Variable Lebesgue norm estimates for BMO functions II

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Abstract

The paper concerns characterization of BMO in terms of Banach function spaces. In particular, we are interested in characterizing BMO by using the variable Lebesgue norm.

1 Introduction

We propose a property of the Hardy-Littlewood maximal operator M here. For a measurable function $f : \mathbf{R}^n \to \mathbf{C}$, we define

$$Mf(x) := \sup_{r>0,\, y\in\mathbf{R}^n\,;\, x\in y+(-r,r)^n} \frac{1}{(2r)^n} \int_{y+[-r,r]^n} |f(z)|\, dz.$$

To state our main results, we need to describe the Lebesgue space $L^{p(\cdot)}(\mathbf{R}^n)$ with variable exponent. For a measurable function $p(\cdot) : \mathbf{R}^n \to [1,\infty)$, the Lebesgue space $L^{p(\cdot)}(\mathbf{R}^n)$ with variable exponent is defined to the set of all measurable functions f on $\mathbf{R}^n$ for which the quantity

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0 \,:\, \int_{\mathbf{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}$$

is finite. We shall prove;

Theorem A Let $p(\cdot) : \mathbf{R}^n \to [1,\infty)$ be a bounded function. Assume that the Hardy-Littlewood maximal operator M is of weak type $(p(\cdot), p(\cdot))$, namely

$$\sup_{\lambda>0} \lambda \|\chi_{\{Mf(x)>\lambda\}}\|_{L^{p(\cdot)}} \le C\|f\|_{L^{p(\cdot)}}$$

holds for all measurable functions f. Then there exists a constant $0 < \delta \le 1$ such that

$$\|f^\delta\|_{L^{p(\cdot)}} \le C\left(\frac{1}{|Q|}\int_Q f(x)\, dx\right)^\delta \|\chi_Q\|_{L^{p(\cdot)}} \tag{1}$$

for all non-negative measurable functions f supported on a cube Q.

The space $\mathrm{BMO}(\mathbf{R}^n)$ is a famous space and it dates back to the paper of John and Nirenberg [12]. Theorem A enables us to characterize $\mathrm{BMO}(\mathbf{R}^n)$, the set of all locally integrable functions with bounded mean oscillation, by means of Banach function spaces.

In the whole paper we will use the following notation:

1. Given a measurable set $S \subset \mathbf{R}^n$, we denote the Lebesgue measure by $|S|$ and the characteristic function by χ_S.
2. Given a measurable set $S \subset \mathbf{R}^n$ such that $0 < |S| < \infty$ and a function f on $\mathbf{R}^n$, we denote the mean value of f on S by f_S, namely $f_S := \frac{1}{|S|}\int_S f(x)\, dx$.
3. A symbol C always stands for a positive constant independent of the main parameters.
4. A cube $Q \subset \mathbf{R}^n$ is always assumed to be open and have sides parallel to the coordinate axes. Namely we can write
$$Q = \prod_{\nu=1}^n (x_\nu - r/2, x_\nu + r/2)$$
using a point $x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$ and a constant $r > 0$.

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5. The BMO space $\mathrm{BMO}(\mathbf{R}^n)$ consists of all $b \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ such that

$$\|b\|_{\mathrm{BMO}} := \sup_{Q:\mathrm{cube}} \frac{1}{|Q|} \int_Q |b(x) - b_Q|\, dx < \infty. \tag{2}$$

We apply Theorem A and investigate the space $\mathrm{BMO}(\mathbf{R}^n)$:

Theorem B. Let $p(\cdot) : \mathbf{R}^n \to [1, \infty)$ be a bounded function. Assume that the Hardy-Littlewood maximal operator M is of weak type $(p(\cdot), p(\cdot))$. Then we have that for all $b \in \mathrm{BMO}(\mathbf{R}^n)$,

$$C^{-1}\|b\|_{\mathrm{BMO}} \le \sup_{Q:\mathrm{cube}} \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}} \|(b - b_Q)\chi_Q\|_{L^{p(\cdot)}} \le C\, \|b\|_{\mathrm{BMO}}. \tag{3}$$

Theorem B gives an example of affirmative answers for the following problem:

Problem Let X be a subset of the set of all measurable functions on $\mathbf{R}^n$. Suppose that X is a Banach function space eqquipped with a norm $\|\cdot\|_X$. We write

$$\|b\|_{\mathrm{BMO}_X} := \sup_{Q:\mathrm{cube}} \frac{1}{\|\chi_Q\|_X} \|(b - b_Q)\chi_Q\|_X.$$

Can we say that there exists a constant $C > 0$ such that

$$C^{-1}\|b\|_{\mathrm{BMO}} \le \|b\|_{\mathrm{BMO}_X} \le C\|b\|_{\mathrm{BMO}}$$

for all $b \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$?

The first author and the second author proved the following results:

Theorem C. Let $p(\cdot) : \mathbf{R}^n \to (0, \infty)$ be a bounded variable exponent.

