_{p,\infty}\|$.
Using a similar argument as above, we can get the estimate $\mathrm{III} \le c\|f\ ;W\dot{K}^{\alpha}_{p,\infty}\|$, which completes the proof of the case (3).

The inequality in the case (2) follows from the interpolation. □

Next proposition says that if we restrict the condition of the weight parameters α and β, then we can improve the right hand side of Proposition 2.1 with respect to the summation parameter q.

Proposition 2.2. *Let $1 < p \le \sigma \le \infty,\ 0 < \delta < q \le \infty$ and $m \ge 0$. Suppose that $\phi \in L^{r,1} \cap L^{\infty}$, with $1+1/\sigma = 1/p+1/r$, satisfies $|\phi(x)| \le C_*|x|^{-m}$ for all $x \ne 0$. If $-n/\sigma < \beta < \alpha < n(1-1/p) < m$, then we obtain the following estimates.*
(i): *In the case $p < \sigma$*

$$\|f * \phi\ ;\dot{K}^{\beta}_{\sigma,\delta}\| \le c\|f\ ;W\dot{K}^{\alpha}_{p,q}\|, \text{ and}$$

(ii): *in the case $p = \sigma$*

$$\|f * \phi\ ;W\dot{K}^{\beta}_{p,\delta}\| \le c\|f\ ;W\dot{K}^{\alpha}_{p,q}\|.$$

Proof. It suffices to prove the estimate in the cases $q = \infty$ and $0 < \delta < 1$. We omit the proof of this proposition, because it is not hard to complete the proof modifying the argument of the proof of Proposition 2.1. □

Using Propositions 2.1 and 2.2, we can investigate the behavior of the heat semigroup $e^{t\Delta}$ and $\nabla e^{t\Delta}\mathbb{P}$ in the weak Herz spaces. We use this corollary many times.

Corollary 2.1. *Let $1 < p \le \sigma \le \infty,\ 0 < q, \delta \le \infty,\ -n/\sigma \le \beta \le \alpha \le n(1-1/p)$ and $j \in \{0,1\}$.*

[I]: *(In the case $q = \delta$) If the exponents verify the first two conditions of one of (1), (2) and (3) in Proposition 2.1, the following inequalities hold.*
(i): *(In the case $p < \sigma$)*

$$\|\nabla^j e^{t\Delta} f\ ;\dot{K}^{\beta}_{\sigma,q}\| \le ct^{-j/2-(\alpha-\beta+n(1/p-1/\sigma))/2}\|f\ ;W\dot{K}^{\alpha}_{p,q}\|.$$

(ii): *(In the case $p = \sigma$)*

$$\|\nabla^j e^{t\Delta} f\ ;W\dot{K}^{\beta}_{p,q}\| \le ct^{-j/2-(\alpha-\beta)/2}\|f\ ;W\dot{K}^{\alpha}_{p,q}\|.$$

(iii): *(In the case $p < \sigma$)*

$$\|\nabla^j e^{t\Delta}\mathbb{P} f\ ;\dot{K}^{\beta}_{\sigma,q}\| \le ct^{-j/2-(\alpha-\beta+n(1/p-1/\sigma))/2}\|f\ ;W\dot{K}^{\alpha}_{p,q}\|.$$

(iv): *(In the case $p = \sigma$) If $j = 1$ or $\beta < \alpha$, it follows*

$$\|\nabla^j e^{t\Delta}\mathbb{P} f\ ;W\dot{K}^{\beta}_{p,q}\| \le ct^{-j/2-(\alpha-\beta)/2}\|f\ ;W\dot{K}^{\alpha}_{p,q}\|.$$

[II]: *(In the case $q = \delta$) When $1 < p < \infty$ and $-n/p < \alpha < n(1-1/p)$, it follows*

$$\|e^{t\Delta}\mathbb{P} f\ ;W\dot{K}^{\alpha}_{p,q}\| \le c\|f\ ;W\dot{K}^{\alpha}_{p,q}\|.$$

[III]: *(In the case $\delta < q$) The following inequalities hold if $-n/\sigma < \beta < \alpha < n(1-1/p)$).*
(i): *(In the case $p < \sigma$)*

$$\|\nabla^j e^{t\Delta} f\ ;\dot{K}^{\beta}_{\sigma,\delta}\| \le ct^{-j/2-(\alpha-\beta+n(1/p-1/\sigma))/2}\|f\ ;W\dot{K}^{\alpha}_{p,q}\|.$$

(ii): *(In the case $p = \sigma$)*

$$\|\nabla^j e^{t\Delta} f\ ;W\dot{K}^{\beta}_{p,\delta}\| \le c\ t^{-j/2-(\alpha-\beta)/2}\|f\ ;W\dot{K}^{\alpha}_{p,q}\|.$$

(iii): *(In the case $p < \sigma$)*

$$\|\nabla^j e^{t\Delta}\mathbb{P} f\ ;\dot{K}^{\beta}_{\sigma,\delta}\| \le c\ t^{-j/2-(\alpha-\beta+n(1/p-1/\sigma))/2}\|f\ ;W\dot{K}^{\alpha}_{p,q}\|.$$

(iv): *(In the case $p = \sigma$)*

$$\|\nabla^j e^{t\Delta}\mathbb{P} f\ ;W\dot{K}^{\beta}_{p,\delta}\| \le c\ t^{-j/2-(\alpha-\beta)/2}\|f\ ;W\dot{K}^{\alpha}_{p,q}\|.$$

Proof. (I-i), (I-ii): The proofs of the inequalities are concluded from Lemma 2.5, Proposition 2.1 and the equation $e^{t\Delta}f = (f_{1/\sqrt{t}} * G)_{\sqrt{t}}$.

(I-iii): It is well known that the operator $e^{t\Delta}\mathbb{P}$ is a convolution operator with the Oseen kernel $\mathbb{K}_t(x) = t^{-n/2}\mathbb{K}(x/\sqrt{t})$ such that $|\nabla^m \mathbb{K}(x)| \le c\langle x\rangle^{-(n+m)}$, see for example [31]. Then the inequality can be verified by applying Proposition 2.1 to $\mathbb{K}$.

(I-iv): If $j = 1$, then the inequality is showed by the same argument as above. We shall consider the case $\beta < \alpha$. When $p < \infty$ and $-n/p < \beta$, from Proposition 2.3 $\mathbb{P}$ is bounded on $W\dot{K}^{\beta}_{p,q}$. Therefore, we have

$$\begin{aligned}\|e^{t\Delta}\mathbb{P}f\ ; W\dot{K}^{\beta}_{p,q}\| &\le c\|e^{t\Delta}f\ ; W\dot{K}^{\beta}_{p,q}\| \\ &\le ct^{-(\alpha-\beta)/2}\|f\ ; W\dot{K}^{\alpha}_{p,q}\|.\end{aligned}$$

When $p < \infty$ and $\beta = -n/p$, for two positive numbers ε and θ with $-n/p + \epsilon < n(1 - 1/p)$ and $-n/p + \varepsilon + \theta = \alpha$, we have

$$\begin{aligned}\|e^{t\Delta}\mathbb{P}f\ ; W\dot{K}^{-n/p}_{p,q}\| &\le ct^{-\varepsilon/2}\|e^{t/2\Delta}\mathbb{P}f\ ; W\dot{K}^{-n/p+\varepsilon}_{p,q}\| \\ &\le ct^{-\varepsilon/2}\|e^{t/2\Delta}f\ ; W\dot{K}^{-n/p+\varepsilon}_{p,q}\| \\ &\le ct^{-(\varepsilon+\theta)/2}\|f\ ; W\dot{K}^{-n/p+\varepsilon+\theta}_{p,q}\|.\end{aligned}$$

When $p = \infty$, there exists $r \in (1, \infty)$ with $-n/r < \beta < n(1 - 1/r)$. Hence, one obtains

$$\begin{aligned}\|e^{t\Delta}\mathbb{P}f\ ; W\dot{K}^{\beta}_{\infty,q}\| &\le ct^{-n/(2r)}\|e^{t/2\Delta}\mathbb{P}f\ ; W\dot{K}^{\beta}_{r,q}\| \\ &\le ct^{-n/(2r)}\|e^{t/2\Delta}f\ ; W\dot{K}^{\beta}_{r,q}\| \\ &\le ct^{-n/(2r)}\|e^{t/2\Delta}f\ ; W\dot{K}^{\beta+n/r}_{\infty,q}\| \\ &\le ct^{-(\alpha+\beta)/2}\|f\ ; W\dot{K}^{\alpha}_{\infty,q}\|.\end{aligned}$$

(II), (III): We omit the details. □

The following proposition is useful to establish estimates on the weak Herz spaces for many operators, for example, the Hardy-Littlewood maximal operator, Calderon-Zygmund operators and fractional integral operators.

