Sharp maximal inequalities and its application to some bilinear estimates

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Abstract

In this note we establish the sharp maximal inequalities for Herz spaces and Morrey spaces by use of good λ-inequality. As an application, we obtain estimates of some bilinear forms which include usual product of functions and the nonlinear term of Euler and Navier-Stokes equations on Herz spaces and Morrey spaces.

Keywords Sharp maximal function, Sharp maximal inequality, Herz space, Morrey space, bilinear estimate.

2000 Mathematics Subject Classification 42B25.

1 Introduction

The present paper has two purposes. The first one is to establish the sharp maximal inequality, so-called Fefferman-Stein's inequality, for Herz and Morrey spaces. The second one is to extend bilinear estimates, which were shown by Kozono and Taniuchi [19], by using the sharp maximal inequality. This article is based on Miyachi's unpublished paper.

The sharp maximal inequality stands for an inequality of the form

$$\|f\|_X \leq c\|f^\sharp\|_X,$$

with some quasi-Banach space X, where $f^\sharp$ denotes the Fefferman-Stein sharp maximal function. It was firstly introduced in [13]. We will show the inequality with a more general sharp maximal function $f^{\sharp(r)}_{l,\tau}$. Indeed, $f^{\sharp(1)}_{0,0} = f^\sharp$. For the precise definition of $f^{\sharp(r)}_{l,\tau}$, see Section 2. Such maximal function were studied by DeVore and Sharpley in [10]. The prototype of the maximal function was introduced in papers of Calderón [5] and Calderón and Scott [6]. The sharp function contain information on the smoothness of functions. For instance, the following equivalence due to Calderón holds: for $1 < p \leq \infty$ and a positive integer m,

$$\|\nabla^m f\|_{L^p} \approx \|f^{\sharp(1)}_{m-1,m}\|_{L^p}.$$

In this connection, Cho [8] studied the inequalities

$$\|f\|_{\dot{A}^s_{r,q}} \leq c\|f^{\sharp(1)}_{0,\alpha}\|_{L^p}$$

for some exponents r, q, s, p and $0 < \alpha \leq 1$, where $\dot{A}^s_{p,q}$ is the Besov space $\dot{B}^s_{p,q}$ or Triebel-Lizorkin space $\dot{F}^s_{p,q}$. Moreover, this maximal function was used for the extension problem of functions belonging to Triebel-Lizorkin spaces [10] and the modified sharp function was used to characterize Besov spaces and Triebel-Lizorkin spaces with smoothness index $s > 0$ [36], [39]. See also [3], [21], [26] and [38], for several variants of the sharp maximal functions.

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The sharp function is useful for real interpolation theory [2] and for pointwise estimates of several operators appearing in harmonic analysis, for example, singular integrals [37], pseudodifferential operators [31], Coifman-Rochberg-Weiss type commutators [15]. And, combining these pointwise estimates with the sharp maximal inequality enable us to estimate these operators. As an application of the sharp maximal function to the theory of partial differential equations, Krylov [20] used pointwise estimates of the sharp function of second-order derivatives to study L^p-theory of divergence and non-divergence elliptic and parabolic equations with discontinuous coefficients. Also, in [32], Rogers and Seeger applied it to the estimate for the Fourier multipliers related to the initial value problem for the dispersive equation $i\partial_t u + (-\Delta)^{\alpha/2}u = 0$ for $\alpha \in (0,1) \cup (1,\infty)$, $u(\cdot,0) = f$.

There are several ways to obtain the sharp maximal inequality for L^p. For example, by the duality argument [21], [37], by using the non-increasing rearrangement [1] and by a good λ-inequality [16], [30]. Similarly, the weighted inequality is shown by these techniques, see, for example [12]. When we use the first or the second, the duality and the Hardy's inequality prevent the exponents to extend to the large range. Meanwhile, when we use the good λ-inequality, the exponents are not restricted, [30]. To enjoy this favor, we use a good λ-inequality. On Morrey spaces, Sawano and Tanaka [35] showed the sharp maximal inequality with non-doubling measures by applying a good λ-inequality. Their good λ-inequality is slightly different from ours. Their statement does not include the exponents p, q less than 1, but it seems that their proof also works even for the case $p, q \le 1$. Our inequalities cover their inequality as special cases. The main results in the first half of this paper are the following maximal inequalities. For the precise definitions of $\dot{K}^{\alpha}_{p,q}$ and $\mathcal{M}^p_q$, see Section 2.

Theorem 1.1. *Let $0 < p, r < \infty$, $0 < p \le \infty$, $-n/p < \alpha < \infty$ and $l \in \mathbb{N}_0$. Then, there exists a constant c such that for $f \in L^r_{loc}(\mathbb{R}^n \backslash \{0\})$ satisfying $\left(\fint_{Q_k} |f|^r dy\right)^{1/r} \to 0$ as $k \to \infty$, where $Q_k = (-2^k, 2^k)^n \backslash (-2^{k-1}, 2^{k-1})^n$,*

$$\|f\|_{\dot{K}^{\alpha}_{p,q}} \le c \|f^{\sharp(r),Q^*_{\cdot}}_{l,0}\|_{\dot{K}^{\alpha}_{p,q}},$$

where

$$\|f^{\sharp(r),Q^*_{\cdot}}_{l,0}\|_{\dot{K}^{\alpha}_{p,q}} := \Big(\sum_{k\in\mathbb{Z}} 2^{k\alpha q} \|f^{\sharp(r),Q^*_k}_{l,0}\|^q_{L^p(Q_k)}\Big)^{1/q}$$

and $Q^*_k := Q_{k-1} \cup Q_k \cup Q_{k+1}$.

Theorem 1.2. *Let $0 < q \le p < \infty$, $0 < r, s < \infty$ and $l \in \mathbb{N}_0$. Then, there exists a constant c such that for $f \in L^r_{loc}$ satisfying that $\left(\fint_{I_k} |f|^r dy\right)^{1/r} \to 0$ as $k \to \infty$, for some cube I, where $I_k = 2^k I$,*

$$\|f\|_{\mathcal{M}^p_q} \le c \|f^{\sharp(r),I_{\cdot}}_{l,0}\|_{\mathcal{M}^p_q},$$

where

$$\|f^{\sharp(r),I_{\cdot}}_{l,0}\|_{\mathcal{M}^p_q} := \sup_I |I|^{1/p-1/q} \|f^{\sharp(r),I}_{l,0}\|_{L^q(I)},$$

where the supremum is taken over all cubes.

Remark 1.1. *1. It is not hard to modify the argument of the proof of Theorem 1.1 to check that the same inequalities hold for the nonhomogeneous Herz space $K^{\alpha}_{p,q}(\mathbb{R}^n)$.*

2. In [17], Komori showed the inequality above in the context of nonhomogeneous Herz space with $\alpha = -n/p$ in the following sense; for $1 < p < \infty$ and $f \in L^p_{loc}$,

$$\|f\|_{CMO^p} \le c \|f^{\sharp(1)}_{0,0}\|_{\dot{K}^{-n/p}_{\infty,p}},$$

where CMO^p is a space of functions of central mean oscillation and equipped with the norm

$$\|f\|_{CMO^p} := \sup_{R>1}\Big(\fint_{B(0,R)} |f - f_{B(0,R)}|^p dy\Big)^{1/p}.$$

3. The decay condition on the average of function in Theorem 1.2 is equivalent to the one for all cubes I.
4. The sharp maximal inequalities above fail for any $P \in \mathbf{P}_l \backslash \{0\}$. Because we must exclude the case, the decay condition seems natural. Indeed,

$$\Big(\fint_{Q_k} |P|^r dx\Big)^{1/r} \approx \|P\|_{L^\infty(Q_k)} \not\to 0, \text{ as } k \to \infty,$$

for every polynomial $P \not\equiv 0$.

The latter half of this article is motivated by Miyachi's unpublished paper, and the basic ideas of this half are based on it. Also, the spirit of Miyachi in the unpublished paper are collected in [33]. Thanks to his idea, we can establish same bilinear estimates as (ii) of Theorems 1.4 and 1.5 for function spaces on which the Fefferman-Stein inequality holds and the Hardy-Littelewood maximal function is bounded.

In the latter half, we consider the estimate of the bilinear form $f\nabla^m g$, $(m \in \mathbb{N} \cup \{0\})$, which includes the nonlinear term $u\nabla u$ of Euler and Navier-Stokes equations. Our main tools are the sharp maximal inequality and pointwise estimate of the sharp maximal function of $f\nabla^m g$.

Our purpose is to estimate products of functions. Many estimates of these are known. It seems that the most famous inequality of product of functions is Hölder's inequality. The inequality says that if $f \in L^q$ and $g \in L^r$ then a product fg belongs to L^p when $1/p = 1/q + 1/r$. In this connection at endpoint $p = q$, the following inequality is well known;

$$\|fg\|_{L^p} \le c\Big(\|f\|_{L^p}\|g\|_{BMO} + \|f\|_{BMO}\|g\|_{L^p}\Big)$$

which is a consequence of a pointwise inequality for the sharp function of fg;

$$(fg)^\sharp(x) \le c\Big(M_r f(x)\|g\|_{BMO} + \|f\|_{BMO} M_r g(x)\Big)$$

where $M_r f = M(|f|^r)^{1/r}$, M is the Hardy-Littlewood maximal function and $1 < r < \infty$. Miyachi showed the above bilinear estimate with $0 < p < \infty$ by using pointwise estimate of the sharp function of fg. There is a similar bilinear inequality with the derivatives. In [19], Kozono and Taniuchi showed the following bilinear inequality by the use of the boundedness of the bilinear Fourier multipliers due to Coifman and Meyer [9]; for $f, g \in W^{1,p}$ with $\nabla f,\ \nabla g \in BMO$ and $1 < p < \infty$,

$$\|f\nabla g\|_{L^p} \le c\Big(\|f\|_{L^p}\|\nabla g\|_{BMO} + \|\nabla f\|_{BMO}\|g\|_{L^p}\Big).$$