1. (Izuki [9]) If $p(\cdot)$ is an exponent for which M is bounded on $L^{p(\cdot)}(\mathbf{R}^n)$, then we have that for all $b \in \mathrm{BMO}(\mathbf{R}^n)$,

$$C^{-1}\|b\|_{\mathrm{BMO}} \le \sup_{Q:\mathrm{cube}} \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}} \|(b - b_Q)\chi_Q\|_{L^{p(\cdot)}} \le C\, \|b\|_{\mathrm{BMO}}. \tag{4}$$

2. (Izuki-Sawano [10]) If $1 \le p_- = \inf p(\cdot)$ and $p(\cdot) \in LH(\mathbf{R}^n)$, then equivalence (4) is also true.

We refer to Subsection 2.1 for the definition of $LH(\mathbf{R}^n)$.

Theorem D[Ho [8]] If the Hardy-Littlewood maximal operator M is bounded on the associate space X', then we have that, for all $b \in \mathrm{BMO}(\mathbf{R}^n)$,

$$C^{-1}\|b\|_{\mathrm{BMO}} \le \|b\|_{\mathrm{BMO}_X} \le C\, \|b\|_{\mathrm{BMO}}.$$

We refer to the book [1] for the definition of Banach function spaces and we recall it in Subsection 2.2.

We note that Theorem D includes Theorem C. However, Theorem B is an outrange of Theorem D.

Here we organize the remaining part of this paper. We clarify some terminology in Section 2. In Section 3, we prove Theorems A and B In Section 4, we give an equivalence norm of $\mathrm{BMO_X}$ under some condition on X. Section 5 contains another characterization of $\mathrm{BMO}(\mathbf{R}^n)$ by using the harmonic extension.

2 Preliminaries

2.1 Lebesgue spaces with variable exponent

Let $\Omega \subset \mathbf{R}^n$ be a measurable set such that $|\Omega| > 0$.

Given a measurable function $p(\cdot) : \Omega \to [1, \infty]$, define the Lebesgue space with variable exponent

$$L^{p(\cdot)}(\Omega) := \{f : \rho_p(f/\lambda) < \infty \text{ for some } \lambda > 0\},$$

where

$$\rho_p(f) := \int_{\{p(x)<\infty\}} |f(x)|^{p(x)}\, dx + \|f\|_{L^\infty(\{p(x)=\infty\})}.$$

We additionally define

$$\|f\|_{L^{p(\cdot)}} := \|f\|_{L^{p(\cdot)}(\Omega)} = \inf\{\lambda > 0 : \rho_p(f/\lambda) \le 1\}.$$

The functional $\|\cdot\|_{L^{p(\cdot)}}$ is a norm of the space $L^{p(\cdot)}(\Omega)$. If a variable exponent $p(\cdot)$ equals to a constant, then $L^{p(\cdot)}(\Omega)$ is the usual Lebesgue space with norm coincidence.

1. Given a variable exponent $p(\cdot) : \Omega \to [1, \infty]$, we define

$$p_+ := \|p\|_{L^\infty(\Omega)}, \; p_- := \left\{ \left(\frac{1}{p} \right)_+ \right\}^{-1}.$$

2. The set $\mathcal{P}(\Omega)$ consists of all variable exponents $p(\cdot)$ such that $1 < p_- \le p_+ < \infty$.
3. The set $\mathcal{B}(\Omega)$ consists of all $p(\cdot) \in \mathcal{P}(\Omega)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\Omega)$.
4. A measurable function $r(\cdot) : \Omega \to (0, \infty)$ is said to be globally log-Hölder continuous if the following two conditions are satisfied:

$$|r(x) - r(y)| \le \frac{C}{-\log(|x-y|)} \qquad (|x-y| \le 1/2),$$
$$|r(x) - r_\infty| \le \frac{C}{\log(e + |x|)} \qquad (x \in \Omega),$$

where r_∞ is a real constant. The set $LH(\Omega)$ consists of all globally log-Hölder continuous functions.

The next proposition ([3, 6]) gives us a sufficient condition for the boundedness of the Hardy-Littlewood maximal operator when a variable exponent $p(\cdot) : \mathbf{R}^n \to [1, \infty]$ satisfies $1 \le p_- \le p_+ \le \infty$ and $1/p(\cdot) \in LH(\mathbf{R}^n)$. Then M is of weak type $(p(\cdot), p(\cdot))$, that is,

$$\|\chi_{\{Mf(x) > \lambda\}}\|_{L^{p(\cdot)}} \le C\lambda^{-1} \|f\|_{L^{p(\cdot)}}$$

holds for all $\lambda > 0$ and all $f \in L^{p(\cdot)}(\mathbf{R}^n)$. Additionally if $1 < p_-$, then M is bounded on $L^{p(\cdot)}(\mathbf{R}^n)$, that is,

$$\|Mf\|_{L^{p(\cdot)}} \le C \, \|f\|_{L^{p(\cdot)}}.$$

We next state some equivalent conditions due to Diening [5]. Below $p'(\cdot)$ means the conjugate exponent of $p(\cdot)$, that is, $1/p(x) + 1/p'(x) = 1$ holds, and $\mathcal{Y}$ consists of all families of disjoint cubes. We recall the result due to Diening[5]. Given a variable exponent $p(\cdot) \in \mathcal{P}(\mathbf{R}^n)$, the next four conditions are equivalent:

(D1) $p(\cdot) \in \mathcal{B}(\mathbf{R}^n)$.

