Pseudo-differential operators of class $S^m_{0,0}$ on the Herz-type spaces

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Abstract

In this paper, we show the boundedness of pseudo-differential operators of class $S^m_{0,0}$ on the Herz spaces $\dot{K}^\alpha_{p,q}$ and the Herz-type Hardy spaces $H\dot{K}^\alpha_{p,q}$.

Keywords Pseudo-differential operators, Herz spaces, Herz-type Hardy spaces.
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1 Introduction

Beurling [3] introduced the space $A^p(\mathbb{R}^n) = K^{n(1-1/q)}_{p,1}(\mathbb{R}^n)$ with $1 < p < \infty$ to study convolution algebras, which are now called Beurling algebras as a special class of the Herz spaces $K^\alpha_{p,q}$, (See Definition 2.1). The Herz spaces can be regarded as one of extensions of $L^p(\mathbb{R}^n)$ and the theory of the Herz space is developed by Feichtinger [9], Herz [15] and Flett [10].

On the other hand, Calderón and Vaillancourt [4] showed that pseudo-differential operators of class $S^0_{0,0}$ are bounded on $L^2(\mathbb{R}^n)$. And Miyachi [22] showed that those of class $S^{-n|1/p-1/2|}_{0,0}$ are bounded from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, $(0 < p < \infty)$.

The aim of this paper is to study the boundedness of pseudo-differential operators of class $S^m_{0,0}$ on the Herz spaces and the Herz-type Hardy spaces. Lu, Yabuta and Yang [21] showed that the operators having a kernel estimate are bounded from the Herz-type Hardy spaces to the Herz spaces:

Theorem 1.1 ([21])**.** *Let $T : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ be a linear and continuous operator. Suppose that the distribution kernel of T coincides in the complement of the diagonal with a locally integrable function $k(x, y)$ satisfying*

$$|k(x, y) - k(x, 0)| \le c\frac{|y|^\delta}{|x|^{n+\delta}}$$

when $2|y| < |x|$ for some $\delta \in (0, 1]$. Let $1 < p < \infty,\ 0 < q < \infty$ and $n(1-1/p) \le \alpha < n(1-1/p)+\delta$. If T is bounded on $L^p(\mathbb{R}^n)$, then T is also bounded from $H\dot{K}^\alpha_{p,q}$ into $\dot{K}^\alpha_{p,q}$.

If $m < -n-1$, pseudo-differential operators of class $S^m_{0,0}$, which is our target class, satisfies the above condition on kernels. But it is not clear that those of class $S^m_{0,0}$ with $m \ge -n-1$ satisfies the above condition or not. The main result in this paper is the following:

Theorem 1.2. *Let $1 < p < \infty,\ 0 < q < \infty,\ n(1-1/p) \le \alpha < \infty$ and $m < -\alpha - n|1/p - 1/2|$. Suppose* (i):

$$1 < p \le 2,\ l > n/2 \text{ and } l' > [-m] + n/2 + 1,$$

or (ii):

$$2 \le p < \infty,\ l > n/p \text{ and } l' > [-m - n(1/2 - 1/p)] + n/2 + 1.$$

Then $S^m_{0,0}(l, l') \subset \mathcal{L}(H\dot{K}^\alpha_{p,q}, \dot{K}^\alpha_{p,q})$.

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In Theorem 3.1, if p is close to 1 or 2 then we can take m close to $-n/2$. Hence we cannot use Theorem A directly in this case.

We explain more about the Herz spaces and the mapping properties of pseudo-differential operators on $L^p(\mathbb{R}^n)$ and $H^p(\mathbb{R}^n)$. Feichtinger [9] gave the different norms of Beurling algebras, which is equivalent to that in Beurling [3]. And the spaces $\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$ and $K^{\alpha}_{p,q}(\mathbb{R}^n)$ were introduced by Herz [15]. Flett [10] gave another equivalent norms on $\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$ and $K^{\alpha}_{p,q}(\mathbb{R}^n)$. These spaces are useful in the analysis of mapping properties of important operators. For example, Baernstein II and Sawyer [1] showed some multiplier theorems on $H^p(\mathbb{R}^n)$ by using a norm of the Herz space as the condition. Many authors studied the boundedness of pseudo-differential operators on the Herz-type Hardy space. For example, Fan and Yang [8] studied pseudo-differential operators of class $S^0_{1,0}$ on the local Herz-type Hardy spaces $h\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$. Also there are many papers which studied several other operators on the Herz spaces and the Herz-type Hardy spaces, [16], [18], [19], [20] etc... It is well-known that Calderón and Vaillancourt [4] showed that pseudo-differential operator of class $S^0_{0,0}$ is bounded on $L^2(\mathbb{R}^n)$. Futhermore, Miyachi [22] showed that pseudo-differential operator of class $S^{-n|1/p-1/2|}_{0,0}$ is bounded from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, $(0 < p < \infty)$, by using the atomic decomposition and the analytic interpolation theory. Here we remark that the index $-n|1/p-1/2|$ is optimal. In the proof of Theorem 3.1, we will follow the argument in [22]. However, to the best of my knowledge, there seems no literature mentioning the result of Theorem 3.1.