In his unpublished paper, Miyachi gave a proof of this bilinear estimate by using pointwise estimate of the sharp maximal function of $f\partial^\beta g$ with $|\beta| = 1$ and $1 < p < \infty$. Their bilinear inequality played an important role in the study of the blow-up phenomena of smooth solutions to the Navier-Stokes equations. It is clear that $\|fg\|_{\dot{W}^{1,p}}$ also can be dominated by the right hand side. Here, $\dot{W}^{1,p}$ is the homogeneous Sobolev space. Moreover, there is a similar estimate due to Christ and Weinstein [7] with the fractional derivative;

$$\begin{aligned}\|fg\|_{\dot{F}^s_{p,2}} &\approx \|D^s(fg)\|_{L^p}\\ &\le c\Big(\|f\|_{L^q}\|D^s g\|_{L^{q'}} + \|D^s f\|_{L^r}\|g\|_{L^{r'}}\Big)\\ &\approx \Big(\|f\|_{\dot{F}^0_{q,2}}\|g\|_{\dot{F}^s_{q',2}} + \|f\|_{\dot{F}^s_{r,2}}\|g\|_{\dot{F}^0_{r',2}}\Big),\end{aligned}$$

where D^s is a Fourier multiplier operator $D^s f = \mathcal{F}^{-1}[|\cdot|^s \mathcal{F} f]$, where $\mathcal{F}$ is the Fourier transform, $\mathcal{F}^{-1}$ is the inverse Fourier transform and $0 < s < 1$. There are other bilinear estimates for fg in several function spaces by using the paraproduct of Bony. For instance, in [18], the authors showed the following bilinear estimate in Triebel-Lizorkin spaces,

$$\|fg\|_{\dot{F}^s_{p,q}} \le c\Big(\|f\|_{\dot{F}^{s+\alpha}_{p_1,q}}\|g\|_{\dot{F}^{-\alpha}_{p_2,\infty}} + \|f\|_{\dot{F}^{-\beta}_{r_1,\infty}}\|g\|_{\dot{F}^{s+\beta}_{r_2,q}}\Big)$$

with some exponents.

Our bilinear estimates are the following. For the precise definitions of $H\dot{K}^\alpha_{p,q}$, $\dot{K}^0_{BMO,\infty}$ and $H\mathcal{M}^p_q$, see Section 2.

Theorem 1.3. *Let $0 < p < \infty$, $0 < q \le \infty$ and $-n/p < \alpha < \infty$.*
(i): *There exists a constant c such that for any $f, g \in \dot{K}^\alpha_{p,q} \cap \dot{K}^0_{BMO,\infty}$,*

$$\|fg\|_{K^\alpha_{p,q}} \le c\Big(\|f\|_{\dot{K}^\alpha_{p,q}}\|g\|_{\dot{K}^0_{BMO,\infty}} + \|f\|_{\dot{K}^0_{BMO,\infty}}\|g\|_{\dot{K}^\alpha_{p,q}}\Big).$$

(ii): *Let $m \in \mathbb{N}$. There exists a constant c such that for any $f \in \dot{K}^\alpha_{p,q} \cap L^1_{loc}$ with $\nabla^m f \in \dot{K}^0_{BMO,\infty}$ and $g \in H\dot{K}^\alpha_{p,q} \cap L^1_{loc}$ with $\nabla^m g \in \dot{K}^0_{BMO,\infty}$,*

$$\|f\nabla^m g\|_{\dot{K}^\alpha_{p,q}} \le c\Big(\|f\|_{\dot{K}^\alpha_{p,q}}\|\nabla^m g\|_{\dot{K}^0_{BMO,\infty}} + \|\nabla^m f\|_{\dot{K}^0_{BMO,\infty}}\|g\|_{H\dot{K}^\alpha_{p,q}}\Big).$$

Theorem 1.4. *Let $0 < q \le p < \infty$.*
(i): *There exists a constant c such that for any $f, g \in \mathcal{M}^p_q \cap BMO$,*

$$\|fg\|_{\mathcal{M}^p_q} \le c\Big(\|f\|_{\mathcal{M}^p_q}\|g\|_{BMO} + \|f\|_{BMO}\|g\|_{\mathcal{M}^p_q}\Big).$$

(ii): *Let $m \in \mathbb{N}$. There exists a constant c such that for any $f \in \mathcal{M}^p_q \cap L^1_{loc}$ with $\nabla^m f \in BMO$ and $g \in H\mathcal{M}^p_q \cap L^1_{loc}$ with $\nabla^m g \in BMO$,*

$$\|f\nabla^m g\|_{\mathcal{M}^p_q} \le c\Big(\|f\|_{\mathcal{M}^p_q}\|\nabla^m g\|_{BMO} + \|\nabla^m f\|_{BMO}\|g\|_{H\mathcal{M}^p_q}\Big).$$

Remark 1.2. *1. The inequalities in Theorem 1.3 also hold for the nonhomogeneous Herz space.*

2. It is clear that the inequality

$$\|f\nabla^m g\|_{\dot{K}^\alpha_{p,q}} \le c\Big(\|f\|_{BMO}\|\nabla^m g\|_{\dot{K}^\alpha_{p,q}} + \|\nabla^m f\|_{H\dot{K}^\alpha_{p,q}}\|g\|_{BMO}\Big)$$

fails for $f \in BMO$ with $\nabla^m f \in \dot{K}^\alpha_{p,q}$ and $g \in BMO$ with $\nabla^m g \in H\dot{K}^\alpha_{p,q}$. In fact, if we take $f = c \in \mathbb{C}\backslash\{0\}$, then the left hand side is not zero in general while the right hand side must be zero.

3. Since (ii) in Theorems 1.3 and 1.4 are proved by using a pointwise inequality, they have some variants. For example, we can obtain the bilinear estimates for the Campanato norm replacing BMO norm.

4. Since these inequalities are derived from a good λ-inequality and pointwise estimate, we also obtain the weighted versions of the ones in the sense of B. Muckenhoupt. See Remarks 3.1, 3.2, 3.3 and 4.1.

5. By the results below, the following local estimate can be shown; for $0 < p, r < \infty$, cube Q and $f, g \in L^p(Q)$ with $\nabla^m f, \nabla^m g \in BMO(Q)$,

$$\|f\nabla^m g\|_{L^p(Q)} \le c\Big(\|f\|_{L^p(Q)}\|\nabla^m g\|_{BMO(Q)} + \|\nabla^m f\|_{BMO(Q)}\|g\|_{L^p(Q)} + |Q|^{1/p-1/r}\|f\nabla^m g\|_{L^r(Q)}\Big)$$

with a constant c independent of f, g and Q.

6. In [40], the above bilinear estimate will be applied to study the blow-up phenomena of solutions to the Navier-Stokes equations with the initial data in the Herz space.

2 Preliminaries

Throughout this paper we use the following notations. Let Ω be a nonempty subset of $\mathbb{R}^n$. $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $x \in \mathbb{R}^n$ and $t > 0$, $B(x,t)$ denotes the ball, centered at x of radius t. By a " cube " Q we mean a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. Its side length and center will be denoted by $l(Q)$ and $c(Q)$. Also, for $a > 0$, aQ means the cube with the same center as Q whose side length is a times that of Q. The set of all dyadic cubes is denoted by $\mathcal{Q}$, and set $Q_k := (-2^k, 2^k)^n \backslash (-2^{k-1}, 2^{k-1})^n$ for $k \in \mathbb{Z}$. Let $\mathbf{P}_l$ be the space of polynomials of degree at most l. For a measurable set E, χ_E denotes the characteristic function of E and the slashed integral $\displaystyle⨍_E f dx$ denotes the average $f_E = \dfrac{1}{|E|}\displaystyle\int_E f dx$, where $|E|$ is the Lebesgue measure of E. $\nabla^m f$ stands for the vector of all derivatives $\partial^\alpha f$ with $|\alpha| = m$ and $\|\nabla^m f\|_X = \sum_{|\alpha|=m} \|\partial^\alpha f\|_X$ with some quasi-Banach space X. In what follows c denotes a constant that is independent of the functions involved, which may differ from line to line.

In this section, we recall the definitions and fundamental properties of function spaces and also collect several lemmas about polynomials which are used to study the sharp maximal function. In particular, Lemmas 2.3 and 2.6 play an important role in Section 4.

Definition 2.1. *Let $0 < r, \tau < \infty$ and $l \in \mathbb{N}_0$ be such that $[\tau] - 1 \le l$. For functions $f \in L^r_{loc}(\Omega)$, one defines the two maximal functions*

$$M_r^{\tau,\Omega} f(x) := \sup_{x \in Q \subset \Omega} |Q|^{\tau/n} \Big(⨍_Q |f|^r dy \Big)^{1/r},$$

and

$$f_{l,\tau}^{\sharp(r),\Omega}(x) := \sup_{x \in Q \subset \Omega} \inf_{P \in \mathbf{P}_l} |Q|^{-\tau/n} \Big(⨍_Q |f - P|^r dy \Big)^{1/r},$$

where the supremum is taken over all cubes Q containing x and included in Ω. In particular, we write $f^\sharp = f_{0,0}^{\sharp(1)}$ and $M_r^\Omega = M_r^{0,\Omega}$. Omit r and Ω when $r = 1$ and $\Omega = \mathbb{R}^n$, respectively. Also, $\widetilde{M}_r^{\tau,\Omega} f$ and $\widetilde{f}_{l,\tau}^{\sharp(r),\Omega}$ mean the dyadic maximal function and the dyadic sharp maximal function of f, respectively. That is

$$\widetilde{M}_r^{\tau,\Omega} f(x) := \sup_{\substack{x \in Q \subset \Omega \\ Q \in \mathcal{Q}}} |Q|^{\tau/n} \Big(⨍_Q |f|^r dy \Big)^{1/r},$$

and

$$\widetilde{f}_{l,\tau}^{\sharp(r),\Omega}(x) := \sup_{\substack{x \in Q \subset \Omega \\ Q \in \mathcal{Q}}} \inf_{P \in \mathbf{P}_l} |Q|^{-\tau/n} \Big(⨍_Q |f - P|^r dy \Big)^{1/r},$$

where the supremums are taken over all dyadic cubes Q containing x and included in Ω. In particular, we write $\widetilde{M}_r^\Omega = \widetilde{M}_r^{0,\Omega}$.