(D2) $p'(\cdot) \in \mathcal{B}(\mathbf{R}^n)$.

(D3) There exists a constant $q \in (1, p_-)$ such that $p(\cdot)/q \in \mathcal{B}(\mathbf{R}^n)$.

(D4) For all $Y \in \mathcal{Y}$ and all $f \in L^{p(\cdot)}(\mathbf{R}^n)$, we have

$$\left\| \sum_{Q \in Y} |f|_Q \chi_Q \right\|_{L^{p(\cdot)}} \le C \, \|f\|_{L^{p(\cdot)}}.$$

If we take an arbitrary cube Q and put $Y = \{Q\}$ and $f = f\chi_Q$ in (D4) above, then we get a weaker condition:

(A1) $|f|_Q \|\chi_Q\|_{L^{p(\cdot)}} \le C \, \|f\chi_Q\|_{L^{p(\cdot)}}$ holds for all cubes Q and all $f \in L^{p(\cdot)}(\mathbf{R}^n)$.

Condition (A1) is a necessary condition for the weak boundedness of M on $L^{p(\cdot)}$ and equivalent to the following (A2) called the Muckenhoupt condition for a variable exponent $p(\cdot)$:

(A2) $\displaystyle \sup_{Q:\text{cube}} \frac{1}{|Q|} \|\chi_Q\|_{L^{p(\cdot)}} \|\chi_Q\|_{L^{p'(\cdot)}} < \infty.$

We will prove those facts in Lemmas H and I in the context of general Banach function spaces.

2.2 Banach function spaces

In this subsection we first recall the definition and fundamental properties of Banach function spaces.

Let $\Omega \subset \mathbf{R}^n$ be a measurable subset with $|\Omega| > 0$ and $\mathcal{M}(\Omega)$ the set of all measurable and complex-valued functions on Ω. A linear space $X \subset \mathcal{M}(\Omega)$ is said to be a Banach function space if there exists a functional $\|\cdot\|_X : \mathcal{M}(\Omega) \to [0, \infty]$ with the following conditions: Let f, g, $f_j \in \mathcal{M}(\Omega)$ $(j = 1, 2, \ldots)$.

1. $f \in X$ if and only if $\|f\|_X < \infty$.

2. (Norm property):
   (a) (Positivity): $\|f\|_X \geq 0$.
   (b) (Strict Positivity): $\|f\|_X = 0$ if and only if $f = 0$ a.e..
   (c) (Homogeneity): $\|\lambda f\|_X = |\lambda| \cdot \|f\|_X$.
   (d) (Triangle inequality): $\|f+g\|_X \leq \|f\|_X + \|g\|_X$.
3. (Symmetry): $\|f\|_X = \|\,|f|\,\|_X$.
4. (Lattice property): If $0 \leq g \leq f$ a.e., then $\|g\|_X \leq \|f\|_X$.
5. (Fatou property): If $0 \leq f_1 \leq f_2 \leq \ldots$ and $\lim_{j\to\infty} f_j = f$, then
$$\lim_{j\to\infty} \|f_j\|_X = \|f\|_X.$$
6. For all measurable sets F with $|F| < \infty$, it follows $\|\chi_F\|_X < \infty$ and
$$\int_F |f(x)|dx \leq C_F \|f\|_X \quad (f \in X)$$
with the constant C_F depending on F.

Next, we recall the notion of the associate space. Let $X \subset \mathcal{M}(\Omega)$ be a Banach function space equipped with a norm $\|\cdot\|_X$. The associate space X' is defined by
$$X' := \{f \in \mathcal{M}(\Omega) : \|f\|_{X'} < \infty\},$$
where
$$\|f\|_{X'} := \sup\left\{\left|\int_\Omega f(x)g(x)\,dx\right| : \|g\|_X \leq 1\right\}.$$
For example the Lebesgue space $L^{p(\cdot)}(\Omega)$ with variable exponent $p(\cdot) : \Omega \to [1,\infty]$ is a Banach function space and the associated space is $L^{p'(\cdot)}(\Omega)$.

The following lemma consists of the generalized Hölder inequality and the norm equivalence for Banach function spaces.

Lemma G. Let $X \subset \mathcal{M}(\Omega)$ be a Banach function space.

1. For all $f \in X$ and all $g \in X'$, we have
$$\int_\Omega |f(x)g(x)|\,dx \leq C\,\|f\|_X \|g\|_{X'}.$$
2. For all $f \in X$ we have
$$C^{-1}\|f\|_X \leq \sup\left\{\left|\int_\Omega f(x)g(x)\,dx\right| : \|g\|_{X'} \leq 1\right\} \leq C\,\|f\|_X.$$
In particular the space $(X')'$ is equal to X.