We also explain the theory of interpolation for families of quasi-Banach spaces. Coifman, Cwikel, Rochberg, Sagher and Weiss [6], [5] discussed the theory of interpolation for families of Banach spaces. This theory is a natural extension of the interpolation for the pair. Hernández [12] and Tabacco Vignati [25], [26] developed the theory for families of quasi-Banach spaces. In [14], Hernández and Yang characterized the intermediate spaces for families of the Herz-type Hardy spaces by using the atomic decomposition $(1 < q < \infty)$ established by Lu and Yang [18].

Finally we explain the structure of this paper. In Section 2, we define the Herz spaces, the Herz-type Hardy spaces and Hörmander's symbol classes and recall tools which will be used in this paper. We use Lipschitz classes on product spaces introduced by Miyachi [22] to describe smoothness of symbols, as a substitute for Hörmander's symbol classes. In Section 3, we will prove the main theorem (Theorem 3.1) by using the tools in Section 2. Also, we state a result in the non-homogeneous case. In Section 4, first we will use the characterization of intermediate spaces for couples of the Herz-type Hardy spaces in [13] and the duality argument to show the boundedness of pseudo-differential operators on wider spaces.

2 Definitions and Tools

For $k \in \mathbb{Z}$, let $B_k = \{x \in \mathbb{R}^n; |x| \le 2^k\}$, $A_k = B_k \backslash B_{k-1}$. We denote the characteristic function of E, measurable subset of $\mathbb{R}^n$, by χ_E and A_k by χ_k. We recall the definitions of the Herz spaces and the Herz-type Hardy spaces.

Definition 2.1. *(Herz space). Let* $0 < p, q \le \infty$ *and* $\alpha \in \mathbb{R}^n$. *We set* (i):

$$\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n) := \{f \in L^q_{loc}(\mathbb{R}^n \backslash \{0\});\ \|f\|_{\dot{K}^{\alpha}_{p,q}} < \infty\} : \text{homogeneous Herz space},$$

where

$$\|f\|_{\dot{K}^{\alpha}_{p,q}} := \Big(\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\|f\chi_k\|^q_{L^p}\Big)^{1/q},$$

and (ii):

$$K^{\alpha}_{p,q}(\mathbb{R}^n) := \{f \in L^p_{loc}(\mathbb{R}^n); \|f\|_{K^{\alpha}_{p,q}} < \infty\} : \text{non-homogeneous Herz space},$$

where

$$\|f\|_{K^{\alpha}_{p,q}} := \Big(\|f\chi_{B(0,1)}\|^q_{L^p} + \sum_{k\in\mathbb{N}} 2^{k\alpha q}\|f\chi_k\|^q_{L^p}\Big)^{1/q}.$$

The usual modifications in the definitions above are made when $q = \infty$.

We take a function $\varphi \in \mathscr{S}$ such that $\int \varphi dx = 1$ and set $\varphi_+^*(f)(x) = \sup_{t>0} |f * \varphi_t(x)|$, where $\varphi_t(x) := t^{-n}\varphi(x/t)$.

Definition 2.2. *(Herz-type Hardy space). Let $0 < p, q \le \infty$ and $\alpha \in \mathbb{R}$. We set* (i):

$$H\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n) := \{f \in \mathscr{S}';\ \varphi_+^*(f) \in \dot{K}^{\alpha}_{p,q}\}\text{: homogeneous Herz-type Hardy space,}$$

where

$$\|f\|_{H\dot{K}^{\alpha}_{p,q}} := \|\varphi_+^*(f)\|_{\dot{K}^{\alpha}_{p,q}},$$

and (ii):

$$HK^{\alpha}_{p,q}(\mathbb{R}^n) := \{f \in \mathscr{S}'; \varphi_+^*(f) \in K^{\alpha}_{p,q}\} \text{ : non-homogeneous Herz-type Hardy space,}$$

where

$$\|f\|_{HK^{\alpha}_{p,q}} := \|\varphi_+^*(f)\|_{K^{\alpha}_{p,q}}.$$

The following basic results are well known [13], [17]: $\dot{K}^0_{p,q} = K^0_{p,p} = L^p$, if $0 < p \le \infty$, $H\dot{K}^0_{p,p} = HK^0_{p,p} = H^p$, if $0 < p < \infty$. The spaces $\dot{K}^{\alpha}_{p,q}$ and $K^{\alpha}_{p,q}$ are quasi-Banach spaces, and if $p, q \ge 1$ then $\dot{K}^{\alpha}_{p,q}$ and $K^{\alpha}_{p,q}$ are Banach spaces. The same is true for $H\dot{K}^{\alpha}_{p,q}$ and $HK^{\alpha}_{p,q}$. $H\dot{K}^{\alpha}p,q$ and $HK^{\alpha}_{p,q}$ are defined independently of the choice of φ. When $1 \le p, q < \infty$ and $\alpha \in \mathbb{R}$, then $(\dot{K}^{\alpha}_{p,q})^* = \dot{K}^{-\alpha}_{p',q'}$ where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. In particular,