Proposition 2.3. *Let $1 < p < n/\gamma,\ 0 < q \le \infty,\ 0 \le \gamma < n$ and $-n/p + \gamma < \alpha < n(1 - 1/p)$. Let $1/p - 1/r = \gamma/n$ and T_γ be a bounded operator from $L^{p,\infty}$ to $L^{r,\infty}$ with $\|T_\gamma\ ; L^{p,\infty} \to L^{r,\infty}\| \le C_1$ satisfying*

$$|T_\gamma(f)(x)| \le C_2 \int \frac{|f(y)|}{|x-y|^{n-\gamma}}dy$$

for $f \in L^1_{loc}$ with $x \notin supp\ f$. Then T_γ is also a bounded operator from $W\dot{K}^{\alpha}_{p,q}$ to $W\dot{K}^{\alpha}_{r,q}$ with $\|T_\gamma\| \le c(C_1 + C_2)$.

Proof. We prove the case $q = \infty$ only;

$$\|T_\gamma(f)\ ; W\dot{K}^{\alpha}_{r,\infty}\| \le c(C_1 + C_2)\|f\ ; W\dot{K}^{\alpha}_{p,\infty}\|.$$

We decompose

$$\begin{aligned}\|T_\gamma(f)\ ; W\dot{K}^{\alpha}_{r,\infty}\| \le\ & c\sup_{k\in\mathbb{Z}} 2^{k\alpha} \sup_{\lambda>0} \lambda|\{x \in A_k;\ \sum_{j=-\infty}^{k-2} |T_\gamma(|f\chi_j|)(x)| > \lambda\}|^{1/r} \\ &+ c\sup_{k\in\mathbb{Z}} 2^{k\alpha} \sup_{\lambda>0} \lambda|\{x \in A_k;\ \sum_{j=k-1}^{k+1} |T_\gamma(|f\chi_j|)(x)| > \lambda\}|^{1/r} \\ &+ c\sup_{k\in\mathbb{Z}} 2^{k\alpha} \sup_{\lambda>0} \lambda|\{x \in A_k;\ \sum_{j=k+2}^{\infty} |T_\gamma(|f\chi_j|)(x)| > \lambda\}|^{1/r} \\ =:\ & \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned}$$

The boundedness of T_γ yields that $\mathrm{II} \le cC_1 \, \|f\ ; W\dot{K}^{\alpha}_{p,\infty}\|$. Let $x \in A_k$. Since from the property of T_γ we have

$$\begin{aligned}
\sum_{j=-\infty}^{k-2} T_\gamma(|f\chi_j|)(x) &\le cC_2 2^{-k(n-\gamma)} \sum_{j=-\infty}^{k-2} \|f\ ; L^1(A_j)\| \\
&\le cC_2 2^{-k(n-\gamma)} \sum_{j=-\infty}^{k-2} 2^{jn(1-1/p)} \|f\ ; L^{(p,\infty)}(A_j)\| =: A,
\end{aligned}$$

and hence, I is dominated by $C_2\|f\ ; W\dot{K}^{\alpha}_{p,\infty}\|$ in the following way;

$$\begin{aligned}
\mathrm{I} &= \sup_{k\in\mathbb{Z}} 2^{k\alpha} \sup_{0<\lambda\le A} \lambda |\{x \in A_k;\ \sum_{j=-\infty}^{k-2} T_\gamma(|f\chi_j|)(x) > \lambda\}|^{1/r} \\
&\le cC_2 \sup_{k\in\mathbb{Z}} 2^{k(\alpha+n/r-n+\gamma)} \sum_{j=-\infty}^{k-2} 2^{jn(1-1/p)} \|f\ ; L^{(p,\infty)}(A_j)\| \\
&\le cC_2 \|f\ ; W\dot{K}^{\alpha}_{p,\infty}\|.
\end{aligned}$$

It is not hard to show that $\mathrm{III} \le cC_2\|f\ ; W\dot{K}^{\alpha}_{p,\infty}\|$ by the same argument as above. □

The final proposition is a generalization of [31, Lemma 27.6] in the setting of the weak Herz spaces and the proof is based on that of [31, Lemma 27.6]. The proposition is applied in the proof of Theorems 1.4 and 1.5 in Section 4.

Proposition 2.4. *Let $n \ge 2$, $1 < p < n$, $1 \le q \le \infty$ and $1 - n/p < \alpha < n(1-1/p)$. For a positive $\beta > 0$ satisfying*

$$0 < \beta < \min\Big(\frac{1}{2},\ \frac{n}{2}\big(\frac{1}{p} - \frac{1}{n}\big),\ \frac{n}{2}\big(1 - \frac{1}{p} + \frac{1}{n}\big),\ \frac{1}{2}\big(\alpha - (1 - \frac{n}{p})\big)\Big),$$

suppose that $1/r_1 = 1/r + 2\beta/n$, $1/r_2 = 1/r - 2\beta/n$ and $1/p - 1/r = 1/n$. Then, we have that for every $j, k \in \{1, \cdots, n\}$

$$\| \int_0^\infty (-\Delta)^{1/2} e^{t\Delta} \mathbb{P}_{j,k} f(t) dt\ ; (W\dot{K}^{\alpha}_{r_1,q},\ W\dot{K}^{\alpha}_{r_2,q})_{1/2,\infty}\| \le c\, \|f\ ; L^\infty(0,\infty;\ W\dot{K}^{\alpha}_{p,q})\|,$$

with a constant c independent of f. In particular, in the case $q = \infty$, we have

$$\| \int_0^\infty (-\Delta)^{1/2} e^{t\Delta} \mathbb{P}_{j,k} f(t) dt\ ; W\dot{K}^{\alpha}_{r,\infty}\| \le c\, \|f\ ; L^\infty(0,\infty;\ W\dot{K}^{\alpha}_{p,\infty})\|.$$

Proof. Let $A > 0$ and we decompose

$$\begin{aligned}
F :&= \int_0^\infty (-\Delta)^{1/2} e^{t\Delta} \mathbb{P}_{j,k} f(t) dt \\
&= \int_0^A \cdots dt + \int_A^\infty \cdots dt \\
&= \int_0^A (-\Delta)^{1-\beta} e^{t\Delta} \mathbb{P}_{j,k} (-\Delta)^{-(1/2-\beta)} f(t) dt \\
&+ \int_A^\infty (-\Delta)^{1+\beta} e^{t\Delta} \mathbb{P}_{j,k} (-\Delta)^{-(\beta+1/2)} f(t) dt \\
&=: G_A + H_A.
\end{aligned}$$

Remark that $1/p - 1/r_1 = 2(1/2-\beta)/n,\ 1/p - 1/r_2 = 2(1/2+\beta)/n$ and $-n/r_i < \alpha < n(1-1/p)$ for $i = 1, 2$. Therefore, applications of Proposition 2.3 give the following estimates;

$$\begin{aligned}
\|G_A\ ; W\dot{K}^{\alpha}_{r_1,q}\| &\le \int_0^A \|(-\Delta)^{1-\beta} e^{t\Delta}\mathbb{P}_{j,k} I_{2(1/2-\beta)} f(t)\ ; W\dot{K}^{\alpha}_{r_1,q}\|\ dt \\
&\le c\int_0^A t^{\beta-1}\ \|e^{\frac{t}{2}\Delta}\mathbb{P}_{j,k} I_{2(1/2-\beta)} f(t)\ ; W\dot{K}^{\alpha}_{r_1,q}\|\ dt \\
&\le c\int_0^A t^{\beta-1}\|I_{2(1/2-\beta)} f(t)\ ; W\dot{K}^{\alpha}_{r_1,q}\|\ dt \\
&\le c\int_0^A t^{\beta-1}\ \|f(t)\ ; W\dot{K}^{\alpha}_{p,q}\|\ dt \\
&\le cA^{\beta}\ \|f\ ; L^{\infty}(0,\infty;\ W\dot{K}^{\alpha}_{p,q})\|,
\end{aligned}$$

where I_s is the fractional integral operator which satisfies the conditions of Proposition 2.3. Here we have used the boundedness of the convolution operator $(\Delta^{1-\beta}G)_{\sqrt{t}} *$ and $e^{t\Delta}\mathbb{P}_{j,k}$ on $W\dot{K}^{\alpha}_{r_1,q}$. Also, by the similar argument to above, we can obtain

$$\|H_A\ ; W\dot{K}^{\alpha}_{r_2,q}\| \le cA^{-\beta}\ \|f\ ; L^{\infty}(0,\infty;\ W\dot{K}^{\alpha}_{p,q})\|.$$