Definition 2.2. *Let $0 < p, q \le \infty$ and $\alpha \in \mathbb{R}$. Define the (homogeneous) Herz space $\dot{K}^\alpha_{p,q}$ as*

$$\dot{K}^\alpha_{p,q}(\mathbb{R}^n) := \{ f \in L^p_{loc}(\mathbb{R}^n \backslash \{0\}); \|f\|_{\dot{K}^\alpha_{p,q}} := \Big(\sum_{k \in \mathbb{Z}} 2^{k\alpha q} \|f\|^q_{L^p(Q_k)} \Big)^{1/q} < \infty \},$$

and

$$\dot{K}^0_{BMO,\infty}(\mathbb{R}^n) := \{ f \in L^1_{loc}(\mathbb{R}^n \backslash \{0\}); \|f\|_{\dot{K}^0_{BMO,\infty}} := \sup_{k \in \mathbb{Z}} \|f\|_{BMO(Q_k^*)} < \infty \},$$

where $Q_k^* := Q_{k-1} \cup Q_k \cup Q_{k+1}$ and

$$\|f\|_{BMO(\Omega)} := \sup_{Q \subset \Omega} \inf_{c \in \mathbb{C}} ⨍_Q |f - c| dx,$$

where the supremum is taken over all cubes Q included in Ω.

It is trivial that $\|f\|_{\dot{K}^0_{BMO,\infty}} \le \|f\|_{BMO}$, i.e. $BMO \hookrightarrow \dot{K}^0_{BMO,\infty}$. Moreover, $\dot{K}^0_{BMO,\infty}$ strictly includes BMO. Indeed, a function $h(x) := -\chi_{x>0}\log|x|$, appearing in [14, pp.121], is not in $BMO(\mathbb{R})$ but in $\dot{K}^0_{BMO,\infty}(\mathbb{R})$. It is well known that $-\log|x|$ is in BMO. Then it is clear from the definition that

$$\|h\|_{\dot{K}^0_{BMO,\infty}} \le \|-\log|x|\ \|_{BMO} < \infty.$$

Definition 2.3. *Let $0 < q \le p < \infty$. Define the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ as*

$$\mathcal{M}^p_q(\mathbb{R}^n) := \{f \in L^q_{loc}; \|f\|_{\mathcal{M}^p_q} := \sup_I |I|^{1/p}\Big(⨍_I |f|^q dy\Big)^{1/q} < \infty\},$$

where the supremum is taken over all cubes.

It is easy to see the equivalence $\|f\|_{\mathcal{M}^p_q} \approx \sup_{I\in\mathcal{Q}} |I|^{1/p}\Big(⨍_I |f|^q dy\Big)^{1/q}$.

Above two function spaces extend Lebesgue spaces; for all $0 < p < \infty$,

$$\dot{K}^0_{p,p} = L^p = \mathcal{M}^p_p.$$

In particular, Herz spaces include also Lebesgue spaces with power weight, i.e.

$$\dot{K}^\alpha_{p,p} = L^p(|x|^{\alpha p}dx).$$

Note that, for nonhomogeneous Herz spaces, we have

$$K^\alpha_{p,p} = L^p(\langle x\rangle^{\alpha p}dx)$$

where $\langle x\rangle = (1+|x|^2)^{1/2}$.

Morrey spaces have the following inclusion property:

$$\mathcal{M}^p_p = L^p \hookrightarrow L^{p,\infty}(\text{Lorentz space}) \hookrightarrow \mathcal{M}^p_{q_1} \hookrightarrow \mathcal{M}^p_{q_2} \quad \text{for } 0 < q_2 \le q_1 < p < \infty.$$

Further, the following inclusion holds; for $0 < q \le p < \infty$

$$\mathcal{M}^p_q \hookrightarrow \dot{K}^{n(1/p-1/q)}_{q,\infty}.$$

To define Hardy type spaces for the above two function spaces, we fix a test function $\phi \in C^\infty_0$ which is supported in the unit ball $B(0,1)$ and whose integral is not zero. For a distribution $f \in \mathscr{D}'$ we define the radial maximal function $\phi_+(f)$ by

$$\phi_+(f)(x) = \sup_{0<t<\infty} |\langle f, \phi_t(x-\cdot)\rangle|$$

where $\phi_t(x) = t^{-n}\phi(x/t)$.

Definition 2.4. *Let p and q be the same as in Definition 2.2 and $-n/p < \alpha < \infty$. Define the Herz-type Hardy space $H\dot{K}^\alpha_{p,q}$ as*

$$H\dot{K}^\alpha_{p,q}(\mathbb{R}^n) := \{f \in \mathscr{D}'; \|f\|_{H\dot{K}^\alpha_{p,q}} := \|\phi_+(f)\|_{\dot{K}^\alpha_{p,q}}\}.$$

Definition 2.5. *Let p and q be the same as in Definition 2.3. Define the Morrey-type Hardy space $H\mathcal{M}^p_q$ as*

$$H\mathcal{M}^p_q(\mathbb{R}^n) := \{f \in \mathscr{D}'; \|f\|_{H\mathcal{M}^p_q} := \|\phi_+(f)\|_{\mathcal{M}^p_q}\}.$$

Remark 2.1. *1. $H\dot{K}^{\alpha}_{p,q}$ and $H\mathcal{M}^p_q$ are independent of the choise of ϕ. The fact about Herz-type Hardy space was shown in [25]. As for $H\mathcal{M}^p_q$, this was shown in [34].*
2. It is known in [23] that if $1 < p < \infty, 0 < q \le \infty, -n/p < \alpha < n(1-1/p)$ then $H\dot{K}^{\alpha}_{p,q}$ coincides with $\dot{K}^{\alpha}_{p,q}$. Similarly, for $1 < q \le p < \infty$, $H\mathcal{M}^p_q = \mathcal{M}^p_q$ with norm equivalence. Indeed, from an inequality $\phi_+(f) \le cMf$, the inclusion $\mathcal{M}^p_q \subset H\mathcal{M}^p_q$ is clear. The reverse inclusion is also deduced from the Banach-Alaoglu theorem, see [22].

Many authors studied the mapping properties of several operators on Herz spaces and Morrey spaces. We use the boundedness of the Hardy-Littlewood maximal operator $M = M_1^{0,\mathbb{R}^n}$ in the sequel.

Proposition 2.1. *M is a bounded operator on $\dot{K}^{\alpha}_{p,q}$ if $1 < p \le \infty,\ 0 < q \le \infty$ and $-n/p < \alpha < n(1-1/p)$.*

Proposition 2.2. *M is a bounded operator on $\mathcal{M}^p_q$ if $1 < q \le p < \infty$.*

The first lemma is well known.

Lemma 2.1 ([28, lemma 3.1])**.** *The following inequalities hold for all cubes $Q \subset \mathbb{R}^n$ and all polynomials $P \in \mathbf{P}_l$.*
(i): *For $0 < r < \infty$,*

$$\Big(⨍_Q |P|^r dy\Big)^{1/r} \le \|P\|_{L^\infty(Q)} \le c\Big(⨍_Q |P|^r dy\Big)^{1/r},$$

where the constant c depends on n, l and r only.
(ii): *For each $a \ge 1$,*

$$\|P\|_{L^\infty(aQ)} \le ca^l \|P\|_{L^\infty(Q)},$$

where the constant c depends on n and l only.
(iii): *(Markov's inequality) For each multi-index β,*

$$\|\partial^\beta P\|_{L^\infty(Q)} \le c|Q|^{-|\beta|/n} \|P\|_{L^\infty(Q)},$$

where the constant c depends on n and l only.

We define the specific polynomial class $\Pi_l(f, L^r(Q))$. In the proofs of Propositions 3.1 and 3.2, we use polynomials in these classes.

Definition 2.6. *Let $0 < r < \infty$, $l \in \mathbb{N}_0$, Q be a cube in $\mathbb{R}^n$ and f a measurable function. Define*

$$\Pi_l(f, L^r(Q)) = \{\pi \in \mathbf{P}_l; \|f-\pi\|_{L^r(Q)} = \inf_{P \in \mathbf{P}_l} \|f-P\|_{L^r(Q)}\}.$$

The next lemma is fundamental to deal the above function spaces with $q \le 1$.

Lemma 2.2 ([28, Lemma 3.3])**.** *Let $0 < r < \infty$, $l \in \mathbb{N}_0$, Q be a cube in $\mathbb{R}^n$ and f a measurable function. Then the following* (i), (ii) *and* (iii) *hold.*
(i): $\Pi_l(f, L^r(Q)) \ne \emptyset$.
(ii): *There exists a constant $c = c(n, l, r)$ such that for all $\pi \in \Pi_l(f, L^r(Q))$,*

$$\|\pi\|_{L^\infty(Q)} \le c\Big(⨍_Q |f|^r dy\Big)^{1/r}.$$

(iii): *For any positive number $\delta \le 1$ there exists $c = c(n, l, r, \delta)$ such that for $Q_i \subset Q$ with $|Q_i| \ge \delta|Q|$ and $\pi_i \in \Pi_l(f, L^r(Q_i))$, $i = 1, 2$,*

$$\|\pi_1 - \pi_2\|_{L^\infty(Q)} \le c \inf_{P \in \mathbf{P}_l} \Big(⨍_Q |f-P|^r dy\Big)^{1/r}.$$

The following property of the sharp maximal functions is important. A similar inequality can be found in [10].

Lemma 2.3. *Let $1 \le r < \infty$, $l \in \mathbb{N}$ and $f \in L^r_{loc}$. If $\partial^\alpha f \in L^r_{loc}$ for all $|\alpha| = l$, then the inequality*

$$f^{\sharp(r),\Omega}_{l,l}(x) \le 2^l \sum_{|\alpha|=l} (\partial^\alpha f)^{\sharp(r),\Omega}_{0,0}(x)$$

holds for any open subset Ω in $\mathbb{R}^n$ and $x \in \Omega$.