$$\|f\|_{\dot{K}^{\alpha}_{p,q}} = \sup\{\Big|\int f(x)g(x)dx\Big|; g \in \dot{K}^{-\alpha}_{p',q'} \text{ with } \|g\|_{\dot{K}^{-\alpha}_{p',q'}} \le 1\},$$

if $1 \le p, q < \infty$ and $-n/p < \alpha < n(1 - 1/p)$. Also $\dot{K}^{\alpha}_{p,q} = H\dot{K}^{\alpha}_{p,q}$, if $1 < p < \infty$, $0 < q < \infty$ and $-n/p < \alpha < n(1 - 1/p)$.

For an integer k, $\mathbf{P}_k$ denotes the set of all polynomial functions on $\mathbb{R}^n$ of degree not exceeding k. If k is a negative integer, we set $\mathbf{P}_k = 0$. We say $f \perp \mathbf{P}_k$ for $f \in L^1_{loc}$, when $fP \in L^1$ and $\int f(x)P(x)dx = 0$ for all $P \in \mathbf{P}_k$. Let $[m]$ denote the integer part of real nunber m.

Proposition 2.1. *[22]. Let $0 < p, q < \infty$ and $-n/p < \alpha < \infty$. Then the following are dense subspaces of $H\dot{K}^{\alpha}_{p,q}$*
(i):

$$X_k := \{f \in C_0^{\infty}; f \perp \mathbf{P}_k\} \text{ with } k \ge [\alpha - (1 - 1/p)],$$

(ii):

$$\mathscr{S}_0 := \{f \in \mathscr{S}; 0 \notin \mathit{supp}\hat{f}\text{: compact}\}.$$

Next,we recall the definition of Hörmander's symbol classes. For $\xi \in \mathbb{R}^n$, we set $\langle\xi\rangle := (1 + |\xi|^2)^{1/2}$.

Definition 2.3. *Let $m \in \mathbb{R}$ and $0 \le \delta \le \varrho \le 1$. We set*

$$S^m_{\varrho,\delta} := \{p \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}^n); \text{for } \alpha \text{ and } \beta,\ |\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m-\varrho|\alpha|+\delta|\beta|}\}.$$

For any $L \in \mathbb{N} \cup \{0\}$ and $p \in S^m_{\varrho,\delta}$, let

$$|p|^m_L := \max_{|\alpha+\beta| \le L} \sup_{x,\xi \in \mathbb{R}^n} \frac{|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)|}{\langle\xi\rangle^{m-\varrho|\alpha|+\delta|\beta|}}.$$

For $p \in S^m_{\varrho,\delta}$ we define pseudo-differential operator $p(X,D)$ whose symbol is p:

$$p(X,D)f(x) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\xi}p(x,\xi)\hat{f}(\xi)d\xi,$$

for any $f \in \mathscr{S}$, where $\hat{f}$ denotes the Fourier transformation of f.

For $m \in \mathbb{R}$ and $L \in \mathbb{N} \cup \{0\}$, we set

$$S^m_{0,0}(L) := \{p \in C^L(\mathbb{R}^n \times \mathbb{R}^n); |\partial^\alpha_\xi \partial^\beta_x p(x,\xi)| \le C_{\alpha,\beta}\langle \xi \rangle^m, \text{ for } |\alpha + \beta| \le L\}.$$

It is trivial that $(S^m_{0,0}(L), |\cdot|^m_L)$ is a Banach space. But we adopt the next Lipschitz classes on product spaces in the main theorem, an extension of Hörmander's symbol classes, [22]. We define the Fourier transform of f, a function on $\mathbb{R}^n \times \mathbb{R}^n$, by

$$\mathscr{F}[f](\xi,\eta) := \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{-i(x\xi + y\eta)} f(x,y) dx dy.$$

Then, the inverse Fourier transform $\mathscr{F}^{-1}$ is given by

$$\mathscr{F}^{-1}[f](x,y) := \frac{1}{(2\pi)^{2n}} \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(x\xi + y\eta)} f(\xi,\eta) d\xi d\eta.$$

Let $\theta \in C^\infty_0(\mathbb{R}^n)$ with $\theta(\xi) = \begin{cases} 1 \text{ , if } |\xi| \le 1 \\ 0 \text{ , if } |\xi| \ge 2, \end{cases}$ $\theta_0 = \theta$ and

$\theta_j(\xi) := \theta(\frac{\xi}{2^j}) - \theta(\frac{\xi}{2^{j-1}})$, for $j \in \mathbb{N}$. Then, $\sum_{j=0}^\infty \theta_j \equiv 1$ and supp $\theta_j \subset A_j$.