Then, we reach the desired inequality in the following way;

$$\begin{aligned}
\text{LHS} &= \sup_{t>0} t^{-1/2} K(t, F;\ W\dot{K}^{\alpha}_{r_1,q},\ W\dot{K}^{\alpha}_{r_2,q}) \\
&\le \sup_{t>0} t^{-1/2}(\|G_{t^{1/(2\beta)}}\ ; W\dot{K}^{\alpha}_{r_1,q}\| + t\ \|H_{t^{1/(2\beta)}}\ ; W\dot{K}^{\alpha}_{r_2,q}\|) \\
&\le c\ \|f\ ; L^{\infty}(0,\infty;\ W\dot{K}^{\alpha}_{p,q})\| = \text{RHS}.
\end{aligned}$$

It remains to show the continuous inclusion $(W\dot{K}^{\alpha}_{r_1,\infty},\ W\dot{K}^{\alpha}_{r_2,\infty})_{1/2,\ \infty} \hookrightarrow W\dot{K}^{\alpha}_{r,\infty}$. Once we know the estimate

$$\|g\chi_k\ ; L^{(r,\infty)}\| \le c2^{-k\alpha}\|g\ ; (W\dot{K}^{\alpha}_{r_1,\infty},\ W\dot{K}^{\alpha}_{r_2,\infty})_{1/2,\ \infty}\|,$$

the inclusion is immediately verified. By using the real interpolation theory for Lorentz spaces, we get the estimate as follows;

$$\begin{aligned}
\|g\chi_k\ ; L^{(r,\infty)}\| &\approx \|g\chi_k\ ; (L^{(r_1,\infty)}, L^{(r_2,\infty)})_{1/2,\infty}\| \\
&= \sup_{\lambda>0} \lambda^{-\frac{1}{2}} \inf_{g\chi_k = g^1+g^2} (\|g^1\ ; L^{(r_1,\infty)})\| + \lambda\|g^2\ ; L^{(r_2,\infty)}\|) \\
&\le \sup_{\lambda>0} \lambda^{-\frac{1}{2}} \inf_{\substack{g\chi_k = g^1+g^2 \\ g^1=g^2=0 \text{ on } A_k^c}} (\|g^1\ ; L^{(r_1,\infty)})\| + \lambda\|g^2\ ; L^{(r_2,\infty)}\|) \\
&= \sup_{\lambda>0} \lambda^{-\frac{1}{2}} \inf_{g = g^1+g^2} (\|g^1\chi_k\ ; L^{(r_1,\infty)})\| + \lambda\|g^2\chi_k\ ; L^{(r_2,\infty)}\|) \\
&\le 2^{-k\alpha} \sup_{\lambda>0} \lambda^{-\frac{1}{2}} \inf_{g = g^1+g^2} (\|g^1\ ; W\dot{K}^{\alpha}_{r_1,\infty}\| + \lambda\|g^2\ ; W\dot{K}^{\alpha}_{r_2,\infty}\|) \\
&= 2^{-k\alpha}\|g\ ; (W\dot{K}^{\alpha}_{r_1,\infty}, W\dot{K}^{\alpha}_{r_2,\infty})_{1/2,\infty}\|
\end{aligned}$$

□

Remark 2.1. *In the proof above, we have used the one side inclusion*

$$(W\dot{K}^{\alpha}_{r_1,\infty},\ W\dot{K}^{\alpha}_{r_2,\infty})_{1/2,\ \infty} \hookrightarrow W\dot{K}^{\alpha}_{r,\infty}$$

only. The author does not know whether the reverse inclusion is true or not. On the other hand, for the interpolation with respect to the weight parameter α, we have

$$(W\dot{K}^{\alpha_1}_{p,\infty},\ W\dot{K}^{\alpha_2}_{p,\infty})_{\theta,\ \infty} = W\dot{K}^{\alpha}_{p,\infty}$$

where $\alpha = (1-\theta)\alpha_1+\theta\alpha_2$ and $0 < p \le \infty$. Then it is natural to try to prove Proposition 2.4 by using the interpolation with respect to the parameter α, but we have not the estimate $W\dot{K}^{\alpha}_{p,\infty} - W\dot{K}^{\beta}_{\sigma,\infty}$ of heat semigroup in the case $\alpha < \beta$.

In order to solve (I.E.), we use the following Picard contraction principle. For example, see [31] for the proof.

Proposition 2.5 (The Picard contraction principle)**.** *Let E be a Banach space and let B be a bounded bilinear transform $E \times E$ to E satisfying*

$$\|B(e,f)\ ; E\| \le C_B \|e\ ; E\|\ \|f\ ; E\|.$$

Then, if $0 < \delta < (4C_B)^{-1}$ and $e_0 \in E$ satisfies $\|e_0\ ; E\| \le \delta$, the equation $e = e_0 - B(e,e)$ has a solution with $\|e\ ; E\| \le 2\delta$. This solution is unique in the closed ball $\bar{B}(0, 2\delta)$. Moreover, the solution continuously depends on e_0; if $\|f_0\ ; E\| \le \delta$, $f = f_0 - B(f,f)$ and $\|f\ ; E\| \le 2\delta$, then $\|e - f\ ; E\| \le (1 - 4C_B\delta)^{-1}\|e_0 - f_0\ ; E\|$.

3 Proof of Theorems 1.1 and 1.2

3.1 Proof of Theorem 1.1

The constant $T > 0$ will be chosen later. To prove Theorem 1.1, we divide the proof into 4 Steps.

Step 1: The bilinear form B is a map from $X_T \times X_T$ to X_T and has the estimate

$$\|B(u,v)\ ; X_T\| \le C_B T^{(1-n/p-\alpha)/2} \|u\ ; X_T\|\ \|v\ ; X_T\|.$$

The claim is an immediate consequence of Corollary 2.1. Indeed, by using the corollary, we have the following three estimates. Firstly, because $1 - n/p - \alpha > 0$, one has that for $0 < t \le T$,

$$\begin{aligned}
\|B(u,v)(t)\ ; \dot{K}^{\alpha}_{p,\infty}\| &\le \int_0^t \|\nabla e^{(t-s)\Delta}\mathbb{P}(u \otimes v)(s)\ ; \dot{K}^{\alpha}_{p,\infty}\| ds \\
&\le c \int_0^t (t-s)^{-(1+n/p+\alpha)/2} \|u \otimes v(s)\ ; W\dot{K}^{2\alpha}_{p/2,\infty}\| ds \\
&\le c\|u\ ; X_{T,1}\|\ \|v\ ; X_{T,1}\| \int_0^t s^{-(1+n/p+\alpha)/2} ds \\
&\le (C_B/2) T^{(1-n/p-\alpha)/2} \|u\ ; X_T\|\ \|v\ ; X_T\|,
\end{aligned}$$

which implies $\|B(u,v)\ ; X_{T,1}\| \le (C_B/2)T^{(1-n/p-\alpha)/2}\|u\ ; X_T\|\ \|v\ ; X_T\|$. Secondly, from the same reason as above, one can show that $\|B(u,v)\ ; X_{T,2}\| \le (C_B/2)T^{(1-\alpha-n/p)/2}\|u\ ; X_T\|\ \|v\ ; X_T\|$ in the following way;

$$\begin{aligned}
\|\nabla B(u,v)(t)\ ; \dot{K}^{\alpha}_{p,\infty}\| &\le \int_0^t \|\nabla e^{(t-s)\Delta}\mathbb{P}(u \cdot \nabla)v(s)\ ; \dot{K}^{\alpha}_{p,\infty}\| ds \\
&\le c \int_0^t (t-s)^{-(1+n/p+\alpha)/2} \|(u \cdot \nabla)v(s)\ ; W\dot{K}^{2\alpha}_{p/2,\infty}\| ds \\
&\le c\|u\ ; X_{T,1}\|\ \|v\ ; X_{T,2}\| \int_0^t (t-s)^{-(1+n/p+\alpha)/2} s^{-1/2} ds \\
&\le (C_B/2) t^{-(n/p+\alpha)/2} \|u\ ; X_T\|\ \|v\ ; X_T\|.
\end{aligned}$$

As a result, we get the desired estimate.