As an application of Lemma G, we show the following equivalence. **Lemma H.** Let $X \subset \mathcal{M}(\mathbf{R}^n)$ be a Banach function space. Then the following two conditions are equivalent:

(I)
$$\sup_{Q:\text{cube}} \frac{1}{|Q|}\|\chi_Q\|_X \|\chi_Q\|_{X'} < \infty.$$
(II) For all cubes Q and all $f \in L^1_{\text{loc}}(\mathbf{R}^n)$ we have
$$|f|_Q \|\chi_Q\|_X \leq C\,\|f\chi_Q\|_X.$$

Proof. Take an open cube Q and $f \in L^1_{\text{loc}}(\mathbf{R}^n)$ arbitarily. The implication (II) $\Rightarrow$ (I) is proved as follows;
$$\begin{aligned}\frac{1}{|Q|}\|\chi_Q\|_X\|\chi_Q\|_{X'} &\leq \frac{C}{|Q|}\|\chi_Q\|_X \sup\left\{\int_{\mathbf{R}^n}|f(x)|\chi_Q(x)\,dx : \|f\|_X \leq 1\right\}\\ &= C\sup\{|f|_Q\|\chi_Q\|_X : \|f\|_X \leq 1\}\\ &\leq C\sup\{\|f\chi_Q\|_X : \|f\|_X \leq 1\}\\ &\leq C.\end{aligned}$$

On the other hand, from (I) and the Hölder inequality, (II) is verified;

$$
\begin{aligned}
|f|_Q \|\chi_Q\|_X &= \frac{1}{|Q|}\int_Q |f(y)|\,dy \cdot \|\chi_Q\|_X \\
&\le C\cdot \frac{1}{|Q|}\|f\chi_Q\|_X \|\chi_Q\|_{X'}\|\chi_Q\|_X \\
&\le C\,\|f\chi_Q\|_X.
\end{aligned}
$$

∘

Lemma H. If the Hardy-Littlewood maximal operator M is weak bounded on X, that is

$$\|\chi_{\{Mg>\lambda\}}\|_X \le C\lambda^{-1}\|g\|_X$$

holds for all $\lambda > 0$ and all $g \in X$, then we have

$$|f|_Q\|\chi_Q\|_X \le C\,\|f\chi_Q\|_X$$

for all cubes Q and all $f \in L^1_{\rm loc}(\mathbf{R}^n)$.

Proof. Take a cube Q and $f \in L^1_{\rm loc}(\mathbf{R}^n)$ arbitrarily. If $|f|_Q = 0$, then the conclusion is obviously true. Below we assume $|f|_Q > 0$ and write $\lambda := |f|_Q/2$. Since $|f|_Q\chi_Q(x) \le C\,M(f\chi_Q)(x)$ one has

$$M(f\chi_Q) > \lambda \quad \text{on } Q.$$

Thus, we get

$$|f|_Q\|\chi_Q\|_X \le |f|_Q\|\chi_{\{M(f\chi_Q)(x)>\lambda\}}\|_X \le |f|_Q\cdot C\lambda^{-1}\|f\chi_Q\|_X = C\,\|f\chi_Q\|_X.$$

This proves the lemma. ∘

3 Proof of Theorems A and B

3.1 Proof of Theorem A

We will use the next lemma in order to prove the theorem above. **Lemma J.**

(i) Let $r(\cdot) : \mathbf{R}^n \to (0,\infty)$ be a bounded measurable function with $r_+ \le 1$. It holds

$$\|f+g\|_{L^{r(\cdot)}} \ge \|f\|_{L^{r(\cdot)}} + \|g\|_{L^{r(\cdot)}}$$

for all positive measurable functions f, g.

(ii) Let $p(\cdot) : \mathbf{R}^n \to (0,\infty)$ be a bounded measurable function with $p_+ \ge 1$. It holds

$$\|f+g\|_{L^{p(\cdot)}}^{p_+} \ge \|f\|_{L^{p(\cdot)}}^{p_+} + \|g\|_{L^{p(\cdot)}}^{p_+}$$

for all positive measurable functions f, g.

Proof. We first prove (i). Note that

$$((1-\theta)a+\theta b)^{r(x)} \ge (1-\theta)a^{r(x)} + \theta b^{r(x)},$$

since $\phi_x(t) := t^{r(x)}$ is concave. Hence we have

$$
\begin{aligned}
&\int_{\mathbf{R}^n}\left(\frac{f(x)+g(x)}{\|f\|_{L^{r(\cdot)}}+\|g\|_{L^{r(\cdot)}}}\right)^{r(x)} dx \\
&= \int_{\mathbf{R}^n}\left(\frac{\|f\|_{L^{r(\cdot)}}}{\|f\|_{L^{r(\cdot)}}+\|g\|_{L^{r(\cdot)}}}\cdot\frac{f(x)}{\|f\|_{L^{r(\cdot)}}} + \frac{\|g\|_{L^{r(\cdot)}}}{\|f\|_{L^{r(\cdot)}}+\|g\|_{L^{r(\cdot)}}}\cdot\frac{g(x)}{\|g\|_{L^{r(\cdot)}}}\right)^{r(x)} dx \\
&\ge \int_{\mathbf{R}^n}\left\{\frac{\|f\|_{L^{r(\cdot)}}}{\|f\|_{L^{r(\cdot)}}+\|g\|_{L^{r(\cdot)}}}\left(\frac{f(x)}{\|f\|_{L^{r(\cdot)}}}\right)^{r(x)} + \frac{\|g\|_{L^{r(\cdot)}}}{\|f\|_{L^{r(\cdot)}}+\|g\|_{L^{r(\cdot)}}}\left(\frac{g(x)}{\|g\|_{L^{r(\cdot)}}}\right)^{r(x)}\right\} dx \\
&= 1.
\end{aligned}
$$