Definition 2.4. *(Lipschitz classes on product spaces) [22]. For $m \in \mathbb{R}^n$ and non-negative integers $l,\ l'$, we set*

$$S^m_{0,0}(l,l') := \{p \in \mathscr{S}'(\mathbb{R}^n \times \mathbb{R}^n); \|p\|_{m;l,l'} < \infty\}.$$

where

$$\|p\|_{m;l,l'} := \sup_{\substack{x,\xi \in \mathbb{R}^n \\ j,k \in \mathbb{N} \cup \{0\}}} 2^{jl} 2^{kl'} \left| \mathscr{F}^{-1}[\theta_j(y)\theta_k(\eta)\mathscr{F}[p](y,\eta)](x,\xi)\langle \xi \rangle^{-m} \right|.$$

For quasi-Banach spaces X, Y, we write $S^m_{0,0}(l,l') \subset \mathcal{L}(X,Y)$ if and only if $\|p(X,D)f\|_Y \le C\|p\|_{m;l,l'}\|f\|_X$, $(p \in \mathscr{S},\ f \in X)$. Also we write $\mathcal{L}(X) = \mathcal{L}(X,X)$. Before stating our result, we recall Miyachi's result [22].

Theorem 2.1 ([22])**.** *Let $m(p) := -n|1/p - 1/2|$. Suppose* (i):

$$0 < p \le 2,\ l > n/2 \text{ and } l' > n/p$$

or (ii):

$$2 < p < \infty,\ l > n/p \text{ and } l' > n/2.$$

Then $S^{m(p)}_{0,0}(l,l') \subset \mathcal{L}(H^p, L^p)$.

Lu and Yang [18] showed the atomic decomposition of Herz-type Hardy spaces, whose statement is similar to that of Hardy spaces.

Theorem 2.2 ([18])**.** *Let $1 < p < \infty,\ 0 < q < \infty,\ n(1-1/p) \le \alpha < \infty$, and $s \ge [\alpha - n(1-1/p)]$, s is a integer. Then $f \in H\dot{K}^\alpha_{p,q}$ if and only if there exist $a_j \in L^p$ and complex numbers λ_j such that $f = \sum_{j \in \mathbb{Z}} \lambda_j a_j$ in the sence $\mathscr{S}'$,* supp $a_j \subset B_j$, $\|a_j\|_{L^p} \le |B_j|^{-\alpha/n}$, $\int a_j(x) x^\beta dx = 0$, $(0 \le |\beta| \le s)$ *and $\{\lambda_j\}_{j \in \mathbb{Z}} \in l^q$. Moreover $\|f\|_{H\dot{K}^\alpha_{p,q}} \approx \inf (\sum_{j \in \mathbb{Z}} |\lambda_j|^q)^{1/q}$.*

Later, we will use theorems C and D to prove the main result in the next section.

3 Pseudo-differential operators on the Herz-type spaces

Here we state the main result.

Theorem 3.1. *Let* $1 < p < \infty,\ 0 < q < \infty,\ n(1-1/p) \le \alpha < \infty,\ m < -\alpha - n|1/p - 1/2|$. *Suppose* (i):

$$1 < p \le 2,\ l > n/2 \text{ and } l' > [-m] + n/2 + 1,$$

or (ii):

$$2 \le p < \infty,\ l > n/p \text{ and } l' > [-m - n(1/2 - 1/p)] + n/2 + 1.$$

Then $S^m_{0,0}(l,l') \subset \mathcal{L}(H\dot{K}^\alpha_{p,q}, \dot{K}^\alpha_{p,q})$.

Proof. The proof follows the idea of Miyachi [22]. We take $p \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$, $f = \sum_{j\in\mathbb{Z}} \lambda_j a_j \in H\dot{K}^\alpha_{p,q} \cap \mathscr{S}_0$. First we prove the case $0 < q \le 1$. Let s be an integer sufficiently large. Then we have

$$\|p(X,D)f\|^q_{\dot{K}^\alpha_{p,q}} \le \sum_{j\in\mathbb{Z}} |\lambda_j|^q \|p(X,D)a_j\|^q_{\dot{K}^\alpha_{p,q}}.$$