Step 2: We verify that

$$\|e^{\cdot\Delta}u_0\ ; X_T\| \le C_0 \|u_0\ ; W\dot{K}^{\alpha}_{p,\infty}\|.$$

Our hypothesis guarantees the following estimates to hold;

$$\|e^{t\Delta}u_0\ ;W\dot{K}^{\alpha}_{p,\infty}\| \le C_0/2\|u_0\ ;W\dot{K}^{\alpha}_{p,\infty}\|$$
$$\|\nabla e^{t\Delta}u_0\ ;W\dot{K}^{\alpha}_{p,\infty}\| \le (C_0/2)t^{-1/2}\|u_0\ ;W\dot{K}^{\alpha}_{p,\infty}\|.$$

Therefore, the above estimate holds.

Now, we define T as

$$\left(\frac{1}{8C_BC_0\|u_0\ ;W\dot{K}^{\alpha}_{p,\infty}\|}\right)^{2/(1-n/p-\alpha)} = \frac{C^*}{\|u_0\ ;W\dot{K}^{\alpha}_{p,\infty}\|^{2/(1-n/p-\alpha)}}.$$

Then, because T satisfies that $C_0\|u_0\ ;W\dot{K}^{\alpha}_{p,\infty}\| < (4C_BT^{(1-n/p-\alpha)/2})^{-1}$, we can find a solution $u \in X_T$ to (I.E.) and the solution continuously depends on the initial data u_0, from the Picard contraction principle. Remark that the constant C^*, in the definition of T above, depends on p, n and α only. Then the solution automatically satisfies the condition (1). In fact, we have

$$\|e^{t\Delta}u_0\ ;L^\infty\| \le ct^{-(\alpha+n/p)/2}\|u_0\ ;W\dot{K}^{\alpha}_{p,\infty}\| \text{ and}$$
$$\begin{aligned}\|B(u,u)(t)\ ;L^\infty\| &\le c\int_0^t (t-s)^{-(\alpha+n/p)}\|u(s)\ ;W\dot{K}^{\alpha}_{p,\infty}\|\|\nabla u(s)\ ;W\dot{K}^{\alpha}_{p,\infty}\|ds\\ &\le c\|u\ ;X_{T,1}\|\|u\ ;X_{T,2}\|\int_0^t (t-s)^{-(\alpha+n/p)}s^{-1/2}ds\\ &\le ct^{1/2-(\alpha+n/p)}\|u\ ;X_T\|^2,\end{aligned}$$

which implies

$$\sup_{0<t\le T} t^{(\alpha+n/p)/2}\|B(u,u)(t)\ ;L^\infty\| \le cT^{(1-\alpha-n/p)/2}\|u\ ;X_T\|^2 < \infty.$$

<u>Step 3:</u> *In this step, we check the properties* (2), (3), (4) *and* (5) *for the solution* u. But, because these assertions can be verified by the same arguments as Steps 3, 4 and 5 in the proof of Theorem 1.3, we omit the details.

<u>Step 4:</u> *Finally, the solution* u *is the only solution of* (I.E.) *in the class* $L^\infty(0,T;W\dot{K}^{\alpha}_{p,\infty})$. Let v also be a solution of (I.E.) with the initial data u_0 in the class $L^\infty(0,T;W\dot{K}^{\alpha}_{p,\infty})$. Put $w = u - v = -B(u,u) + B(v,v) = -B(u,w) - B(w,v)$. An application of Corollary 2.1 and our condition $-n/p \le 0 \le \alpha < 1 - n/p$ yield that for $0 < t \le T_0 < T$,

$$\begin{aligned}\|w(t)\ ;W\dot{K}^{\alpha}_{p,\infty}\| &\le c\int_0^t (t-s)^{-(1+n/p+\alpha)/2}\|u\otimes w(s) + w\otimes v(s)\ ;W\dot{K}^{2\alpha}_{p/2,\infty}\|ds\\ &\le c\Big(\sup_{0<s\le T_0}\|u(s)\ ;W\dot{K}^{\alpha}_{p,\infty}\| + \sup_{0<s\le T_0}\|v(s)\ ;W\dot{K}^{\alpha}_{p,\infty}\|\Big)\sup_{0<s\le T_0}\|w(s)\ ;W\dot{K}^{\alpha}_{p,\infty}\|\ T_0^{(1-n/p-\alpha)/2}.\end{aligned}$$

Here, if we take T_0 as

$$c(\sup_{0<s\le T_0}\|u(s)\ ;W\dot{K}^{\alpha}_{p,\infty}\| + \sup_{0<s\le T_0}\|v(s)\ ;W\dot{K}^{\alpha}_{p,\infty}\|)T_0^{2/(1-n/p-\alpha)} \le 1/2,$$

then we can get $w = 0$ on $(0, T_0]$, i.e., $u = v$ on $(0, T_0]$. By the similar method as Proposition 3.1 in [27], we have $u = v$ on $(0, T)$ and the proof of Theorem 1.1 is completed.

□

3.2 Proof of Theorem 1.2

Next we give the proof of Theorem 1.2. In the proof, the following bilinear estimate plays an important role.

Theorem 3.1 ([47])**.** *Let $1 < p < \infty$, $1 \le q \le \infty$, $-n/p < \alpha < \infty$ and $m \in \mathbb{N}$. There exists a constant c such that for any $f \in \dot{K}^\alpha_{p,q}$ with $\nabla^m f \in \dot{K}^0_{BMO,\infty}$ and $g \in \dot{K}^\alpha_{p,q}$ with $\nabla^m g \in \dot{K}^0_{BMO,\infty}$,*

$$\|f\nabla^m g\ ; \dot{K}^\alpha_{p,q}\| \le c\Big(\|f\ ; \dot{K}^\alpha_{p,q}\|\ \|\nabla^m g\ ; \dot{K}^0_{BMO,\infty}\| + \|\nabla^m f\ ; \dot{K}^0_{BMO,\infty}\|\ \|g\ ; \dot{K}^\alpha_{p,q}\|\Big).$$

We can conclude that $\sup_{T^*/2\le t<T^*} \|u(t)\ ; \dot{K}^\alpha_{p,\infty}\|$ is finite by applying Theorem 3.1 to u and ∇u, as follows; for $T^*/2 \le t < T^*$

$$\begin{aligned}\|u(t)\ ; \dot{K}^\alpha_{p,\infty}\| &\le c\|u_0\ ; \dot{K}^\alpha_{p,\infty}\| + c\int_0^t \|(u\cdot\nabla)u(s)\ ; \dot{K}^\alpha_{p,\infty}\|ds \\ &\le c\|u_0\ ; \dot{K}^\alpha_{p,\infty}\| + c\int_0^t \|u(s)\ ; \dot{K}^\alpha_{p,\infty}\|\ \|\nabla u(s)\ ; \dot{K}^0_{BMO,\infty}\|ds,\end{aligned}$$

and from the Gronwall's inequality,

$$\|u(t)\ ; \dot{K}^\alpha_{p,\infty}\| \le c\|u_0\ ; \dot{K}^\alpha_{p,\infty}\|\ \exp\Big(c\int_0^{T^*} \|\nabla u(s)\ ; \dot{K}^0_{BMO,\infty}\|ds\Big) < \infty.$$

Now we suppose that for every $\tilde{T} > T^*$, u would not be a mild solution of (N-S) with initial data u_0 in the class $L^\infty(0,\tilde{T}; \dot{K}^\alpha_{p,\infty})$. Then, from Remark 1.1 we know that for any positive number $\tau < T^*$,

$$\frac{C}{\|u(\tau)\ ; \dot{K}^\alpha_{p,\infty}\|^{2/(1-n/p-\alpha)}} \le T^* - \tau$$

with a constant C independent of τ. Therefore, positivity of $1 - n/p - \alpha$ yields

$$\limsup_{\tau\nearrow T^*} \|u(\tau)\ ; \dot{K}^\alpha_{p,\infty}\| \ge \limsup_{\tau\nearrow T^*}\Big(\frac{C}{T^*-\tau}\Big)^{(1-n/p-\alpha)/2} = \infty,$$

which contradicts that $u \in L^\infty(0,T^*; \dot{K}^\alpha_{p,\infty})$.

□

4 Proof of Theorems 1.3, 1.4 and 1.5

4.1 Proof of Theorem 1.3

We divide the proof into 5 steps and begin with an estimate of the bilinear form B.