Miyachi proved Lemma 2.3 with $r = 1$ and $l = 1$ in his unpublished paper. To verify Lemma 2.3 in that case, he used the following two lemmas. Once these are proved, it is easy to check Lemma 2.3.

We write $f \perp \mathbf{P}_m$ if $\int fP dx = 0$ for all $P \in \mathbf{P}_m$, for $f \in L^1_{loc}$.

The first lemma follows easily from the duality.

Lemma 2.4. *Let Q be a cube, $m \in \mathbb{N}$, $1 \le r < \infty$, $1/r + 1/r' = 1$, $f \in L^r_{loc}$ and*

$$A^m(Q) = \{\varphi \in C^\infty_0(Q); \varphi \perp \mathbf{P}_m, \text{ and } \|\varphi\|_{L^{r'}} \le |Q|^{-1/r-m/n}\}.$$

Then, the equality

$$\inf_{P \in \mathbf{P}_m} |Q|^{-1/r-m/n} \|f - P\|_{L^r(Q)} = \sup_{\varphi \in A^m(Q)} |\int f\varphi dx|$$

holds.

The proof of the second lemma is similar to that of [27, Lemma 2].

Lemma 2.5 ([27, Lemma 2])**.** *Let Q be a cube, $m \in \mathbb{N}$, $1 \le r < \infty$, $1/r + 1/r' = 1$ and*

$$B^m(Q) = \{\varphi \in C^\infty_0(Q); \varphi = \sum_{j=1}^n \partial_j \psi_j,$$
$$\text{where } \psi_j \in C^\infty_0(Q),\ \psi_j \perp \mathbf{P}_{m-1} \text{ and } \|\psi_j\|_{L^{r'}} \le 2|Q|^{-1/r-(m-1)/n}\}.$$

Then, the inclusion $A^m(Q) \subset B^m(Q)$ holds.

We end this section with the following lemma which is a key to the proof of (ii) in Proposition 4.1 and is interesting of its own right in itself. To state the lemma, we define the grand maximal function.

Definition 2.7. *Let $k \in \mathbb{N}_0$. For $x \in \mathbb{R}^n$ and $0 < t < \infty$, define $\mathcal{T}_k(x,t)$ as the set of all functions $\phi \in C^\infty_0$ so that supp $\phi \subset B(x,t)$ and $\|\partial^\alpha \phi\|_{L^\infty} \le t^{-n-|\alpha|}$ for $|\alpha| \le k$. For $f \in \mathscr{D}'$, set*

$$f^*_k(x) = \sup\{|\langle f, \phi\rangle|; \phi \in \cup_{0<t<\infty} \mathcal{T}_k(x,t)\}.$$

Lemma 2.6. *Let $1 \le p \le \infty$, Q be a cube, β a non zero multi-index and k a non-negative integer. Suppose that $g \in \mathscr{D}'$ and $\partial^\beta g \in L^p(Q)$. Then, there exists a constant A_Q satisfying both*

$$\|\partial^\beta g - A_Q\|_{L^p(Q)} \le c \inf_{c \in \mathbb{C}} \|\partial^\beta g - c\|_{L^p(Q)} \quad \text{and}$$
$$|A_Q| \le c|Q|^{-|\beta|/n} \inf_{\xi \in Q} g^*_k(\xi),$$

where the constant c depends on n, p, β and k only. Moreover, if $g \in L^1_{loc}$, then we also have

$$|A_Q| \le c|Q|^{-|\beta|/n} \inf_{P \in \mathbf{P}_{|\beta|-1}} \fint_Q |g - P| dy.$$

This lemma improves the result by Miyachi in the case $|\beta| = 1$. The following proof is based on the idea due to [11], and differs from that of Miyachi. Naturally enough, the average $(\partial^\beta g)_Q$ does not satisfy the second estimate. Since A_Q is defined by integral of $\partial^\beta g$ and the bound of $|A_Q|$ is dominated by $C|Q|^{-|\beta|/n} \fint_Q |g| dy$, the second estimate is stronger than Markov's inequality for polynomials.

Proof. Let $\rho \in C_0^\infty(\mathbb{R}^n)$ be such that supp $\rho \subset B(0, 1/2)$ and $\int \rho dx = 1$. For a function f, we write

$$f^Q(x) := |Q|^{-1} f(\frac{x - c(Q)}{l(Q)}).$$

It is obvious that supp $\rho^Q \subset B(c(Q), l(Q)/2) \subset Q$. We define A_Q by

$$\int_Q \partial^\beta g\ \rho^Q dx = (-1)^{|\beta|} |Q|^{-|\beta|/n} \langle g, (\partial^\beta \rho)^Q \rangle).$$

Since the integral of ρ^Q equals to 1, it is easy to check that A_Q satisfies the first property. Let $X := n^{(n+k)/2} \max_{|\alpha| \le k} \|\partial^{\alpha+\beta} \rho\|_{L^\infty}$. Then we have $|\partial^\alpha((\partial^\beta \rho)^Q)(x)| \le X(\sqrt{n} l(Q))^{-(n+|\alpha|)}$ for all $|\alpha| \le k$. Because supp $(\partial^\beta \rho)^Q \subset Q \subset B(\xi, \sqrt{n} l(Q))$ and $(\partial^\beta \rho)^Q / X \in \mathcal{T}_k(\xi, \sqrt{n} l(Q))$ for all $\xi \in B(c(Q), \frac{\sqrt{n}}{2} l(Q))$, we obtain the require inequality in the following way;

$$\begin{aligned} |A_Q| &= X|Q|^{-|\beta|/n} |\langle g, (\partial^\beta \rho)^Q / X \rangle| \\ &\le X|Q|^{-|\beta|/n} \inf_{\xi \in B(c(Q), \frac{\sqrt{n}}{2} l(Q))} g_k^*(\xi) \\ &\le X|Q|^{-|\beta|/n} \inf_{\xi \in Q} g_k^*(\xi). \end{aligned}$$

For every $P \in \mathbf{P}_{|\beta|-1}$, we can write

$$A_Q = \int \rho_Q(x) \partial^\beta (g(x) - P(x)) dx.$$

Therefore, in the case $g \in L^1_{loc}$, the desired inequality follows from integration by parts. □

3 Proof of Theorems 1.1 and 1.2

The following lemma is our good λ-inequality that we mentioned in the Section 1 and is the key to the proofs of Theorems 1.1 and 1.2. Miyachi proved Lemma 3.1 in his unpublished paper. For convenience for readers, we give his proof.

Lemma 3.1. *Let $0 < r < \infty$ and $l \in \mathbb{N}_0$ Then, there exist $B = B(n, r, l) > 1$, $C_0 = C_0(n, r, l) > 0$ so that for each $Q \in \mathcal{Q}$, $f \in L^r(Q)$, $\lambda > \left(\fint_Q |f|^r dy \right)^{1/r}$ and $0 < \delta \le 1$*

$$|\{x \in Q; \widetilde{M}_r^Q f(x) > B\lambda,\ \widetilde{f}_{l,0}^{\sharp(r),Q} \le \delta\lambda\}| \le C_0 \left(\frac{\delta}{B}\right)^r |\{x \in Q; \widetilde{M}_r^Q f(x) > \lambda\}|. \tag{1}$$

Proof. The constant $B > 1$ will be chosen later. We put

$$E_\lambda := \{x \in Q; \widetilde{M}_r^Q f(x) > B\lambda,\ \widetilde{f}_{l,0}^{\sharp(r),Q} \le \delta\lambda\}$$

and

$$\Omega_\lambda := \{x \in Q; \widetilde{M}_r^Q f(x) > \lambda\}.$$

We may assume that both E_λ and Ω_λ are not empty. Then we can find a family of dyadic cubes $\mathcal{R} = \{R\}$ satisfying $R \neq Q$ and $\left(\fint_R |f|^r dy\right)^{1/r} > \lambda$. We collect maximal dyadic cubes $\{R_j\}_{j\in\mathbb{N}}$ from $\mathcal{R}$. It is clear that $\bigcup_{j\in\mathbb{N}} R_j = \Omega_\lambda$. Therefore, it suffices to prove

$$|\{x \in Q; \widetilde{M}_r^Q f(x) > B\lambda\} \cap R_j| \leq C_0\Big(\frac{\delta}{B}\Big)^r |R_j| \tag{2}$$

assuming $X_j := \{x \in Q; \widetilde{f}^{\sharp(r),Q}_{l,0}(x) \leq \delta\lambda\} \cap R_j \neq \emptyset$.

Let $\tilde{R}_j$ be the dyadic double of R_j and we take $\pi \in \Pi_l(f, L^r(\tilde{R}_j))$ and $x \in R_j$ such that $\widetilde{M}_r^Q f(x) > B\lambda$. Hölder's inequality yields that

$$\begin{aligned} B\lambda < \widetilde{M}_r^Q f(x) &\leq \max(1, 3^{1/r-1}) \\ &\quad \times \Big(\widetilde{M}_r^Q((f-\pi)\chi_{\tilde{R}_j})(x) + \widetilde{M}_r^Q(f\chi_{\mathbb{R}^n\setminus\tilde{R}_j})(x) + \widetilde{M}_r^Q(\pi\chi_{\tilde{R}_j})(x)\Big). \end{aligned}$$

Because of the maximal property of R_j, we have that

$$\begin{aligned} \widetilde{M}_r^Q(f\chi_{\mathbb{R}^n\setminus\tilde{R}_j})(x) &= \sup_{\substack{x\in I\subset Q\\ I\in\mathcal{Q}}} \Big(\fint_I |f\chi_{\mathbb{R}^n\setminus\tilde{R}_j}|^r dy\Big)^{1/r} \\ &= \sup_{\substack{x\in I, \tilde{R}_j\subset I\subset Q\\ I\in\mathcal{Q}}} \Big(\fint_I |f\chi_{\mathbb{R}^n\setminus\tilde{R}_j}|^r dy\Big)^{1/r} \\ &\leq \lambda, \end{aligned}$$

and by $\sigma < n/r$ and by (ii) of Lemma 2.2,

$$\begin{aligned} \widetilde{M}_r^Q(\pi\chi_{\tilde{R}_j})(x) &= \sup_{\substack{x\in I\subset Q\\ I\in\mathcal{Q}}} |I|^{-1/r}\|\pi\chi_{\tilde{R}_j}\|_{L^r(I)} \\ &= \sup_{\substack{x\in I\subset\tilde{R}_j\\ I\in\mathcal{Q}}} |I|^{-1/r}\|\pi\|_{L^r(I)} \\ &\leq \|\pi\|_{L^\infty(\tilde{R}_j)} \\ &\leq c\Big(\fint_{\tilde{R}_j} |f|^r dy\Big)^{1/r} \\ &\leq c\lambda. \end{aligned}$$