On the other hand, we have

$$\begin{aligned}\|p(X,D)a_j\|^q_{\dot{K}^\alpha_{p,q}} &= \sum_{k\in\mathbb{Z}} 2^{k\alpha q}\|p(X,D)a_j\chi_k\|^q_{L^p}\\ &= \sum_{k=-\infty}^{j+1} 2^{k\alpha q}\|p(X,D)a_j\chi_k\|^q_{L^p} + \sum_{k=j+2}^{\infty} 2^{k\alpha q}\|p(X,D)a_j\chi_k\|^q_{L^p}\\ &=: A_1 + A_2.\end{aligned}$$

By Theorem C,

$$\begin{aligned}A_1 &\le \sum_{k=-\infty}^{j+1} 2^{k\alpha q}\|p(X,D)a_j\|^q_{L^p}\\ &\lesssim \|p\|^q_{m;l,l'} \sum_{k=-\infty}^{j+1} 2^{(k-j)\alpha q}\\ &\lesssim \|p\|^q_{m;l,l'}.\end{aligned}$$

To estimate A_2 we decompose the symbol p by using the above partition of unity $\{\theta_t\}_{t=0}^\infty$ in ξ -space:

$$\begin{aligned}p(x,\xi) &= \sum_{t=0}^\infty p(x,\xi)\theta_t(\xi)\\ &= \sum_{t=0}^\infty p_t(x,\xi),\end{aligned}$$

where $p_t(x,\xi) := p(x,\xi)\theta_t(\xi)$. Also, let $K(x,y)$, $(K_t(x,y), \text{resp.})$ be the kernel of the pseudo-differential operator $p(X,D)$, $(p_t(X,D), \text{resp.})$: $K(x,y) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{iy\xi}p(x,\xi)d\xi$ $(K_t(x,y) := (2\pi)^{-n}\int_{\mathbb{R}^n} e^{iy\xi}p_t(x,\xi)d\xi, \text{resp.})$.

· In the case (i): Let $k \ge j+2$. We consider the case $j \le 0$. Let γ be a multi-index such that

$|\gamma| = [-m]+1$ if $k \geq 0$ and $|\gamma| = [-m]$ if $k < 0$. By using vanishing moments of order s we have

$$
\begin{aligned}
\|p_t(x,\xi)a_j\chi_k\|_{L^p} &= \Big(\int_{A_k} \Big|\int_{B_j} K_t(x,x-y)a_j(y)dy\Big|^p dx\Big)^{1/p} \\
&= \Big(\int_{A_k} \Big|\int_{B_j} \sum_{|\beta|=s+1} \frac{(-1)^{s+1}}{\beta!}\partial_2^\beta K_t(x,x-\theta y)y^\beta a_j(y)dy\Big|^p dx\Big)^{1/p} \\
&\lesssim 2^{j(s+1)} \sum_{|\beta|=s+1} \Big(\int_{A_k} \Big(\int_{B_j} |\partial_2^\beta K_t(x,x-\theta y)||a_j(y)|dy\Big)^p dx\Big)^{1/p}.
\end{aligned}
$$

We use the notation $\langle x\rangle = (1+|x|^2)^{1/2}$. By Minkowski's inequality and Hölder's inequality, we have

$$
\begin{aligned}
\|p_t(x,\xi)a_j\chi_k\|_{L^p} &\lesssim 2^{-j(\alpha-s-1)} \sum_{|\beta|=s+1} \Big(\int_{B_j} \Big(\int_{A_k} |\partial_2^\beta K_t(x,x-\theta y)|^p dx\Big)^{p'/p} dy\Big)^{1/p'} \\
&\lesssim 2^{-j(\alpha-s-1)}2^{kn(1/p-1/2)}2^{-k|\gamma|} \\
&\quad \times \sum_{|\beta|=s+1} \Big(\int_{B_j} \Big(\int_{A_k} \langle x\rangle^{2|\gamma|}|\partial_2^\beta K_t(x,x-\theta y)|^2 dx\Big)^{p'/2} dy\Big)^{1/p'}.
\end{aligned}
$$

To estimate the integral, we write

$$p_{\gamma'}(x,\xi) = \langle\xi\rangle^{-m}\partial_\xi^{\gamma'} p(x,\xi) \in S^0_{0,0}(l, l'-|\gamma'|),]$$

$$\psi_{y,\gamma',t}(\xi) = e^{-i\theta y\xi}\langle\xi\rangle^m \partial_\xi^{\gamma-\gamma'}\big((i\xi)^\beta\theta_t(\xi)\big)$$

and

$$g_{y,\gamma',t}(x) = \mathscr{F}^{-1}[\psi_{y,\gamma',t}](x).$$

Integration by parts gives

$$(-i(x-\theta y))^\gamma \partial_2^\beta K_t(x,x-\theta y) = \sum_{\gamma'\leq\gamma} \binom{\gamma}{\gamma'} p_{\gamma'}(X,D)g_{y,\gamma',t}(x).$$