<u>*Step 1:*</u> *The bilinear form B is the map from $X \times X$ to X and has the estimate*

$$\|B(u,v)\ ; X\| \le C_B\|u\ ; X\|\ \|v\ ; X\|.$$

From Corollary 2.1, we know that our hypothesis guarantees the following inequalities to hold; for $1/\sigma = 1/n + 1/p$

$$\begin{aligned}\|B(u,v)(t)\ ; \dot{K}^0_{n,\infty}\| &\le c\int_0^t \|e^{(t-s)\Delta}(u\cdot\nabla)v(s)\ ; \dot{K}^0_{n,\infty}\|ds \\ &\le c\int_0^t (t-s)^{-n/2p}\|(u\cdot\nabla)v(s)\ ; W\dot{K}^0_{\sigma,\infty}\|ds \\ &\le c\|u\ ; X_2\|\ \|v\ ; X_4\|\int_0^t (t-s)^{-n/2p}s^{-(2-n/p)/2}ds \\ &\le C_B/4\|u\ ; X\|\ \|v\ ; X\|,\end{aligned}$$

which implies $\|B(u,v)\ ;X_1\| \le C_B/4\|u\ ;X\|\ \|v\ ;X\|$. As for the X_2 norm of $B(u,v)$, an application of the same corollary yields

$$\begin{aligned}\|B(u,v)(t)\ ;\dot K^0_{p,\infty}\| &\le \int_0^t \|\nabla e^{(t-s)\Delta}\mathbb{P}(u\otimes v)(s)\ ;\dot K^0_{p,\infty}\|ds\\ &\le c\int_0^t (t-s)^{-1/2-n(2/p-1/p)/2}\|u\otimes v(s)\ ;W\dot K^0_{p/2,\infty}\|ds\\ &\le c\|u\ ;X_2\|\ \|v\ ;X_2\|\int_0^t (t-s)^{-1/2-n/2p}s^{-(1-n/p)}ds\\ &\le (C_B/4)t^{-(1-n/p)/2}\|u\ ;X\|\ \|v\ ;X\|,\end{aligned}$$

and as a consequence, we have $\|B(u,v)\ ;X_2\| \le C_B/4\|u\ ;X\|\ \|v\ ;X\|$. Furthermore, to estimate the X_3 norm of $B(u,v)$, we use Corollary 2.1 again and obtain

$$\begin{aligned}\|B(u,v)(t)\ ;L^\infty\| &\le \int_0^t \|e^{(t-s)\Delta}\mathbb{P}(u\cdot\nabla)v(s)\ ;L^\infty\|ds\\ &\le c\int_0^t \min\Big((t-s)^{-n/2\sigma}\|(u\cdot\nabla)v(s)\ ;W\dot K^0_{\sigma,\infty}\|,\ (t-s)^{-1/2}\|u\otimes v(s)\ ;L^\infty\|\Big)ds\\ &\le c\|u\ ;X\|\|v\ ;X\|\int_0^t \min\Big((t-s)^{-n/2\sigma}s^{-1+n/2p},\ (t-s)^{-1/2}s^{-1}\Big)ds\\ &\le (C_B/4)t^{-1/2}\|u\ ;X\|\|v\ ;X\|.\end{aligned}$$

Finally, the 4th term consisting in X norm of $B(u,v)$ can be bounded by the product of the X norms of u and v as follows;

$$\begin{aligned}\|\nabla B(u,v)(t)\ \dot K^0_{n,\infty}\| &\le \int_0^t \|\nabla e^{(t-s)\Delta}\mathbb{P}(u\cdot\nabla)v(s)\ ;\dot K^0_{n,\infty}\|ds\\ &\le c\int_0^t (t-s)^{-1/2-n(1/\sigma-1/n)/2}\|(u\cdot\nabla)v(s)\ W\dot K^0_{\sigma,\infty}\|ds\\ &\le c\|u\ X_2\|\ \|v\ ;X_4\|\int_0^t (t-s)^{-(1+n/p)/2}s^{-(2-n/p)/2}ds\\ &\le (C_B/4)t^{-1/2}\|u\ ;X\|\ \|v\ ;X\|,\end{aligned}$$

hence one has $\|B(u,v)\ ;X_4\| \le C_B/4\|u\ ;X\|\ \|v\ ;X\|$. By putting three estimates above together, the desired norm inequality is verified and we finish Step 1.

Step 2: In this step, we shall check

$$\|e^{\cdot\Delta}u_0\ ;X\| \le C_0\|u_0\ ;W\dot K^0_{n,\infty}\|.$$

From our assumption and applications of Corollary 2.1, we conclude the claim in the following way;

$$\begin{aligned}\|e^{t\Delta}u_0\ ;W\dot K^0_{n,\infty}\| &\le C_0/4\|u_0\ ;W\dot K^0_{n,\infty}\|,\\ \|e^{t\Delta}u_0\ ;\dot K^0_{p,\infty}\| &\le (C_0/4)\ t^{-(1-n/p)/2}\|u_0\ ;W\dot K^0_{n,\infty}\|,\\ \|e^{t\Delta}u_0\ ;L^\infty\| &\le (C_0/4)t^{-1/2}\|u_0\ ;W\dot K^0_{n,\infty}\|\ \textit{and}\\ \|\nabla e^{t\Delta}u_0\ ;W\dot K^0_{n,\infty}\| &\le (C_0/4)\ t^{-1/2}\|u_0\ ;W\dot K^0_{n,\infty}\|.\end{aligned}$$

Therefore, from the Picard contraction principle, we can find a solution $u \in X$ to (I.E.) which continuously depends on u_0.

Step 3: In this step, we show the continuity of u.
Since $e^{(t\pm\tau)\Delta}u_0 - e^{t\Delta}u_0 = u_0 * (G_{\sqrt{t\pm\tau}} - G_{\sqrt{t}})$, by using Proposition 2.3, one has

$$\|e^{(t\pm\tau)\Delta}u_0 - e^{t\Delta}u_0\ ;W\dot K^0_{n,\infty}\| \le c|G_{\sqrt{t\pm\tau}} - G_{\sqrt{t}}|_{\mathcal{S}}\ \|u_0\ ;W\dot K^0_{n,\infty}\|$$

which implies that $e^{t\Delta}u_0 \in C((0,\infty); W\dot{K}^0_{n,\infty})$, where $|\cdot|_{\mathcal{S}}$ denotes a seminorm of $\mathcal{S}$. To show the right continuity of B in $\dot{K}^0_{n,\infty}$ on $(0,\infty)$, we write, for $t>0$ and $\tau>0$,

$$\begin{aligned} B(u,u)(t+\tau)-B(u,u)(t) &= \int_0^t (e^{(t+\tau-s)\Delta}\mathbb{P}(u\cdot\nabla)u(s) - e^{(t-s)\Delta}\mathbb{P}(u\cdot\nabla)u(s))ds \\ &\quad + \int_t^{t+\tau} e^{(t+\tau-s)\Delta}\mathbb{P}(u\cdot\nabla)u(s)ds \\ &=: \mathrm{I}_+ + \mathrm{II}_+. \end{aligned}$$

Because we have, for sufficiently small $\tau>0$,

$$\|e^{(t+\tau-s)\Delta}\mathbb{P}(u\cdot\nabla)u(s)\ ;\dot{K}^0_{n,\infty}\| \le c(t-s)^{-n/2p}s^{-1+n/2p}\|u\ ;X_2\|\ \|u\ ;X_4\| \in L^1(0,t),$$

the Lebesgue's convergence theorem yields that $\lim_{\tau\searrow 0}\|\mathrm{I}_+\ ;\dot{K}^0_{n,\infty}\| = 0$. And the second term also tends to 0 as $\tau\searrow 0$;

$$\begin{aligned} \|\mathrm{II}_+\ ;\dot{K}^0_{n,\infty}\| &\le \int_t^{t+\tau} \|e^{(t+\tau-s)\Delta}\mathbb{P}(u\cdot\nabla)u(s)\ ;\dot{K}^0_{n,\infty}\|ds \\ &\le c\|u\ ;X_2\|\ \|u\ ;X_4\| \int_t^{t+\tau}(t+\tau-s)^{-n/2p}s^{-1+n/2p}ds \\ &\to 0 \quad \text{as } \tau\searrow 0. \end{aligned}$$

Therefore, (9) is proved. Next, we check the left continuity of B in $W\dot{K}^0_{n,\infty}$ on $(0,\infty)$. To this end, we write, for $t>0$ and $\tau>0$,

$$\begin{aligned} B(u,u)(t-\tau)-B(u,u)(t) &= \int_0^{t-\tau} e^{(t-\tau-s)\Delta}\mathbb{P}(u\cdot\nabla)u(s) - e^{(t-s)\Delta}\mathbb{P}(u\cdot\nabla)u(s)ds \\ &\quad + \int_{t-\tau}^{t} e^{(t-s)\Delta}\mathbb{P}(u\cdot\nabla)u(s)ds \\ &=: \mathrm{I}_- + \mathrm{II}_-. \end{aligned}$$