Hence we obtain

$$\widetilde{M}_r^Q((f-\pi)\chi_{\tilde{R}_j})(x) > \Big(\frac{B}{\max(1, 3^{1/r-1})} - 1 - c\Big)\lambda.$$

Here if B satisfies

$$\Big(\frac{B}{\max(1, 3^{1/r-1})} - 1 - c\Big)\lambda > \frac{B}{3^{1/r-1}+1}\lambda,$$

then we have

$$\{x \in Q; \widetilde{M}_r^Q f(x) > B\lambda\} \cap R_j \subset \{x \in Q; \widetilde{M}_r^Q((f-\pi)\chi_{\tilde{R}_j})(x) > B/(3^{1/r-1}+1)\lambda\}.$$

From this inclusion and the $L^r - L^{r,\infty}$ boundedness of M_r, we have

$$\begin{aligned} |\{x \in Q; \widetilde{M}_r^Q f(x) > B\lambda\} \cap R_j| &\leq c\Big(\frac{\|(f-\pi)\chi_{\tilde{R}_j}\|_{L^r}}{B/(3^{1/r-1}+1)\lambda}\Big)^r \\ &\leq c\Bigg(\frac{|\tilde{R}_j|^{1/r} \inf_{\xi\in\tilde{R}_j} \widetilde{f}^{\sharp(r),Q}_{l,0}(\xi)}{B\lambda}\Bigg)^r \\ &\leq C_0\Big(\frac{\delta}{B}\Big)^r |R_j|, \end{aligned}$$

which is exactly the inequality (2). Here we have used the assumption $X_j \neq \emptyset$ for the above last inequality. $\square$

Remark 3.1. *By using* (2) *and the reverse doubling inequality, we can obtain that for all j and $w \in A_\infty \cap RH_{1+\varepsilon}$,*

$$w(\{x \in Q; \widetilde{M}_r^Q f(x) > B\lambda\} \cap R_j) \leq c(C_0\Big(\frac{\delta}{B}\Big)^r)^{\varepsilon/(1+\varepsilon)} w(R_j)$$

with a constant c independent of j. Here, A_∞ is the Muckenhoupt weight class and RH_ε the reverse Hölder class, that is, for a positive locally integrable function w, $w \in A_\infty$ if and only if

$$\sup_Q(⨍_Q w dx) \exp(⨍_Q \log w^{-1} dx) < \infty$$

and $w \in RH_\varepsilon$ if and only if $(⨍_Q w^\varepsilon dx)^{1/\varepsilon} \leq c ⨍_Q w dx$ for all cubes $Q \subset \mathbb{R}^n$. Therefore, we have the weighted version of (1)*;*

$$w(E_\lambda) \leq c(C_0\Big(\frac{\delta}{B}\Big)^r)^{\varepsilon/(1+\varepsilon)} w(\Omega_\lambda).$$

3.1 Proof of Theorem 1.1

In this subsection, we prove Theorem 1.1 by using Lemma 3.1. But, since it is clear that Corollary 3.1 and Proposition 3.1 below imply Theorem 1.1, we omit the details of the proof of Theorem 1.1.

Corollary 3.1. *Let $0 < p, r < \infty$, $0 < q \leq \infty$, $\alpha \in \mathbb{R}$ and $l \in \mathbb{N}_0$. Then, there exists a constant c such that for every $f \in L^r_{loc}(\mathbb{R}^n \backslash \{0\})$,*

$$\|f\|_{\dot{K}^\alpha_{p,q}} \leq c\Big(\|f^{\sharp(r),Q^*_\cdot}_{l,0}\|_{\dot{K}^\alpha_{p,q}} + \|f\|_{\dot{K}^{\alpha+n(1/q-1/r)}_{p,r}}\Big).$$

Proof. Once we prove the following inequality, the proof is completed;

$$\|\widetilde{M}_r^{Q_k} f\|_{L^p(Q_k)} \leq c\Big(\|\widetilde{f}^{\sharp(r),Q_k}_{l,0}\|_{L^p(Q_k)} + |Q_k|^{1/p}\Big(⨍_{Q_k} |f|^r dy\Big)^{1/r}\Big). \tag{3}$$

Let $\lambda > (2^n(2^n-1))^{1/r}\Big(⨍_{Q_k} |f|^r dy\Big)^{1/r}$. We decompose Q_k into $2^n(2^n-1)$ disjoint congruent cubes whose sidelengths are 2^{k-1}. We denote by $\{Q_k^j\}_{j=1}^{2^n(2^n-1)}$ these disjoint cubes. Since the volume of each Q_k^j equals to $2^{-n}(2^n-1)^{-1}|Q_k|$, we see that $\lambda > \Big(⨍_{Q_k^j} |f|^r dy\Big)^{1/r}$ for all j. Then, from Lemma 3.1 there exist constants $B > 1$ and $C_0 > 0$ so that for all j the inequality (1) with $Q = Q_k^j$ holds. Now we remark that $\widetilde{M}_r^{Q_k^j} f$ coincides with $\widetilde{M}_r^{Q_k} f$, and $\widetilde{f}^{\sharp(r),Q_k^j}_{l,0}$ coincides with $\widetilde{f}^{\sharp(r),Q_k}_{l,0}$ on Q_k^j. Hence, we obtain that

$$\begin{aligned}|\{x \in Q_k^j; \widetilde{M}_r^{Q_k} f(x) > B\lambda,\ \widetilde{f}^{\sharp(r),Q_k}_{l,0}(x) \leq \delta\lambda\}|\\ \leq C_0\Big(\frac{\delta}{B}\Big)^r |\{x \in Q_k^j; \widetilde{M}_r^{Q_k} f(x) > \lambda\}|\end{aligned}$$

for all j. By taking the sum of these inequalities over j, we have the following good λ-inequality for Q_k:

$$\begin{aligned}|\{x \in Q_k; \widetilde{M}_r^{Q_k} f(x) > B\lambda,\ \widetilde{f}^{\sharp(r),Q_k}_{l,0}(x) \leq \delta\lambda\}|\\ \leq C_0\Big(\frac{\delta}{B}\Big)^r |\{x \in Q_k; \widetilde{M}_r^{Q_k} f(x) > \lambda\}|.\end{aligned}$$

The same computation as [35, Theorem 1.3] gives us the norm inequality (3). The rest of the proof is easy. $\square$

Remark 3.2. *By the last inequality in Remark 3.1, we have the weighted version of* (3)*;*

$$\|\widetilde{M}_r^{Q_k} f\|_{L^p_w(Q_k)} \leq c\Big(\|\widetilde{f}_{l,0}^{\sharp(r),Q_k}\|_{L^p_w(Q_k)} + w(Q_k)^{1/p}\Big(\fint_{Q_k}|f|^r dy\Big)^{1/r}\Big),$$

where, of course, the constant c is independent of k. Then, it is easy to see that for $w_i \in A_\infty$, $(i = 1, 2)$,

$$\|f\|_{\dot{K}^\alpha_{p,q}(w_1,w_2)} \leq c\Bigg(\|f_{l,0}^{\sharp(r),Q^*_\cdot}\|_{\dot{K}^\alpha_{p,q}(w_1,w_2)} + \Big(\sum_{k\in\mathbb{Z}} w_1(Q_k)^{q\alpha/n} w_2(Q_k)^{q/p}\Big(\fint_{Q_k}|f|^r dy\Big)^{q/r}\Big)^{1/q}\Bigg),$$

where $\dot{K}^\alpha_{p,q}(w_1, w_2)$ is the weighted Herz space, see [24], for the definition of weighted Herz spaces.

Proposition 3.1. *Let $0 < p, r < \infty$, $0 < q \leq \infty$, $-n/p < \alpha$ and $l \in \mathbb{N}_0$. Then, there exists a constant c so that for $f \in L^r_{loc}(\mathbb{R}^n\backslash\{0\})$ satisfying that $\Big(\fint_{Q_j}|f|^r dy\Big)^{1/r} \to 0$ as $j \to \infty$,*

$$\|f\|_{\dot{K}^{\alpha_0}_{p,r}} \leq c\|f_{l,0}^{\sharp(r),Q^*_\cdot}\|_{\dot{K}^\alpha_{p,q}},$$

where $\alpha_0 = \alpha + n(1/p - 1/r)$.