By Plancherel's theorem, we get $\|g_{y,\gamma',t}\|_{L^2} \lesssim 2^{t(m+s+1+n/2)}$. Therefore an easy computation and Theorem C yield

$$
\begin{aligned}
&\Big(\int_{B_j} \Big(\int_{A_k} |(-i(x-\theta y))^\gamma|^2 |\partial_2^\beta K_t(x,x-\theta y)|^2 dx\Big)^{p'/2} dy\Big)^{1/p'} \\
&\qquad\qquad \lesssim \|p\|_{m;l,l'} 2^{t(m+s+1+n/2)}2^{jn(1-1/p)}.
\end{aligned}
$$

Now we remark that $k \geq j+2,\ x \in A_k$ and $y \in B_j$ implies $\langle x-\theta y\rangle \sim \langle x\rangle$. Hence we have

$$
\begin{aligned}
&\Big(\int_{B_j} \Big(\int_{A_k} \langle x\rangle^{2|\gamma|}|\partial_2^\beta K_t(x,x-\theta y)|^2 dx\Big)^{2/p'} dy\Big)^{1/p'} \\
&\qquad\qquad \lesssim \|p\|_{m;l,l'} 2^{t(m+s+1+n/2)}2^{jn(1-1/p)},
\end{aligned}
$$

and

$$\|p_t(x,\xi)a_j\chi_k\|_{L^p} \lesssim \|p\|_{m;l,l'} 2^{-j(\alpha-s-1-n(1-1/p)}2^{kn(1/p-1/2)}2^{-k|\gamma|}2^{t(m+s+1+n/2)}. \tag{1}$$

We also have the following estimate by repeating the above argument with $\beta = 0$;

$$\|p_t(x,\xi)a_j\chi_k\|_{L^p} \lesssim \|p\|_{m;l,l'} 2^{-j(\alpha-n(1-1/p))}2^{kn(1/p-1/2)}2^{-k|\gamma|}2^{t(m+n/2)}. \tag{2}$$

For each $j \le 0$, there exists a $t_j \in \mathbb{N}$ such that $2^{t_j-1}2^j \le 1 < 2^{t_j}2^j$. Using estimates (1),(2), we deduce the desired estimate in the following way:

$$
\begin{aligned}
\|p(X,D)a_j\chi_k\|_{L^p} &\le \sum_{t=0}^{t_j-1} \|p_t(X,D)a_j\chi_k\|_{L^p} + \sum_{t=t_j}^{\infty} \|p_t(X,D)a_j\chi_k\|_{L^p} \\
&\lesssim \|p\|_{m;l,l'} 2^{-j(\alpha-s-1-n(1-1/p))} 2^{kn(1/p-1/2)} 2^{-k|\gamma|} \sum_{t=0}^{t_j-1} 2^{t(m+s+1+n/2)} \\
&+ \|p\|_{m;l,l'} 2^{-j(\alpha-n(1-1/p))} 2^{kn(1/p-1/2)} 2^{-k|\gamma|} \sum_{t=t_j}^{\infty} 2^{t(m+n/2)} \\
&\lesssim \|p\|_{m;l,l'} 2^{-j(\alpha+m-n(1/2-1/p))} 2^{kn(1/p-1/2)} 2^{-k|\gamma|} \\
&\lesssim \|p\|_{m;l,l'} 2^{-j(\alpha+m-n(1/2-1/p))} 2^{kn(1/p-1/2)} 2^{-km}.
\end{aligned}
$$

Thus, $A_2 = \sum_{k=j+2}^{\infty} 2^{k\alpha q} \|p(X,D)a_j\chi_k\|_{L^p}^q \lesssim \|p\|_{m;l,l'}^q \sum_{k=j+2}^{\infty} 2^{q(k-j)(\alpha+m-n(1/2-1/p))} \lesssim \|p\|_{m;l,l'}^q$. As a result, we obtain $\|p(X,D)f\|_{\dot{K}_{p,q}^{\alpha}} \lesssim \|p\|_{m;l,l'} \|f\|_{H\dot{K}_{p,q}^{\alpha}}$.

Next we consider the case $j > 0$. In this case, we do not use the vanishing moment condition or docomposition of symbol. Let γ be a multi-index such that $|\gamma| > \alpha + n(1/p - 1/2) + 1$.

$$
\begin{aligned}
&\|p(X,D)a_j\chi_k\|_{L^p} \\
&\qquad \le 2^{-j\alpha} 2^{kn(1/p-1/2)} 2^{-k|\gamma|} \Big(\int_{B_j} \Big(\int_{A_k} \langle x \rangle^{2|\gamma|} \big| K(x, x-y) \big|^2 dx \Big)^{1/p'} dy \Big)^{1/p'}.
\end{aligned}
$$