This is the desired result. Next we prove (ii) by applying (i) with $r(\cdot) = p(\cdot)/p_+$. Let h be a positive measurable function. Observe that

$$\|h\|_{L^{p(\cdot)}}^{p_+} = \|h^{p_+}\|_{L^{p(\cdot)/p_+}}$$

and $(f+g)^{p_+} \geq f^{p_+} + g^{p_+}$. Therefore, we obtain

$$\begin{aligned} \|f\|_{L^{p(\cdot)}}^{p_+} + \|g\|_{L^{p(\cdot)}}^{p_+} &= \|f^{p_+}\|_{L^{p(\cdot)/p_+}} + \|g^{p_+}\|_{L^{p(\cdot)/p_+}} \\ &\leq \|f^{p_+} + g^{p_+}\|_{L^{p(\cdot)/p_+}} \\ &\leq \|(f+g)^{p_+}\|_{L^{p(\cdot)/p_+}} \\ &= \|f+g\|_{L^{p(\cdot)}}^{p_+}. \end{aligned}$$

Thus, the proof is complete.

Proof of Theorem A.

1. To prove Theorem A, we invoke the following preliminary observations: We set $Q_{0,0} := E_0 := Q = x_Q + [0,r]^n$. By a dyadic cube of Q we mean the set

$$\{x_Q + 2^{-m}z + 2^{-m}w : w \in [0,r]^n,\, m = 0,1,2,\cdots,\, z \in \{0,1,2,\cdots,2^m-1\}\}.$$

First of all, we let

$$E_k = \left\{x \in Q : 2^{(n+1)(k-1)} f_Q < M^{d,Q} f(x)\right\}, \quad k = 0,1,2,\ldots$$

where $M^{d,Q}$ denotes the dyadic maximal operator with respect to Q, namely,

$$M^{d,Q} f(x) = \sup\left\{\chi_R(x)\left(\frac{1}{|R|}\int_R |f(z)|\,dz\right) : R \text{is a dyadic cube of } Q\right\}.$$

By the definition of the dyadic maximal operator $M^{d,Q}$, we obtain a family of non-overlapping cubes $\{Q_{k,l}\}_{l\in L_k}$ such that

$$\bigcup_{l\in L_k} Q_{k,l} = E_k,$$

and that

$$\frac{1}{|Q_{k,l}|}\int_{Q_{k,l}} f(y)\,dy > 2^{(n+1)(k-1)} f_Q \geq \frac{1}{2^n|Q_{k,l}|}\int_{Q_{k,l}} f(y)\,dy. \tag{5}$$

Note that $(\bigcup_{k=1}^{\infty} E_k \setminus E_{k+1})$ differs from Q by a set of measure zero. Hence,

$$\begin{aligned} f(x) &\leq M^{d,Q} f(x) \qquad (6) \\ &= M^{d,Q} f(x)\chi_{E_0\setminus E_1}(x) + \sum_{k=1}^{\infty} M^{d,Q} f(x)\chi_{E_k\setminus E_{k+1}}(x) \\ &\leq \sum_{k=0}^{\infty} 2^{(n+1)k} f_Q \chi_{E_k}(x) \end{aligned}$$

as we did in [15]. Here for the last inequality, we have used the fact that $E_0 \supset E_1$.

2. About the structure of E_k, we can prove

$$|E_{k+1} \cap Q_{k,l}| \leq \frac{1}{2}|Q_{k,l}|$$

by way of (5) and the decomposition

$$E_{k+1} \cap Q_{k,l} = \bigcup_{l' \in L_{k,l}} Q_{k+1,l'}$$

with $L_{l,k} \subset L_k$. See [15, p.3688] for details. Hence we have

$$|E_{k+1} \cap Q_{k,l}| \leq \frac{1}{2}|Q_{k,l}|.$$

By virtue of the weak boundedness of M, we have

$$\begin{aligned} \frac{1}{2}\|\chi_{E_k}\|_{L^{p(\cdot)}} &= \frac{1}{2}\|\chi_{\{\sum_{l\in L_k}(2\chi_{E_k\setminus E_{k+1}})_{Q_{k,l}}\chi_{Q_{k,l}}>1\}}\|_{L^{p(\cdot)}} \\ &\leq \frac{1}{2}\|\chi_{\{M(2\chi_{E_k\setminus E_{k+1}})>1\}}\|_{L^{p(\cdot)}} \\ &\leq C\|\chi_{E_k\setminus E_{k+1}}\|_{L^{p(\cdot)}}. \end{aligned}$$

Consequently,

$$\|\chi_{E_k}\|_{L^{p(\cdot)}} \le 2C\|\chi_{E_k\setminus E_{k+1}}\|_{L^{p(\cdot)}}. \tag{7}$$