Since one obtains the estimate

$$\begin{aligned} &\|e^{(t-\tau-s)\Delta}\mathbb{P}(u\cdot\nabla)u(s)\ ;W\dot{K}^0_{n,\infty}\| \\ &\qquad \le c\min\Big(\|(u\cdot\nabla)u(s)\ ;W\dot{K}^0_{n,\infty}\|,\ (t-\tau-s)^{-1/2}\|u\otimes u(s)\ ;W\dot{K}^0_{n,\infty}\|\Big) \\ &\qquad \le c\min\Big(s^{-1},\ (t-\tau-s)^{-1/2}s^{-1/2}\Big)\|u\ ;X\|^2, \end{aligned}$$

the convergence $\lim_{\tau\searrow 0}\|\mathrm{I}_-\ ;W\dot{K}^0_{n,\infty}\| = 0$ is verified from the Lebesgue convergence theorem. It is not hard to show that $\lim_{\tau\searrow 0}\|\mathrm{II}_-\ ;W\dot{K}^0_{n,\infty}\| = 0$. As a result, we obtain the continuity of the solution u in $W\dot{K}^0_{n,\infty}$ on $(0,\infty)$, and $u(t)\in W\dot{\mathcal{K}}^0_{n,\infty}$ for $t>0$, that is (6).

Step 4: *Here we check the properties* (7) *and* (8) *for the solution u.*
Since the property (8) is proved in Steps 1 and 3, we have to verify (7). To do this, thanks to Theorem 1.2 in [42], it suffices to show

$$\lim_{t\searrow 0}|\langle\int_0^t e^{(t-s)\Delta}\mathbb{P}(u\cdot\nabla)u(s)ds,\ \phi\rangle| = 0$$

for any $\phi\in C_0^\infty$. For $n/2<r<n$, the equality can be verified by integration by parts;

$$\begin{aligned} |\langle\int_0^t e^{(t-s)\Delta}\mathbb{P}(u\cdot\nabla)u(s)ds,\ \phi\rangle| &\le \int_0^t \|e^{(t-s)\Delta}\mathbb{P}(u\otimes u)(s)\ ;W\dot{K}^0_{r,\infty}\|\ \|\nabla\phi\ ;Z\|ds \\ &\le c\int_0^t (t-s)^{-1+n/2r}\|u\otimes u(s)\ ;W\dot{K}^0_{n/2,\infty}\|\ \|\nabla\phi\ ;Z\|ds \\ &\le ct^{(n/r-1)/2}\|u\ ;X\|^2\|\nabla\phi\ ;Z\| \to 0,\ \textit{as}\ t\searrow 0, \end{aligned}$$

where

$$\|f\ ;Z\| := \sum_{k\in\mathbb{Z}} \|f\ ;L^{r',1}(A_k)\|.$$

Step 5: Finally, we consider the case $u_0 \in W\dot{K}^0_{n,\infty}$.
In this case, by the same argument as above, we can find a solution u in

$$Y := \{u \in X; \lim_{t\searrow 0} t^{(1-n/p)/2}\|u(t)\ ;\dot{K}^0_{p,\infty}\| = \lim_{t\searrow 0} t^{1/2}\|u(t)\ ;L^\infty\| = 0\}.$$

This means that the required convergences are true.

□

4.2 Proof of Theorem 1.4

Next, we shall show Theorem 1.4 by the method due to Meyer [39] and [31]. The main ingredient is Proposition 2.4 which yields the critical estimate of the bilinear form B.

Proof of Theorem 1.4:

Let $w(t) := u(t) - v(t)$, then we have that $w \in L^\infty(0,T;W\dot{K}^0_{n,\infty})$ and

$$w(t) = B(e^{\cdot\Delta}u_0 - u, w)(t) + B(w, e^{\cdot\Delta}u_0 - v)(t) - B(e^{\cdot\Delta}u_0, w)(t) - B(w, e^{\cdot\Delta}u_0)(t).$$

Remark that

$$\begin{aligned} B(u,v)(t) &= \int_0^t \nabla e^{(t-s)\Delta}\mathbb{P}(u\otimes v)(s)ds \\ &= \int_0^\infty \nabla e^{s\Delta}\mathbb{P}(u\otimes v)\chi_{(0,t)}(t-s)ds. \end{aligned}$$

By using Proposition 2.4 with $r_1 = 2n/3$ and $r_2 = 2n$, i.e. $\beta = 1/4$, one obtains

$$\begin{aligned} \|B(e^{\cdot\Delta}u_0 - u, w)(t)\ ;W\dot{K}^0_{n,\infty}\| &= \|\int_0^\infty \nabla e^{s\Delta}\mathbb{P}((e^{\cdot\Delta}u_0 - u)\otimes w)\chi_{(0,t)}(t-s)ds\ ;W\dot{K}^0_{n,\infty}\| \\ &\le c\|((e^{\cdot\Delta}u_0 - u)\otimes w)\chi_{(0,t)}(t-\cdot)\ ;L^\infty(0,\infty;W\dot{K}^0_{n/2,\infty})\| \\ &\le c\|(e^{\cdot\Delta}u_0 - u)\otimes w\ ;L^\infty(0,t;W\dot{K}^0_{n/2,\infty})\| \\ &\le c\Big(\sup_{0<s\le t}\|e^{s\Delta}u_0 - u(s)\ ;W\dot{K}^0_{n,\infty}\|\Big)\Big(\sup_{0<s\le t}\|w(s)\ ;W\dot{K}^0_{n,\infty}\|\Big). \end{aligned}$$

$B(w, e^{\Delta}u_0 - v)$ also has a similar bound as above. On the other hand, the bilinear form $B(e^{\cdot\Delta}u_0, w)$ is estimated as follows;

$$\begin{aligned} \|B(e^{\cdot\Delta}u_0, w)(t)\ ;W\dot{K}^0_{n,\infty}\| &\le \int_0^t \|\nabla e^{\frac{(t-s)}{2}\Delta}e^{\frac{(t-s)}{2}\Delta}\mathbb{P}(e^{s\Delta}u_0\otimes w)(s)\ ;W\dot{K}^0_{n,\infty}\|ds \\ &\le c\int_0^t (t-s)^{-1/2}\|(e^{s\Delta}u_0\otimes w)(s)\ ;W\dot{K}^0_{n,\infty}\|ds \\ &\le c\Big(\sup_{0<s\le t} s^{1/2}\|e^{s\Delta}u_0\ ;W\dot{K}^0_{\infty,\infty}\|\Big)\Big(\sup_{0<s\le t}\|w(s)\ ;W\dot{K}^0_{n,\infty}\|\Big). \end{aligned}$$

We now claim that

$$\sup_{0<s\le t}\|e^{s\Delta}u_0 - u(s)\ ;W\dot{K}^0_{n,\infty}\| \to 0, \text{ as } t\searrow 0, \tag{21}$$

and

$$\sup_{0<s\le t} s^{1/2}\|e^{s\Delta}u_0\ ;W\dot{K}^0_{\infty,\infty}\| \to 0, \text{ as } t\searrow 0. \tag{22}$$

The claim (21) follows from the continuity of u at $t=0$ and the definition of $W\dot{\mathcal{K}}^0_{n,\infty}$. To show the claim (22), we take a small $\tau>0$ such that $\|u_0-\tilde{u_0}\ ; W\dot{K}^0_{n,\infty}\|$ is sufficiently small, where $\tilde{u_0}=e^{\tau\Delta}u_0$. Applying Corollary 2.1 to $\tilde{u_0}$, one has $\|e^{s\Delta}\tilde{u_0}\ ; W\dot{K}^0_{\infty,\infty}\|\le cs^{-n/(2(n+1))}\|\tilde{u_0}\ ; W\dot{K}^0_{n+1,\infty}\|$, which implies the claim (22). Then, there exists $T_0>0$ such that

$$\sup_{0<s\le T_0}\|w(s)\ ; W\dot{K}^0_{n,\infty}\|\le\frac{1}{2}\sup_{0<s\le T_0}\|w(s)\ ; W\dot{K}^0_{n,\infty}\|,$$

which implies $u=v$ on $[0,T_0]$. Next, we put

$$T^*:=\sup\{0<t\le T\ ; u=v \text{ on } [0,t]\}.$$

Then we assume that $T^*<\infty$, otherwise the proof of Theorem 1.4 is completed. By the continuity of solutions, we have $u=v$ on $[0,T^*]$. Then $\tilde{u}(x,t):=u(x,T^*+t)$ and $\tilde{v}(x,t):=v(x,T^*+t)$ are mild solutions of (N-S) in the class $C([0,T-T^*];W\dot{K}^0_{n,\infty})$ with initial data $u(\cdot,T^*)=v(\cdot,T^*)\in W\dot{\mathcal{K}}^0_{n,\infty}$. Therefore, from the above argument there exists $\tau>0$ such that $\tilde{u}=\tilde{v}$ on $[0,\tau]$, i.e.,

$$u=v \text{ on } [0,T^*+\tau]$$

which contradicts the maximal property of T^*.