Proof. From the same decomposition as Corollary 3.1, we can write

$$\|f\|_{\dot{K}^{\alpha_0}_{p,r}} \leq c \sum_{j=1}^{2^n(2^n-1)} (\sum_{k\in\mathbb{Z}} 2^{k\alpha_0 q}\|f\|^q_{L^r(Q^j_k)})^{1/q}.$$

Then it suffices to prove

$$(\sum_{k\in\mathbb{Z}} 2^{k\alpha_0 q}\|f\|^q_{L^r(Q^j_k)})^{1/q} \leq c\|f_{l,0}^{\sharp(r),Q^*_\cdot}\|_{\dot{K}^\alpha_{p,q}}.$$

We take polynomials π_k belonging to $\Pi_l(f, L^r(Q^j_k))$ for every j and k. From a chain of inequalities

$$\begin{aligned}\|f - \pi_k\|_{L^r(Q^j_k)} &= \inf_{P\in\mathbf{P}_l}\|f - P\|_{L^r(Q^j_k)} \\ &\leq |Q^j_k|^{1/r}\inf_{\xi\in Q^j_k} f_{l,0}^{\sharp(r),Q_k}(\xi) \\ &\leq |Q^j_k|^{1/r-1/p}\|f_{l,0}^{\sharp(r),Q^*_k}\|_{L^p(Q^j_k)},\end{aligned}$$

we have

$$\Big(\sum_{k\in\mathbb{Z}} 2^{k\alpha_0 q}\|f - \pi_k\|^q_{L^r(Q^j_k)}\Big)^{1/q} \leq c\|f_{l,0}^{\sharp(r),Q^*_\cdot}\|_{\dot{K}^\alpha_{p,q}}.$$

Therefore, we have only to show the inequality

$$(\sum_{k\in\mathbb{Z}} 2^{k\alpha_0 q}\|\pi_k\|^q_{L^r(Q^j_k)})^{1/q} \leq c\|f_{l,0}^{\sharp(r),Q^*_\cdot}\|_{\dot{K}^\alpha_{p,q}}. \tag{4}$$

From our assumption, for every integer k, we can take a sufficiently large integer $N_k > k + 1$ satisfying

$$|Q^j_k|^{(\alpha_0+n/r\pm1)/n}\Big(\fint_{Q^j_{N_k}}|f|^r dy\Big)^{1/r} \leq \|f_{l,0}^{\sharp(r),Q^*_\cdot}\|_{\dot{K}^\alpha_{p,q}}.$$

We decompose in the following way;

$$\begin{aligned}\|\pi_k\|_{L^r(Q_k^j)} &\le |Q_k^j|^{1/r}\|\pi_k\|_{L^\infty(Q_k^j)}\\ &\le |Q_k^j|^{1/r}\sum_{i=0}^{N_k-1-k}\|\pi_{k+i}-\pi_{k+i+1}\|_{L^\infty(Q_k^j)}+|Q_k^j|^{1/r}\|\pi_{N_k}\|_{L^\infty(Q_k^j)}\\ &=: \mathrm{I}+\mathrm{II}.\end{aligned}$$

We shall prove

$$(\sum_{k\in\mathbb{Z}}2^{k\alpha_0 q}\ \mathrm{I}^q)^{1/q}\le c\|f_{l,0}^{\sharp(r),Q_\cdot^*}\|_{\dot{K}_{p,q}^\alpha}. \tag{5}$$

A geometric observation shows that

$$\begin{aligned}Q_{k+i}^j &\subset 2Q_{k+i+1}^j \ \subset\ Q_{k+i}\ \cup\ Q_{k+i+1}\ \cup\ Q_{k+i+2}\ =: Q_{k+i+1}^*, \text{and}\\ Q_k^j &\subset 3Q_{k+i+1}^j,\end{aligned}$$

for every $k\in\mathbb{Z}$ and $i=0,\cdots,N_k-1-k$. Then, by using (ii) in Lemma 2.1 we have

$$\begin{aligned}\|P\|_{L^\infty(Q_k^j)} &\le \|P\|_{L^\infty(3Q_{k+i+1}^j)}\\ &\le c\|P\|_{L^\infty(2Q_{k+i+1}^j)},\end{aligned}$$

for any polynomial P, $k\in\mathbb{Z}$ and $i=0,\cdots,N_k-1-k$. From this inequality and (iii) in Lemma 2.2, we obtain

$$\begin{aligned}\|\pi_{k+i}-\pi_{k+i+1}\|_{L^\infty(Q_k^j)} &\le c\|\pi_{k+i}-\pi_{k+i+1}\|_{L^\infty(2Q_{k+i+1}^j)}\\ &\le c\inf_{P\in\mathbf{P}_l}\Big(\fint_{2Q_{k+i+1}^j}|f-P|^r dy\Big)^{1/r}\\ &\le c\inf_{\xi\in Q_{k+i+1}^j} f_{l,0}^{\sharp(r),2Q_{k+i+1}^j}(\xi)\\ &\le c|Q_{k+i+1}^j|^{-1/p}\|f_{l,0}^{\sharp(r),2Q_{k+i+1}^j}\|_{L^p(Q_{k+i+1}^j)}\\ &\le c2^{-(k+i)n/p}\|f_{l,0}^{\sharp(r),Q_{k+i+1}^*}\|_{L^p(Q_{k+i+1}^j)}.\end{aligned}$$

Hence in order to prove (5), it suffices to prove that

$$\Big(\sum_{k\in\mathbb{Z}}2^{k(\alpha_0+n/r)q}(\sum_{i=k}^{\infty}2^{-in/p}\|f_{l,0}^{\sharp(r),Q_i^*}\|_{L^p(Q_i^j)})^q\Big)^{1/q}\le c\Big(\sum_{k\in\mathbb{Z}}2^{k\alpha q}\|f_{l,0}^{\sharp(r),Q_k^*}\|_{L^p(Q_k)}^q\Big)^{1/q}. \tag{6}$$

If $0<q\le 1$, then (6) can be shown as follows

$$\begin{aligned}&\Big(\sum_{k\in\mathbb{Z}}2^{k(\alpha_0+n/r)q}(\sum_{i=k}^{\infty}2^{-in/p}\|f_{l,0}^{\sharp(r),Q_i^*}\|_{L^p(Q_i^j)})^q\Big)^{1/q}\\ &\qquad\le\Big(\sum_{i\in\mathbb{Z}}2^{i\alpha q}\|f_{l,0}^{\sharp(r),Q_i^*}\|_{L^p(Q_i^j)}^q 2^{-i(\alpha+n/p)q}\sum_{k=-\infty}^{i}2^{k(\alpha_0+n/r)q}\Big)^{1/q}\\ &\qquad\le c\Big(\sum_{k\in\mathbb{Z}}2^{k\alpha q}\|f_{l,0}^{\sharp(r),Q_k^*}\|_{L^p(Q_k)}^q\Big)^{1/q}.\end{aligned}$$

Here we have used the assumption $-n/p<\alpha$. If $q=\infty$, then (6) can be seen immediately as follows;

$$
\begin{aligned}
&2^{k(\alpha_0+n/r)}\sum_{i=k}^{\infty}2^{-in/p}\|f_{l,0}^{\sharp(r),Q_i^*}\|_{L^p(Q_i^j)}\\
&\qquad\le\Big(\sup_{i\in\mathbb{Z}}2^{i\alpha}\|f_{l,0}^{\sharp(r),Q_i^*}\|_{L^p(Q_i)}\Big)2^{k(\alpha_0+n/r)}\sum_{i=k}^{\infty}2^{-i(\alpha+n/p)}\\
&\qquad\le c\sup_{k\in\mathbb{Z}}2^{k\alpha}\|f_{l,0}^{\sharp(r),Q_k^*}\|_{L^p(Q_k)}.
\end{aligned}
$$

Here we have again used the assumption $-n/p<\alpha$. Finally, the case $1<q<\infty$ can be proved by the use of interpolation.

Next, we shall prove

$$
(\sum_{k\in\mathbb{Z}}2^{k\alpha_0 q}\ \mathrm{II}^q)^{1/q}\le c\|f_{l,0}^{\sharp(r),Q_\cdot^*}\|_{\dot K_{p,q}^{\alpha}}.
$$

Since $Q_k^j\subset 3Q_{N_k}^j$, we obtain

$$
\begin{aligned}
\|\pi_{N_k}\|_{L^\infty(Q_k^j)}&\le\|\pi_{N_k}\|_{L^\infty(3Q_{N_k}^j)}\\
&\le c\|\pi_{N_k}\|_{L^\infty(Q_{N_k}^j)}\\
&\le c\Big(\fint_{Q_{N_k}^j}|f|^r dy\Big)^{1/r}\\
&\le c2^{-k(\alpha_0+n/r\pm1)}\|f_{l,0}^{\sharp(r),Q_\cdot^*}\|_{\dot K_{p,q}^{\alpha}},
\end{aligned}
$$

which imply the inequality above and completes the proof of Proposition 3.1. □

Remark 3.3. *From the last inequality in Remark 3.2, we obtain the following result. We omit the detail of the calculation.*

Let $p,q,\alpha,\alpha_0,r,s,\tau$ be the same as in Proposition 3.1 and $\sigma=0$. Let $w_i\in A_{\rho_i}$, $(i=1,2)$ with $1\le\rho_i<\infty$ and $w_1\in RH_{1+\varepsilon}$. And we put $\delta(\alpha)=\rho_1$ if $\alpha\ge0$, $=\varepsilon/(1+\varepsilon)$ if $-n/p<\alpha<0$. In the case $w_1=w_2$, we have that

$$
\Big(\sum_{k\in\mathbb{Z}}w_1(Q_k)^{q\alpha/n}w_2(Q_k)^{q/p}\Big(\fint_{Q_k}|f|^r dy\Big)^{q/r}\Big)^{1/q}\le c\|f_{l,0}^{\sharp(r),Q_\cdot^*}\|_{\dot K_{p,q}^{\alpha}(w_1,w_2)}.
$$

On the other hand, in the case $w_1\ne w_2$, if $\alpha\delta(\alpha)>-n\rho_2/p$, then we have the same inequality.

3.2 Proof of Theorem 1.2

In the subsection, we consider Theorem 1.2. Sawano and Tanaka proved similar results to Corollary 3.2 and Proposition 3.2 with $1<q$ and $r=1$. It is trivial that Corollary 3.2 and Proposition 3.2 complete the proof of Theorem 1.2. The arguments in this subsection are similar to those in [35] and are more simple than the one in the case Herz space. But we shall give the proof of Corollary 3.2 and Proposition 3.2 for the sake of completeness.