Hence by using Lemma J, we have

$$\|f+g\|_{L^{p(\cdot)}}^{p_+} \ge \|f\|_{L^{p(\cdot)}}^{p_+} + \|g\|_{L^{p(\cdot)}}^{p_+},$$

which holds for all positive measurable functions f, g. In particular,

$$\|\chi_{E_k}\|_{L^{p(\cdot)}}^{p_+} \ge \|\chi_{E_{k+1}}\|_{L^{p(\cdot)}}^{p_+} + \|\chi_{E_k\setminus E_{k+1}}\|_{L^{p(\cdot)}}^{p_+}. \tag{8}$$

Thus, combining (7) and (8), we obtain

$$\|\chi_{E_k}\|_{L^{p(\cdot)}}^{p_+} \ge \|\chi_{E_{k+1}}\|_{L^{p(\cdot)}}^{p_+} + \left(\frac{1}{2C}\|\chi_{E_k}\|_{L^{p(\cdot)}}\right)^{p_+}$$

we conclude that

$$\|\chi_{E_{k+1}}\|_{L^{p(\cdot)}} \le \left(1-\left(\frac{1}{2C}\right)^{p_+}\right)^{1/p_+}\|\chi_{E_k}\|_{L^{p(\cdot)}}, \quad k=0,1,2,\ldots.$$

Thus, we have

$$\|\chi_{E_k}\|_{L^{p(\cdot)}} \le \left(1-\left(\frac{1}{2C}\right)^{p_+}\right)^{k/p_+}\|\chi_Q\|_{L^{p(\cdot)}}, \quad k=0,1,2,\ldots. \tag{9}$$

3. If we combine (6) and (9), then we take a positive constant $\delta \le 1$ so that

$$2^{(n+1)\delta/p_-}\left(1-\left(\frac{1}{2C}\right)^{p_+}\right)^{1/(p_+p_-)} < 1$$

and obtain

$$\begin{aligned}
\|f^\delta\|_{L^{p(\cdot)}}^{1/p_-} &\le \left\|\sum_{k=0}^{\infty} 2^{(n+1)k\delta}(f_Q)^\delta \chi_{E_k}\right\|_{L^{p(\cdot)}}^{1/p_-} \\
&\le \sum_{k=0}^{\infty} 2^{(n+1)k\delta/p_-}\left\|(f_Q)^\delta\chi_{E_k}\right\|_{L^{p(\cdot)}}^{1/p_-} \\
&\le \sum_{k=0}^{\infty} 2^{(n+1)k\delta/p_-}(f_Q)^{\delta/p_-}\left(1-\left(\frac{1}{2C}\right)^{p_+}\right)^{k/(p_+p_-)}\|\chi_Q\|_{L^{p(\cdot)}}^{1/p_-} \\
&\le C(f_Q)^{\delta/p_-}\|\chi_Q\|_{L^{p(\cdot)}}^{1/p_-}.
\end{aligned}$$

This is the desired result.

∘

3.2 Proof of Theorem B

As an application of Theorem A we prove Theorem B.

Proof. We give the proof based on [10]. Take a cube Q and $b \in \mathrm{BMO}(\mathbf{R}^n)$ arbitrarily. By virtue of Lemma **??** we see that

$$|g|_Q\|\chi_Q\|_{L^{p(\cdot)}} \le C\,\|g\chi_Q\|_{L^{p(\cdot)}}$$

holds for all $g \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$. By putting $g := b - b_Q$, we can immediately get the left hand side inequality of (3). Applying Theorem A, with $f := |b-b_Q|^{1/\delta}\chi_Q$ with $\delta \in (0,1]$, the other implication is verified as follows;

$$\begin{aligned}
\|(b-b_Q)\chi_Q\|_{L^{p(\cdot)}} &\le C\left(\frac{1}{|Q|}\int_Q |b(x)-b_Q|^{1/\delta}\,dx\right)^\delta\|\chi_Q\|_{L^{p(\cdot)}} \\
&\le C\,\|b\|_{\mathrm{BMO}}\|\chi_Q\|_{L^{p(\cdot)}}.
\end{aligned}$$

Thus, Theorem B is proved. ∘

4 Another characterization of $\mathrm{BMO_X}(\mathbf{R}^{\mathrm{n}})$

We know several equivalence norms of $\mathrm{BMO}(\mathbf{R}^n)$. It is well known that

$$\sup_{Q:\text{cube}} \inf_{c\in\mathbf{C}} \frac{1}{\|\chi_Q\|_{L^p}} \|(b-c)\chi_Q\|_{L^p} \tag{10}$$

with $p \in [1,\infty)$ is equivalent to the original one $\|b\|_{\mathrm{BMO}}$. In [14] Muckenhoupt and Wheeden proved that, for the weight w belonging to Muckenhoupt class A_∞,

$$\|b\|_{\mathrm{BMO}(w)} = \sup_Q \frac{1}{w(Q)} \int_Q |b(x) - b_{Q;w}| w(x)\, dx,$$