□

Remark 4.1. *In the proof above, we showed that the bilinear operator B is bounded from $\tilde{\mathcal{E}}\times\tilde{\mathcal{E}}$ to $\tilde{\mathcal{E}}$ where*

$$\tilde{\mathcal{E}}:=\{f\in L^\infty(0,\infty;W\dot{K}^0_{n,\infty});\ \sup_{t>0}\|f(t)\ ; W\dot{K}^0_{n,\infty}\|<\infty\}.$$

This is similar to the Meyer's estimate [39]; B is continuous from $\mathcal{E}\times\mathcal{E}$ to $\mathcal{E}$ where

$$\mathcal{E}:=\{f\in L^\infty(0,\infty;L^{n,\infty});\ \sup_{t>0}\|f(t)\ ; L^{n,\infty}\|<\infty\}.$$

The discontinuity of B from $C([0,T];L^n)\times C([0,T];L^n)$ to $C([0,T];L^n)$ was showed by Oru [43].

4.3 Proof of Theorem 1.5

We make use of the boundedness of B above to get the stability result. As we mentioned at first section, the reverse also holds without the smallness condition on initial data. Because the proof is not difficult, we omit the detail.

Proof of Theorem 1.5: From the smallness assumption on u_0 and v_0, we have the estimate for solutions u and v; $\|u\ ; L^\infty(0,\infty;W\dot{K}^0_{n,\infty})\|$, $\|v\ ; L^\infty(0,\infty;W\dot{K}^0_{n,\infty})\|\le c_*\delta$. Hence the difference of solutions is dominated by itself with a small constant by applying the boundedness of B mentioned in Remark 4.1 as follows;

$$\begin{aligned}\|u(t)-v(t)\ ; W\dot{K}^0_{n,\infty}\| &\le \|e^{t\Delta}(u_0-v_0)\ ; W\dot{K}^0_{n,\infty}\|\\ &\quad+\|B(u,u-v)(t)\ ; W\dot{K}^0_{n,\infty}\|+\|B(v,u-v)(t)\ ; W\dot{K}^0_{n,\infty}\|\\ &\le \|e^{t\Delta}(u_0-v_0)\ ; W\dot{K}^0_{n,\infty}\|+\|B\|\Big(\sup_{0<s\le t}\|u(s)\ ; W\dot{K}^0_{n,\infty}\|+\sup_{0<s\le t}\|v(s)\ ; W\dot{K}^0_{n,\infty}\|\Big)\\ &\quad\times\sup_{0<s\le t}\|u(s)-v(s)\ ; W\dot{K}^0_{n,\infty}\|\\ &\le \|e^{t\Delta}(u_0-v_0)\ ; W\dot{K}^0_{n,\infty}\|+2c_*\|B\|\delta\sup_{0<s\le t}\|u(s)-v(s)\ ; W\dot{K}^0_{n,\infty}\|.\end{aligned}$$

Therefore, we obtain $\sup_{0<s\le t}\|u(s)-v(s)\ ; W\dot{K}^0_{n,\infty}\|\le\frac{1}{1-2c_*\|B\|\delta}\|e^{t\Delta}(u_0-v_0)\ ; W\dot{K}^0_{n,\infty}\|$, which implies $\limsup_{t\nearrow\infty}\|u(t)-v(t)\ ; W\dot{K}^0_{n,\infty}\|=0$.

□

5 Embeddings of Herz spaces into Besov spaces

5.1 Embedding results

We prove Theorem 1.6.

(i): In this case, $-n/\sigma < 0 < \alpha < n(1-1/p)$. Then, by using Corollary 2.1, one has

$$\begin{aligned}\|f\ ;\dot{B}^{-(\alpha+n(1/p-1/\sigma))}_{\sigma,\infty}\| &= \sup_{k\in\mathbb{Z}} 2^{-k(\alpha+n(1/p-1/\sigma))}\|(f_{2^k}*\mathcal{F}^{-1}\phi)_{2^{-k}}\ ;L^\sigma\| \\ &\approx \sup_{k\in\mathbb{Z}} 2^{-k(\alpha+n(1/p-1))}\|f_{2^k}*\mathcal{F}^{-1}\phi\ ;\dot{K}^0_{\sigma,\sigma}\| \\ &\le \sup_{k\in\mathbb{Z}} 2^{-k(\alpha+n(1/p-1))}\|f_{2^k}\ ;W\dot{K}^\alpha_{p,\infty}\| \\ &\approx \|f\ ;W\dot{K}^\alpha_{p,\infty}\|.\end{aligned}$$

(ii): In this case, $-n/p < 0 < \alpha < n(1-1/p)$. The inequality $\|f\ ;\dot{B}^{-\alpha}_{p,\infty}\| \lesssim \|f\ ;\dot{K}^\alpha_{p,\infty}\|$ can be verified by the same argument as (i).

(iii): In this case, $-n/\sigma < 0 < n(1-1/p)$. By replacing the estimate $\|f_{2^k}*\mathcal{F}^{-1}\phi\ ;\dot{K}^0_{\sigma,\sigma}\| \lesssim \|f_{2^k}\ ;W\dot{K}^\alpha_{p,\infty}\|$ in the proof of (i) by the estimate $\|f_{2^k}*\mathcal{F}^{-1}\phi\ ;\dot{K}^0_{\sigma,\sigma}\| \lesssim \|f_{2^k}\ ;W\dot{K}^0_{p,\sigma}\|$, we can get the inequality $\|f\ ;\dot{B}^{-n(1/p-1/\sigma)}_{\sigma,\infty}\| \lesssim \|f\ ;W\dot{K}^0_{p,\sigma}\|$.

(iv): In this case, $0 \le \alpha < n(1-1/p)$. We omit the details.

(v): In this case, $0 \le \alpha < n$. We omit the details.

(vi): If $\alpha + n/p = 0$, i.e. $\alpha = 0$ and $p = \infty$, then we have

$$W\dot{K}^0_{\infty,1} = \dot{K}^0_{\infty,1} \hookrightarrow \dot{K}^0_{\infty,\infty} \hookrightarrow \dot{B}^0_{\infty,\infty}.$$

To show the case $\alpha + n/p > 0$, we use the following lemma with $\Psi = G$.

Lemma 5.1. *Let $0 < p, q \le \infty$. Then, for any $\Psi \in \mathcal{S}$ there exists a constant $C_\Psi > 0$, depending on p, q, α, n and Ψ, such that*

$$\sup_{y\in\mathbb{R}^n}\|\Psi(\cdot+y)\ ;\dot{K}^\alpha_{(p,1),q}\| \le C_\Psi,\ \ if\ \ p<\infty$$

and

$$\sup_{y\in\mathbb{R}^n}\|\Psi(\cdot+y)\ ;\dot{K}^\alpha_{p,q}\| \le C_\Psi,$$

provided that $q < \infty$ and $-n/p < \alpha \le 0$ or $q = \infty$ and $-n/p \le \alpha \le 0$, where

$$\|f\ ;\dot{K}^\alpha_{(p,r),q}\| := \Big(\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\|f\chi_k\ ;L^{p,r}\|^q\Big)^{1/q}.$$

We verify the first inequality of Lemma 5.1 after the proof of Theorem 1.6 is completed. An application of the lemma yields

$$\begin{aligned}|e^{t\Delta}f(x)| &\le t^{-n/2}\|f_{1/\sqrt{t}}\ ;W\dot{K}^\alpha_{p,1}\|\|G(\cdot-\frac{x}{\sqrt{t}})\ ;\dot{K}^{-\alpha}_{(p',1),\infty}\| \\ &\le c\ t^{-(\alpha+n/p)/2}\|f\ ;W\dot{K}^\alpha_{p,1}\|,\end{aligned}$$

where $p' = \frac{p}{p-1}$ is the conjugate exponent. Here we have used that $-n/p' \le -\alpha \le 0$, i.e., $0 \le \alpha \le n(1-1/p)$. As a consequence, we have

$$\sup_{t>0} t^{(\alpha+n/p)/2}\|e^{t\Delta}f\ ;L^\infty\| \le c\|f\ ;W\dot{K}^\alpha_{p,1}\|.$$

(vii): Since $L^1 \hookrightarrow \dot{B}^{-n}_{\infty,\infty}$, we have $\|f\ ;\dot{B}^{-n}_{\infty,\infty}\| \lesssim \|f\ ;\dot{K}^0_{1,1}\|$.