Corollary 3.2. *Let $0<q\le p<\infty$, $0<r<\infty$ and $l\in\mathbb{N}_0$. Then, there exists a constant c so that for any $f\in L_{loc}^r$,*

$$
\|f\|_{\mathcal{M}_q^p}\le c\Big(\|f_{l,0}^{\sharp(r),I_\cdot}\|_{\mathcal{M}_q^p}+\|f\|_{\mathcal{M}_r^p}\Big).
$$

Proof. Since we have a " locally " good λ-inequality in Lemma 3.1, the argument in [35, Theorem 1.3] implies that for all dyadic cubes I,

$$\|\widetilde{M}_r^I f\|_{L^q(I)} \le c\Big(\|f_{l,0}^{\sharp(r),I}\|_{L^q(I)} + |I|^{1/q}\Big(\fint_I |f|^r dy\Big)^{1/r}\Big),$$

which says that the required inequalities hold. Here we have used the equivalence of the norm of Morrey space mentioned after Definition 2.3. $\square$

Proposition 3.2. *Let $0 < q \le p < \infty$, $0 < r \le p$ and $l \in \mathbb{N}_0$. Then, there exists a constant c so that for $f \in L^r_{loc}$ satisfying $\Big(\fint_{I_k} |f|^r dy\Big)^{1/r} \to 0$ as $k \to \infty$, $I_k = 2^k I$, for some cube $I \in \mathbb{R}^n$,*

$$\|f\|_{\mathcal{M}_r^p} \le c\|f_{l,0}^{\sharp(r),I.}\|_{\mathcal{M}_q^p}.$$

Proof. As a mentioned in Remark 1.2, it follows from the decay condition that for all cubes I, we have

$$\Big(\fint_{I_k} |f|^r dy\Big)^{1/r} \to 0 \text{ as } k \to \infty.$$

Let $\pi_R \in \Pi_l(f, L^r(R))$ for any cubes R. Going through a similar argument as the proof of Proposition 3.1, we obtain

$$|R|^{1/p}\Big(\fint_R |f - \pi_R|^r dy\Big)^{1/r} \le c\|f_{l,0}^{\sharp(r),I.}\|_{\mathcal{M}_q^p},$$

for every cube R. Then it only remains to verify for each R

$$|R|^{1/p}\Big(\fint_R |\pi_R|^r dy\Big)^{1/r} \le c\|f_{l,0}^{\sharp(r),I.}\|_{\mathcal{M}_q^p}.$$

To do this, we decompose

$$\Big(\fint_R |\pi_R|^r dy\Big)^{1/r} \le \sum_{i=1}^{N-1} \|\pi_{R_i} - \pi_{R_{i+1}}\|_{L^\infty(R)} + \|\pi_{R_N}\|_{L^\infty(R)},$$

with a large integer N satisfying

$$|R|^{1/p}\Big(\fint_{R_N} |f|^r dy\Big)^{1/r} \le \|f_{l,0}^{\sharp(r),I.}\|_{\mathcal{M}_q^p}.$$

Now because that $R \subset R_i \subset R_{i+1}$, it readily follows that

$$\begin{aligned}\|\pi_{R_i} - \pi_{R_{i+1}}\|_{L^\infty(R)} &\le \inf_{P \in \mathbf{P}_l}\Big(\fint_{R_{i+1}} |f - P|^r dy\Big)^{1/r} \\ &\le c|R_{i+1}|^{-1/q}\|f_{l,0}^{\sharp(r),R_{i+1}}\|_{L^q(R_{i+1})}.\end{aligned}$$

Therefore, we can conclude the desired inequality as follows;

$$\begin{aligned}|R|^{1/p}\sum_{i=1}^{N-1}\|\pi_{R_i} - \pi_{R_{i+1}}\|_{L^\infty(R)} &\le c|R|^{1/p}\sum_{i=1}^{N-1}|R_{i+1}|^{-1/q}\|f_{l,0}^{\sharp(r),R_{i+1}}\|_{L^q(R_{i+1})} \\ &\le c\|f_{l,0}^{\sharp(r),I.}\|_{\mathcal{M}_q^p}\end{aligned}$$

and

$$\begin{aligned}|R|^{1/p}\|\pi_{R_N}\|_{L^\infty(R)} &\le c|R|^{1/p}\Big(\fint_{R_N} |f|^r dy\Big)^{1/r} \\ &\le c\|f_{l,0}^{\sharp(r),I.}\|_{\mathcal{M}_q^p}.\end{aligned}$$

$\square$

4 Proof of Theorems 1.3 and 1.4

In this section, we prove the bilinear estimates in Theorems 1.3 and 1.4 by using Theorems 1.1 and 1.2 and the following pointwise estimate for the sharp maximal function of $f\nabla^m g$. Miyachi showed the inequality (ii) of Proposition 4.1 with $m=1$ in his unpublished paper. The statement (iii) is an analogy of (ii) which is not used in the present paper. Let $p_+ = \max(p,1)$.

Proposition 4.1. *Let $0<r<s<\infty$, $1/r = 1/s + 1/s'$, $m\in\mathbb{N}$ and β be a multi-index with $|\beta| = m$. Then, there exists a constant c such that the following* (i), (ii) *and* (iii) *hold.*
(i): *If $f\in L^s_{loc}(\Omega)$ and $g\in L^{s'}_{loc}(\Omega)$, then for $x\in\Omega$,*

$$(fg)^{\sharp(r),\Omega}_{0,0}(x) \le c\Big(M^\Omega_s f(x) g^{\sharp(s'),\Omega}_{0,0}(x) + f^{\sharp(s),\Omega}_{0,0}(x) M^\Omega_{s'} g(x)\Big).$$

(ii): *If $f\in L^s_{loc}(\Omega)$ and $g\in\mathscr{D}'$ with $\partial^\beta g\in L^{s'_+}_{loc}(\Omega)$, then for $x\in\Omega$ and $k\in\mathbb{N}_0$,*

$$(f\partial^\beta g)^{\sharp(r),\Omega}_{m,0}(x) \le c\Big(M^\Omega_s f(x)(\partial^\beta g)^{\sharp(s'_+),\Omega}_{0,0}(x) + f^{\sharp(r),\Omega}_{m,m}(x) g^*_k(x)\Big).$$

(iii): *If $f\in L^s_{loc}(\Omega)$, $g\in L^1_{loc}(\Omega)$ with $\partial^\beta g\in L^{s'}_{loc}(\Omega)$ and $\nabla^{m+1} g\in BMO$, and $s'\le 1$, then for $x\in\Omega$,*

$$(f\partial^\beta g)^{\sharp(r),\Omega}_{m+1,0}(x) \le c\Big(M^{1,\Omega}_s f(x)\|\nabla^{m+1} g\|_{BMO(\Omega)} + f^{\sharp(s),\Omega}_{m,m}(x) M^\Omega_{s'} g(x)\Big).$$

Proof. (i): Let Q be a cube in Ω with $x\in Q$. Let $\pi_Q(f)\in\Pi_0(f, L^s(Q))$ and $\pi_Q(g)\in\Pi_0(g, L^{s'}(Q))$. From the estimate

$$\Big(\fint_Q |fg - \pi_Q(f)\pi_Q(g)|^r dy\Big)^{1/r} \le c\Bigg(\Big(\fint_Q |f-\pi_Q(f)|^s dy\Big)^{1/s}\Big(\fint_Q |g|^{s'} dy\Big)^{1/s'} + \Big(\fint_Q |f|^s dy\Big)^{1/s}\Big(\fint_Q |g-\pi_Q(g)|^{s'} dy\Big)^{1/s'}\Bigg)^{1/p},$$

we see that

$$\sup_{x\in Q\subset\Omega}\Big(\fint_Q |fg - \pi_Q(f)\pi_Q(g)|^r dy\Big)^{1/r} \le c\Big(M^\Omega_s f(x) g^{\sharp(s'),\Omega}_{0,0}(x) + f^{\sharp(s),\Omega}_{0,0}(x) M^\Omega_{s'} g(x)\Big).$$

(ii): Let Q be such a cube. For any polynomial $P\in\mathbf{P}_m$, we shall estimate $\Big(\fint_Q |f\partial^\beta g - A_Q P|^r dy\Big)^{1/r}$ where A_Q is the constant appearing in Lemma 2.6. Hölder's inequality and the properties of A_Q give us that

$$\begin{aligned}\Big(\fint_Q |f\partial^\beta g - A_Q P|^r dy\Big)^{1/r} &\le \Big(\fint_Q |f|^s dy\Big)^{1/s}\Big(\fint_Q |\partial^\beta g - A_Q|^{s'} dy\Big)^{1/s'} + \Big(\fint_Q |f-P|^r dy\Big)^{1/r}|A_Q| \\ &\le c\Big(M^\Omega_s f(x)\inf_{c\in\mathbb{C}}|Q|^{-1/s'_+}\|\partial^\beta g - c\|_{L^{s'_+}(Q)} + \Big(\fint_Q |f-P|^r dy\Big)^{1/r}|Q|^{-m/n} g^*_k(x)\Big).\end{aligned}$$

Therefore, we have

$$(f\partial^\beta g)^{\sharp(r),\Omega}_{m,0}(x) \le c\Big(M^\Omega_s f(x)(\partial^\beta g)^{\sharp(s'_+),\Omega}_{0,0}(x) + f^{\sharp(r),\Omega}_{m,m}(x) g^*_k(x)\Big).$$

(iii): Let Q be a cube in Ω with $x\in Q$. We take $\pi\in\Pi_{m+1}(g, L^{s'}(Q))$. Remark that, in this setting, we see that

$$\begin{aligned}&g^{\sharp(1),Q}_{m,m+1}\in L^1(Q) \quad and \\ &g^{\sharp(s'),Q}_{m+1,m+1}(x) < \infty \quad a.e.\ x\in Q.\end{aligned} \tag{7}$$

In fact, by Theorem 6.2 in [10], we have a chain of inequalities

$$\begin{aligned}\|g^{\sharp(1),Q}_{m,m+1}\|_{L^1(Q)} &\le c|Q|^{1/2}\|g^{\sharp(1),Q}_{m,m+1}\|_{L^2(Q)}\\ &\le c|Q|^{1/2}\|\nabla^{m+1}g\|_{L^2(Q)} < \infty.\end{aligned}$$

On the other hand, because $g^{\sharp(s'),Q}_{m+1,m+1}$ is dominated by $g^{\sharp(1),Q}_{m,m+1}$, the second assertion follows from the first one. (7) imply the inequality

$$|\partial^\beta g(y) - \partial^\beta \pi(y)| \le c|Q|^{1/n} g^{\sharp(s'),Q}_{m+1,m+1}(y),\ a.e.\ y \in Q. \tag{8}$$

For the proof of this estimate, see [10, Lemma 5.2 and Corollary 5.7].