where $w(Q) := \int_Q w(x)\, dx$ and

$$b_{Q;w} := \frac{1}{w(Q)} \int_Q b(x) w(x)\, dx,$$

is also equivalent to $\|b\|_{\mathrm{BMO}}$. Moreover, owing to the John-Nirenberg inequality in the context of non-doubling measures by Mateu, Mattila, Nicolau and Orobitg [13], we see that for the same weight w above

$$C^{-1} \sup_Q \langle b - b_Q\rangle_{\exp L(Q;w)} \le \|b\|_{\mathrm{BMO}} \le C \sup_Q \langle b - b_Q\rangle_{\exp L(Q;w)},$$

where

$$\langle f\rangle_{\exp L(Q;w)} = \inf\left\{\lambda > 0 \,:\, \left(\exp\left(\frac{|f|}{\lambda}\right) - 1\right)_{Q;w} \le 1\right\}.$$

In this subsection, we establish the same equivalence with "$\inf_{c\in\mathbf{C}}$" instead of the average b_Q in the context of Banach function spaces under a condition.

Theorem E. If Banach function space X satisfies

$$\sup_{Q:\text{cube}} \frac{1}{|Q|} \|\chi_Q\|_X \|\chi_Q\|_{X'} < \infty,$$

then it follows

$$C\|b\|_{\mathrm{BMO}_X} \le \sup_{Q:\text{cube}} \inf_{c\in\mathbf{C}} \frac{1}{\|\chi_Q\|_X} \|(b-c)\chi_Q\|_X \le \|b\|_{\mathrm{BMO}_X}$$

for all measurable functions b.

Proof. The right-hand side inequality is obvious. Applying Lemma G, we can verify the left-hand one as follows; for a cube Q and $c \in \mathbf{C}$,

$$\begin{aligned} \frac{1}{\|\chi_Q\|_X} \|(b-b_Q)\chi_Q\|_X &\le \frac{1}{\|\chi_Q\|_X} \{\|(b-c)\chi_Q\|_X + \|(c-b_Q)\chi_Q\|_X\} \\ &= \frac{1}{\|\chi_Q\|_X} \|(b-c)\chi_Q\|_X + |c - b_Q| \\ &\le \frac{1}{\|\chi_Q\|_X} \|(b-c)\chi_Q\|_X + \frac{1}{|Q|} \int_Q |b-c| dx \\ &\le \frac{1}{\|\chi_Q\|_X} \|(b-c)\chi_Q\|_X + C \frac{1}{|Q|} \|(b-c)\chi_Q\|_X \|\chi_Q\|_{X'} \\ &\le \frac{C}{\|\chi_Q\|_X} \|(b-c)\chi_Q\|_X. \end{aligned} \tag{11}$$

A couple of helpful remarks may be in order.

1. For example, $X = \exp L(\mathbf{R}^n)$ satisfies the condtion in Theorem E where $\exp L(\mathbf{R}^n)$ denotes the set of all functions f such that

$$\|f\|_{\exp L} = \inf\left\{\lambda > 0 \,:\, \int_{\mathbf{R}^n} \left\{\exp\left(\frac{|f(x)|}{\lambda}\right) - 1\right\} dx \le 1\right\} < \infty.$$

In fact, it holds that $\|\chi_Q\|_{\exp L} = \frac{1}{\log(1+1/|Q|)}$ and that

$$\|\chi_Q\|_{(\exp L)'} \le c|Q| \log(1 + 1/|Q|).$$

2. The same argument with fundamental fact $|b|_Q \le 2\langle b\rangle_{\exp L(Q)}$, see [16] for the proof, yields the equivalence

$$\sup_Q \langle b - b_Q\rangle_{\exp L(Q)} \sim \sup_Q \inf_{c\in\mathbf{C}} \langle b - c\rangle_{\exp L(Q)},$$

with $\langle f\rangle_{\exp L(Q)} = \langle f\rangle_{\exp L(Q;1)}$.

Combining Theorems B and E we get another equivalence norm of $\mathrm{BMO}(\mathbf{R}^n)$ by means of variable exponent spaces.

Corollary. Let $p(\cdot) : \mathbf{R}^n \to [1,\infty)$ be a bounded function. Assume that M is of weak type $(p(\cdot), p(\cdot))$. Then we have that for $b \in \mathrm{BMO}(\mathbf{R}^n)$,

$$C^{-1}\ \|b\|_{\mathrm{BMO}} \le \sup_{Q:\mathrm{cube}} \inf_{c\in\mathbf{C}} \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}} \|(b-c)\chi_Q\|_{L^{p(\cdot)}} \le C\|b\|_{\mathrm{BMO}}.$$

5 A characterization by way of harmonic extension

Let $1 \le p < \infty$ be a constant. The $\mathrm{BMO}(\mathbf{R}^n)$ norm $\|b\|_{\mathrm{BMO}}$ is equivalent to

$$\sup_{(x,t)\in\mathbf{R}^n\times(0,\infty)} \left(\int_{\mathbf{R}^n} |b(y) - u(y,t)|^p P_t(x-y)\, dy \right)^{1/p},$$

where P_t is the Poisson kernel given by

$$P_t(x) := \frac{1}{(|x|^2 + t^2)^{\frac{n+1}{2}}} \quad (x \in \mathbf{R}^n,\ t > 0)$$

and $u(x,t) = (b * P_t)(x)$.