(viii): From (iv), it readily follows that

$$
\begin{aligned}
\|f\ ; bmo^{-1}\| &= \sup_{|Q|\le 1}\Big(\fint_Q \int_0^{l(Q)^2} |e^{t\Delta} f(x)|^2 dt dx\Big)^{1/2} \\
&\le c \sup_{|Q|\le 1}\Big(\int_0^{l(Q)^2} t^{-(\alpha+n/p)} \|f\ ; W\dot{K}^{\alpha}_{p,\infty}\|^2 dt\Big)^{1/2} \\
&\le c\|f\ ; W\dot{K}^{\alpha}_{p,\infty}\|.
\end{aligned}
$$

□

We shall verify Lemma 5.1. To do this, it suffices to prove the following lemma;

Lemma 5.2. *Let $0 < p, q \le \infty$ and $y \in \mathbb{R}^n$. Then, for any $\Psi \in \mathcal{S}$ the inequalities*

$$\|\Psi(\cdot + y)\ ; \dot{K}^{\alpha}_{(p,1),q}\| \le C_\Psi(1 + \min(|y|^{\alpha},\ |y|^{\alpha+n/p})),\ if\ p < \infty$$

and

$$\|\Psi(\cdot + y)\ ; \dot{K}^{\alpha}_{p,q}\| \le C_\Psi(1 + \min(|y|^{\alpha},\ |y|^{\alpha+n/p}))$$

hold with a constant C_Ψ depending on p, q, α, n and Ψ, provided that $q < \infty$ and $-n/p < \alpha < \infty$ or $q = \infty$ and $-n/p \le \alpha < \infty$.

Proof. We show the first inequality in the case $q < \infty$ and $-n/p < \alpha < \infty$ only. The other case can be showed by the same argument. Let k_0 be an integer satisfying $2^{k_0-1} \le |y| < 2^{k_0}$, i.e. $y \in A_{k_0}$. We decompose the left hand side;

$$
\begin{aligned}
\|\Psi(\cdot + y)\ ; \dot{K}^{\alpha}_{(p,1),q}\| &\le c\Big(\sum_{k=-\infty}^{k_0-2} 2^{k\alpha q}\|\Psi(\cdot + y)\ ; L^{p,1}(A_k)\|^q\Big)^{1/q} \\
&\quad + c\Big(\sum_{k=k_0-1}^{k_0+1} 2^{k\alpha q}\|\Psi(\cdot + y)\ ; L^{p,1}(A_k)\|^q\Big)^{1/q} \\
&\quad + c\Big(\sum_{k=k_0+2}^{\infty} 2^{k\alpha q}\|\Psi(\cdot + y)\ ; L^{p,1}(A_k)\|^q\Big)^{1/q} \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}
$$

Each term are dominated by the right hand side of the statement in the following way;

$$
\begin{aligned}
\mathrm{I} &\le C_\Psi\Big(\sum_{k=-\infty}^{k_0-2} 2^{k\alpha q}\|\langle \cdot + y\rangle^{-N}\ ; L^{p,1}(A_k)\|^q\Big)^{1/q} \\
&\le C_\Psi\ 2^{-k_0 N}|y|^{\alpha+n/p} \\
&\le C_\Psi,
\end{aligned}
$$

where N is a positive integer with $|N - (\alpha + n/p)| \le 1$,

$$
\begin{aligned}
\mathrm{II} &\le c\min(\|\Psi\ ; L^{p,1}\||y|^{\alpha}\ ,\ \|\Psi\ ; L^{\infty}\||y|^{\alpha+n/p}) \\
&\le C_\Psi \min(|y|^{\alpha}\ ,\ |y|^{\alpha+n/p})
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{III} &\le c\Big(\sum_{k=k_0}^{\infty} 2^{k\alpha q}\|\Psi\ ; L^{p,1}(A_k)\|^q\Big)^{1/q} \\
&\le c\|\Psi\ ; \dot{K}^{\alpha}_{(p,1),q}\| \\
&\le C_\Psi.
\end{aligned}
$$

□

5.2 Negative results

Next, we shall prove Theorem 1.7.

(i): Let

$$f(x) := \sum_{k\in\mathbb{Z}} \frac{2^{-k\alpha}}{|x-x_k|^{n/p}}\chi_k(x),$$

where $x_k := (\frac{3}{2}2^{k-1}, 0, \cdots, 0) \in \mathbb{R}^n$. Since $\|f\ ; L^{p,\infty}(A_k)\| = \sup_{\lambda>0} \lambda |A_k \cap B(x_k, \lambda^{-p/n})|^{1/p}$, we see $f \in W\dot{K}^{\alpha}_{p,\infty}$. On the other hand, by using (10), we have

$$\begin{aligned}
\|f\ ; \dot{B}^{-(\alpha+n(1/p-1/\sigma))}_{\sigma,\infty}\| &\approx \sup_{t>0} t^{(\alpha+n(1/p-1/\sigma))/2}\|e^{t\Delta}f\ ; L^{\sigma}\| \\
&\geq \|e^{\Delta}f\ ; L^{\sigma}\| \\
&= \Big(\sum_{k\in\mathbb{Z}} \|e^{\Delta}f\ ; L^{\sigma}(A_k)\|^{\sigma}\Big)^{1/\sigma}.
\end{aligned}$$

For a sufficiently large k, the term $\|e^{\Delta}f\ ; L^{\sigma}(A_k)\|$ has the lower bound $2^{-k\alpha}$. In fact,

$$\begin{aligned}
\|e^{\Delta}f\ ; L^{\sigma}(A_k)\| &\geq \frac{1}{(4\pi)^{n/2}}\Big(\int_{A_k}(\int_{A_k}\frac{2^{-k\alpha}}{|y-x_k|^{n/p}}e^{-|x-y|^2/4}dy)^{\sigma}dx\Big)^{1/\sigma} \\
&\geq \frac{1}{(4\pi)^{n/2}}\Big(\int_{B(x_k,1)}(\int_{2\leq|y-x_k|\leq 3}\frac{2^{-k\alpha}}{|y-x_k|^{n/p}}e^{-|x-y|^2/4}dy)^{\sigma}dx\Big)^{1/\sigma} \\
&\gtrsim 2^{-k\alpha}.
\end{aligned}$$

Hence, since $\alpha \leq 0$, we have $\|f\ ; \dot{B}^{-(\alpha+n(1/p-1/\sigma))}_{\sigma,\infty}\| = \infty$.

(ii): From the same argument as above, one has

$$\begin{aligned}
\|f\ ; \dot{B}^{-(\alpha+n/p)}_{\infty,\infty}\| &\gtrsim \sup_{k\in\mathbb{Z}} \|f * G\ ; L^{\infty}(B(x_k,1))\| \\
&\geq \sup_{k\gg 0} 2^{-k\alpha} = \infty.
\end{aligned}$$

□

Furthermore, there is no embedding relation between $\dot{K}^{-1}_{n,\infty}$ and BMO, which include L^{∞}, that is,

Proposition 5.1. *There is no embedding relation between $\dot{K}^{-1}_{n,\infty}$ and BMO. Moreover, there is no embedding relation between $\dot{K}^{-1}_{n,\infty}$ and BMO^d, where BMO^d stands for dyadic BMO.*

Proof. Because $\log|x| \notin \dot{K}^{-1}_{n,\infty}$, the problem is whether $\dot{K}^{-1}_{n,\infty} \hookrightarrow BMO$ is true or not. Let us consider a function $h(x) := (-\log|x-2|)\chi_{(2,4)}(x)$ on $\mathbb{R}$. By an easy computation, we obtain $h \in \dot{K}^{-1}_{n,\infty}(\mathbb{R})$ and $h \notin BMO(\mathbb{R}) \cap BMO^d(\mathbb{R})$, which imply the embedding does not hold in general. □

Acknowledgements. The author would like to express thank to Prof. T. Nishitani for valuable comments. The author is grateful to Prof. A. Miyachi for indicating him a proof of the inclusion $W\dot{K}^{0}_{n,\infty} \hookrightarrow BMO^{-1}$. The author also would like to thank Prof. N. Tomita, Prof. H. Miura and Dr. T. Yoneda for valuable conversations. The author would like to thank the referees for many helpful suggestions.

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