Hence, (ii) in Lemma 2.2, Lemma 2.3, (8) and Markov's inequality yield that for any $P \in \mathbf{P}_m$,

$$\begin{aligned}&\Big(\fint_Q |f\partial^\beta g - P\partial^\beta \pi|^r dy\Big)^{1/r}\\ &\quad\le \Big(\fint_Q |f|^s dy\Big)^{1/s}\Big(\fint_Q |\partial^\beta g - \partial^\beta \pi|^{s'} dy\Big)^{1/s'} + \Big(\fint_Q |f-P|^s dy\Big)^{1/s}\Big(\fint_Q |\partial^\beta \pi|^{s'} dy\Big)^{1/s'}\\ &\quad\le c\Big(M^{1,\Omega}_s f(x) \sum_{|\alpha|=m+1} \|(\partial^\alpha g)^{\sharp(s'_+),Q}_{0,0}\|_{L^\infty(Q)} + |Q|^{-m/n}\Big(\fint_Q |f-P|^s dy\Big)^{1/s} M^\Omega_{s'} g(x)\Big).\end{aligned}$$

Thus, we obtain

$$(f\partial^\beta g)^{\sharp(r),\Omega}_{m+1,0}(x) \le c\Big(M^{1,\Omega}_s f(x)\|\nabla^{m+1}g\|_{BMO(\Omega)} + f^{\sharp(s),\Omega}_{m,m}(x) M^\Omega_{s'} g(x)\Big).$$

This is what we desired. □

4.1 Proof of Theorem 1.3

Proof. (i): Let $0 < r < \min(1/2, p/2)$. Since

$$\Big(\fint_{Q_j} |f^2|^r dy\Big)^{1/r} \le 2^{-2j(\alpha+n/p)}\|f\|^2_{\dot K^\alpha_{p,q}} \to 0, as\ j \to \infty,$$

we can use Theorem 1.1 and hence, we have

$$\begin{aligned}\|f^2\|_{\dot K^\alpha_{p,q}} &\le c\|(f^2)^{\sharp(r),Q^*}_{0,0}\|_{\dot K^\alpha_{p,q}},\\ \|g^2\|_{\dot K^\alpha_{p,q}} &\le c\|(g^2)^{\sharp(r),Q^*}_{0,0}\|_{\dot K^\alpha_{p,q}}.\end{aligned}$$

Then we obtain the required inequality in the following way;

$$\begin{aligned}\|fg\|_{\dot K^\alpha_{p,q}} &\le \|f^2\|^{1/2}_{\dot K^\alpha_{p,q}}\ \|g^2\|^{1/2}_{\dot K^\alpha_{p,q}}\\ &\le c\|(f^2)^{\sharp(r),Q^*}_{0,0}\|^{1/2}_{\dot K^\alpha_{p,q}}\ \|(g^2)^{\sharp(r),Q^*}_{0,0}\|^{1/2}_{\dot K^\alpha_{p,q}}\\ &\le c\|M_{2r}f\ f^{\sharp(2r),Q^*}_{0,0}\|^{1/2}_{\dot K^\alpha_{p,q}}\ \|M_{2r}g\ g^{\sharp(r),Q^*}_{0,0}\|^{1/2}_{\dot K^\alpha_{p,q}}\\ &\le c\|f\|^{1/2}_{\dot K^\alpha_{p,q}}\ \|f\|^{1/2}_{\dot K^0_{BMO,\infty}}\ \|g\|^{1/2}_{\dot K^\alpha_{p,q}}\ \|g\|^{1/2}_{\dot K^0_{BMO,\infty}}\\ &\le c\Big(\|f\|_{\dot K^\alpha_{p,q}}\ \|g\|_{\dot K^0_{BMO,\infty}} + \|f\|_{\dot K^0_{BMO,\infty}}\ \|g\|_{\dot K^\alpha_{p,q}}\Big).\end{aligned}$$

(ii): Firstly we shall show that if $h \in \dot K^0_{\infty,BMO}$, then we have

$$\fint_{Q_k} |h| dx = \mathcal{O}(k),\ \text{as}\ k \to \infty, \tag{9}$$

We remark the equivalence

$$\fint_{Q_k} |h|dx \approx \max_{1\le j\le 2^n(2^n-1)} \fint_{Q_k^j} |h|dx.$$

Recall that each of Q_k^j is a subcube of Q_k, appearing in the proof of Corollary 3.1 and Proposition 3.1. Let $\pi_i \in \Pi_0(h, L^1(Q_i^j))$. Since $Q_i^j,\ Q_{i+1}^j \subset 2Q_{i+1}^j \subset Q_{i+1}^*$, we obtain

$$\begin{aligned}\fint_{Q_k^j} |h-\pi_1|dx &\le \fint_{Q_k^j} |h-\pi_k|dx + \sum_{i=1}^{k-1} |\pi_i - \pi_{i+1}| \\ &\le c \inf_{P\in \mathbf{P}_l} \fint_{Q_k^j} |h-P|dx + c\sum_{i=1}^{k-1} \inf_{P\in\mathbf{P}_l} \fint_{2Q_{i+1}^j} |h-P|dx \\ &\le ck\|h\|_{\dot K^0_{BMO,\infty}},\end{aligned}$$

which imply (9). Applying (9) to $\nabla^m g$ one obtains that for any $\varepsilon > 0$, $2^{-\varepsilon k}\fint_{Q_k} |\nabla^m g|dy \to 0$ as $k\to\infty$, and as a consequence, we have

$$\Big(\fint_{Q_k} |f\nabla^m g|^r dy\Big)^{1/r} \to 0$$

as $k \to \infty$ for all $0 < r < p/(p+1)$. Hence, from Theorem 1.1, we obtain Fefferman-Stein's inequality;

$$\|f\nabla^m g\|_{\dot K^\alpha_{p,q}} \le c\|(f\nabla^m g)^{\sharp(r),Q^*}_{m,0}\|_{\dot K^\alpha_{p,q}}$$

with sufficiently small r. Combining the above inequality with (ii) of Proposition 4.1 and Lemma 2.3 leads us to the following estimate; for sufficiently large k,

$$\|f\nabla^m g\|_{\dot K^\alpha_{p,q}} \le c\Big(\|f\|_{\dot K^\alpha_{p,q}}\|\nabla^m g\|_{\dot K^0_{BMO,\infty}} + \|\nabla^m f\|_{\dot K^0_{BMO,\infty}}\|g_k^*\|_{\dot K^\alpha_{p,q}}\Big).$$

Finally, the estimate $\|g_k^*\|_{\dot K^\alpha_{p,q}} \le c\|g\|_{H\dot K^\alpha_{p,q}}$ follows from Uchiyama's pointwise estimate in [41];

$$g_k^*(x) \le cM_{n/(n+k)}(\phi_+(g))(x).$$

□

Remark 4.1. *As we mentioned in 4 of Remark 1.1, Remarks 3.2, 3.3 and Proposition 4.1 yield the weighted version of Theorem 1.3. See [24] for the boundedness of M on the weighted Herz spaces.*

4.2 Proof of Theorem 1.4

Proof. (i): Let $0 < r < \min(1/2, q/2)$. Since $f, g \in \mathcal{M}^p_q$, f^2 and g^2 satisfy the decay assumption in Thoerem 1.2. Then we obtain

$$\|f^2\|_{\mathcal{M}^p_q} \le c\|(f^2)^{\sharp(r),I.}_{0,0}\|_{\mathcal{M}^p_q}$$
$$\|g^2\|_{\mathcal{M}^p_q} \le c\|(g^2)^{\sharp(r),I.}_{0,0}\|_{\mathcal{M}^p_q}.$$

By these inequalities and (i) of Proposition 4.1, we have

$$\begin{aligned}
\|fg\|_{\mathcal{M}^p_q} &\le \|f^2\|^{1/2}_{\mathcal{M}^p_q} \, \|g^2\|^{1/2}_{\mathcal{M}^p_q} \\
&\le c\|(f^2)^{\sharp(r),I.}_{0,0}\|^{1/2}_{\mathcal{M}^p_q} \, \|(g^2)^{\sharp(r),I.}_{0,0}\|^{1/2}_{\mathcal{M}^p_q} \\
&\le c\|f^{\sharp(2r),I.}_{0,0} \; M_{2r}f\|^{1/2}_{\mathcal{M}^p_q} \, \|g^{\sharp(2r),I.}_{0,0} \; M_{2r}g\|^{1/2}_{\mathcal{M}^p_q} \\
&\le c\|f\|^{1/2}_{\mathcal{M}^p_q} \, \|f\|^{1/2}_{BMO} \, \|g\|^{1/2}_{\mathcal{M}^p_q} \, \|g\|^{1/2}_{BMO} \\
&\le c\Big(\|f\|_{\mathcal{M}^p_q} \, \|g\|_{BMO} + \|f\|_{BMO} \, \|g\|_{\mathcal{M}^p_q}\Big).
\end{aligned}$$

(ii), (iii): The same arguments as the proof of Theorem 1.4 complete the proof. □

Remark 4.2. *In the proof of Theorems 1.3 and 1.4, we showed the following interpolation inequalities;*

$$\begin{aligned}
\|f\|_{\dot K^{\alpha/2}_{2p,2q}} &\le c\|f\|^{1/2}_{\dot K^{\alpha}_{p,q}} \, \|f\|^{1/2}_{\dot K^{0}_{BMO,\infty}}, \\
\|f\|_{\mathcal{M}^{2p}_{2q}} &\le c\|f\|^{1/2}_{\mathcal{M}^p_q} \, \|f\|^{1/2}_{BMO}.
\end{aligned}$$

Acknowledgement The author would like to express his gratitude to Prof. K. Yabuta, Prof. E. Nakai, Prof. T. Nishitani and Prof. N. Tomita for encouragement and valuable comments. The author would like to thank Prof. A. Miyachi for showing them his unpublished paper and to Prof. H. Kozono for indicating him [33]. He is grateful to Prof. S. Doi for teaching him a simple proof of Lemma 2.6. The author also would like to thanks to the referees for their useful comments and suggestions.

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