Chen-Lau [2] proved the equivalence replacing P_t by a more general function. Here for the sake of convenience, we include the proof and provide an alternative interpretation. By virtue of [4, Theorem 3.2], we know that

$$\sup_{(x,t)\in\mathbf{R}^n\times(0,\infty)} \left(\frac{1}{t^n} \int_{B(x,t)} |b(y) - u(y,t)|^p\, dy \right)^{1/p}$$

is an equivalent norm for $b \in \mathrm{BMO}(\mathbf{R}^n)$. That is,

$$\sup_{(x,t)\in\mathbf{R}^n\times(0,\infty)} \left(\frac{1}{t^n} \int_{B(x,t)} |b(y) - u(y,t)|^p\, dy \right)^{1/p} \sim \|b\|_{\mathrm{BMO}}.$$

By the definition of poisson kernel, we have

$$\begin{aligned} &\sup_{(x,t)\in\mathbf{R}^n\times(0,\infty)} \left(\frac{1}{t^n} \int_{B(x,t)} |b(y) - u(y,t)|^p\, dy \right)^{1/p} \\ \le\ & C \sup_{(x,t)\in\mathbf{R}^n\times(0,\infty)} \left(\int_{\mathbf{R}^n} |b(y) - u(y,t)|^p P_t(x-y)\, dy \right)^{1/p}. \end{aligned}$$

Meanwhile,

$$\begin{aligned} &\sup_{(x,t)\in\mathbf{R}^n\times(0,\infty)} \left(\int_{\mathbf{R}^n} |b(y) - u(y,t)|^p P_t(x-y)\, dy \right)^{1/p} \\ \le\ & \sup_{(x,t)\in\mathbf{R}^n\times(0,\infty)} \sum_{k\in\mathbf{Z}^n} \left(\int_{B(x+kt,2nt)} |b(y) - u(y,t)|^p P_t(x-y)\, dy \right)^{1/p} \\ \le\ & C \sup_{(x,t)\in\mathbf{R}^n\times(0,\infty)} \sum_{k\in\mathbf{Z}^n} (1+|k|)^{-n-1} \left(\int_{B(x+kt,2nt)} |b(y) - u(y,t)|^p\, dy \right)^{1/p} \\ \le\ & C \sup_{(x,t)\in\mathbf{R}^n\times(0,\infty)} \left(\frac{1}{t^n} \int_{B(x,t)} |b(y) - u(y,t)|^p\, dy \right)^{1/p} \sim \|b\|_{\mathrm{BMO}}. \end{aligned}$$

We can generalize the result from the viewpoint of variable exponent.

Theorem F Let $p(\cdot) : \mathbf{R}^n \to [1,\infty)$ be a variable exponent such that

$$1 \le p_- \le p_+ < \infty.$$

Then we have that for all $b \in \mathrm{BMO}(\mathbf{R}^n)$,

$$C^{-1}\|b\|_{\mathrm{BMO}} \le \|b\|_{\mathrm{BMO}_{p(\cdot)}} \le C\,\|b\|_{\mathrm{BMO}},$$

where

$$\|b\|_{\mathrm{BMO}_{p(\cdot)}} := \inf\left\{\lambda > 0 : \sup_{(x,t)\in\mathbf{R}^n\times(0,\infty)} \int_{\mathbf{R}^n} \left|\frac{b(y)-u(x,t)}{\lambda}\right|^{p(y)} P_t(x-y)\,dy \le 1\right\}.$$

Proof. As we have mentioned, the result is known when $p(\cdot)$ is a constant. Since

$$u^{p(\cdot)} \le u^{p_-} + u^{p_+} \quad \text{for all}\, u > 0,$$

one inequality is obvious. To prove $\|b\|_{\mathrm{BMO}} \le C\|b\|_{\mathrm{BMO}_{p(\cdot)}}$, we let λ satisfy

$$\int_{\mathbf{R}^n} \left|\frac{b(y)-u(x,t)}{\lambda}\right|^{p(y)} P_t(x-y)\,dy \le 1$$

for all $(x,t) \in \mathbf{R}^n \times (0,\infty)$. Then

$$\int_{\mathbf{R}^n} \left(\frac{1}{2}\left|\frac{b(y)-u(x,t)}{\lambda}\right| + \frac{1}{2}\right)^{p(y)} P_t(x-y)\,dy \le \frac{1}{2}\int_{\mathbf{R}^n} \left(\left|\frac{b(y)-u(x,t)}{\lambda}\right|^{p(y)} + 1\right) P_t(x-y)\,dy \le 1,$$

since

$$\int_{\mathbf{R}^n} P_t(x-y)\,dy = 1.$$

Since

$$\frac{t}{2} \le \left(\frac{t+1}{2}\right)^{p(x)}, \quad t > 0$$

it follows that

$$\int_{\mathbf{R}^n} \left|\frac{b(y)-u(x,t)}{2\lambda}\right| P_t(x-y)\,dy \le 1.$$

Thus, the proof is complete.

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