Going throught a similar argument as above, we obtain,

$$
\Big(\int_{B_j} \Big(\int_{A_k} \big| (-i(x-y))^{\gamma} \big|^2 \big| K(x,x-y) \big|^2 dx \Big)^{p'/2} dy \Big)^{1/p'} \lesssim \|p\|_{m;l,l'} 2^{jn(1-1/p)}.
$$

Since $\langle x - y \rangle \sim \langle x \rangle$, we have

$$
\Big(\int_{B_j} \Big(\int_{A_k} \langle x \rangle^{2|\gamma|} \big| K(x,x-y) \big|^2 dx \Big)^{p'/2} dy \Big)^{1/p'} \lesssim \|p\|_{m;l,l'} 2^{jn(1-1/p)}.
$$

Hence we have

$$
\|p(X,D)a_j\chi_k\|_{L^p} \lesssim \|p\|_{m;l,l'} 2^{-j(\alpha-n(1-1/p))} 2^{kn(1/p-1/2)} 2^{-k|\gamma|}. \tag{3}
$$

Now we write $|\gamma| = \alpha + n(1/p - 1/2) + \varepsilon$. Then, for $j > 0$,

$$
\begin{aligned}
A_2 &= \sum_{k=j+2}^{\infty} 2^{k\alpha q} \|p(X,D)a_j\chi_k\|_{L^p}^q \\
&\lesssim \|p\|_{m;l,l'}^q \sum_{k=j+2}^{\infty} 2^{-jq(\alpha-n(1-1/p))} 2^{kq(\alpha+n(1/p-1/2)-|\gamma|)} \\
&= \|p\|_{m;l,l'}^q \sum_{k=j+2}^{\infty} 2^{-q(k-j)\varepsilon} 2^{-jq(\alpha-n(1-1/p)+\varepsilon)} \\
&\lesssim \|p\|_{m;l,l'}^q.
\end{aligned}
$$

We remark that $[-m] + n/2 + 1$ is larger than $\alpha + n/p$.

· In the case (ii): Let $k \geq j+2$. We consider the case $j<0$. Let γ be a multi-index such that $|\gamma| = [-m+n(1/p-1/2)]+1$ if $k \geq 0$ and $|\gamma| = [-m+n(1/p-1/2)]$ if $k<0$.

$$\begin{aligned}\|p(X,D)a_j\chi_k\|_{L^p} &= \Big(\int_{A_k} \Big|\int_{B_j} K_t(x,x-y)a_j(y)dy\Big|^p dx\Big)^{1/p} \\ &\lesssim 2^{-j(\alpha-s-1)} \sum_{|\beta|=s+1} \int_{B_j} \Big(\int_{A_k} \big|\partial_2^\beta K_t(x,x-\theta y)\big|^p dx\Big)^{p'/p} dy\Big)^{1/p'} \\ &\lesssim 2^{-j(\alpha-s-1)} 2^{-k|\gamma|} \sum_{|\beta|=s+1} \Big(\int_{B_j} \Big(\int_{A_k} \langle x\rangle^{q|\gamma|} \big|\partial_2^\beta K_t(x,x-\theta y)\big|^p dx\Big)^{p'/p} dy\Big)^{1/p'}.\end{aligned}$$

Since

$$(-i(x-\theta y))^\gamma \partial_2^\beta K_t(x,x-\theta y) = \sum_{\gamma' \leq \gamma} \binom{\gamma}{\gamma'} p_{\gamma'}(X,D) g_{y,\gamma',t}(x),$$

where

$$p_{\gamma'}(x,\xi) = \langle\xi\rangle^{-m-n(1/2-1/p)} \partial_\xi^{\gamma'} p(x,\xi) \in S_{0,0}^{-n(1/2-1/p)}(l,l'-\gamma') \subset \mathcal{L}(L^p),$$

$$\psi_{y,\gamma',t}(\xi) = e^{-i\theta y\xi} \langle\xi\rangle^{m+n(1/2-1/p)} \partial_\xi^{\gamma-\gamma'} \big((i\xi)^\beta \theta_t(\xi)\big)$$

and

$$g_{y,\gamma',t}(x) = \mathscr{F}^{-1}[\psi_{y,\gamma',t}](x),$$

and

$$\|g_{y,\gamma',t}\|_{L^p} \lesssim 2^{t(m+s+1+n(3/2-2/p))},$$

As a consequence, we have

$$\begin{aligned}&\Big(\int_{B_j} \Big(\int_{A_k} \langle x\rangle^{q|\gamma|} \big|\partial_2^\beta K_t(x,x-\theta y)\big|^p dx\Big)^{p'/p} dy\Big)^{1/p'} \\ &\qquad\qquad \lesssim \|p\|_{m;l,l'} 2^{jn(1-1/p)} 2^{t(m+s+1+n(3/2-2/p))}.\end{aligned}$$

Therefore we have

$$\|p_t(X,D)a_j\chi_k\|_{L^p} \lesssim \|p\|_{m;l,l'} 2^{-j(\alpha-s-1-n(1-1/p))} 2^{-k|\gamma|} 2^{t(m+n(3/2-2/p))}. \tag{4}$$

We also have the following estimate by repeating the above argument with $\beta=0$;

$$\|p_t(X,D)a_j\chi_k\|_{L^p} \lesssim \|p\|_{m;l,l'} 2^{-j(\alpha-n(1-1/p))} 2^{-k|\gamma|} 2^{t(m+n(3/2-2/p))}. \tag{5}$$

Since the above two estimates give $\|p(X,D)a_j\chi_k\|_{L^p} \lesssim \|p\|_{m;l,l'} 2^{-j(m+\alpha-n(1/p-1/2))} 2^{-k|\gamma|}$, we conclude $A_2 \lesssim \|p\|_{m;l,l'}^q$.

We consider the last case, $j>0$. Let $|\gamma|>\alpha$ then we set $|\gamma| = \alpha+\varepsilon$. It is easy to see

$$\begin{aligned}&\|p(X,D)a_j\chi_k\|_{L^p} \\ &\qquad \lesssim 2^{-j\alpha} 2^{-k|\gamma|} \Big(\int_{B_j} \Big(\int_{A_k} \langle x\rangle^{p|\gamma|} \big|K(x,x-y)\big|^p dx\Big)^{p'/pdy}\Big)^{1/p'}\end{aligned}$$

and

$$(-i(x-y))^\gamma K(x,x-y) = p_\gamma(X,D) g_y(x),$$

where

$$p_\gamma(x,\xi) = \langle\xi\rangle^{-m-n(1/2-1/p)} \partial_\xi^\gamma p(x,\xi) \in S_{0,0}^{-n(1/2-1/p)}(l,l'-|\gamma|),$$

$$\psi_y(\xi) = e^{-iy\xi} \langle\xi\rangle^{m+n(1/2-1/p)}$$

and

$$g_y(x) = \mathscr{F}^{-1}[\psi_y](x).$$

We have

$$\Big(\int_{B_j}\Big(\int_{A_k}\langle x\rangle^{p|\gamma|}\big|K(x,x-y)\big|^p dx\Big)^{p'/p}dy\Big)^{1/p'}\lesssim \|p\|_{m;l,l'}2^{jn(1-1/p)},$$

that is

$$\|p(X,D)a_j\chi_k\|_{L^p}\lesssim \|p\|_{m;l,l'}2^{-j(\alpha-n(1-1/p))}2^{-k|\gamma|}. \tag{6}$$

We obtain the following estimates of A_2 without difficulty, $A_2\lesssim \|p\|^q_{m;l,l'}$. As a result, when $0<q\le 1$, we get $\|p(X,D)f\|_{\dot K^\alpha_{p,q}}\lesssim \|p\|_{m;l,l'}\|f\|_{H\dot K^\alpha_{p,q}}$.

Finally we consider the case $1<q<\infty$. In this case, we use the following decomposition, and each term can be easily estimated by (1), (4) and (6).

$$\begin{aligned}\|p(X,D)f\|_{\dot K^\alpha_{p,q}}\lesssim&\Big(\sum_{k\in\mathbb{Z}}2^{k\alpha q}\Big(\sum_{j=k-1}^{\infty}|\lambda_j|\|p(X,D)a_j\chi_k\|_{L^p}\Big)^q\Big)^{1/q}\\&+\Big(\sum_{k=-\infty}^{2}2^{k\alpha q}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|\|p(X,D)a_j\chi_k\|_{L^p}\Big)^q\Big)^{1/q}\\&+\Big(\sum_{k=3}^{\infty}2^{k\alpha q}\Big(\sum_{j=-\infty}^{-1}|\lambda_j|\|p(X,D)a_j\chi_k\|_{L^p}\Big)^q\Big)^{1/q}\\&+\Big(\sum_{k=3}^{\infty}2^{k\alpha q}\Big(\sum_{j=0}^{k-2}|\lambda_j|\|p(X,D)a_j\chi_k\|_{L^p}\Big)^q\Big)^{1/q}.\end{aligned}$$

This completes the proof of theorem. □

We show some results in the non-homogeneous case. We remak that we can take m to be larger than that of Theorem 3.1 and $HK^\alpha_{p,q}\subsetneq K^\alpha_{p,q}$ if $1<p\le\infty,\ 0<q<\infty,\ -\infty<\alpha\le -n/p$ or $n(1-1/p)\le\alpha<\infty$ [13].

Theorem 3.2. *Let* $1<p\le\infty,\ 0<q<\infty$ *and* $n(1-1/p)\le\alpha<\infty$. *Suppose* (i):

$$1<q\le 2,\ m<-n/2,\ l>n/2 \text{ and } l'>\alpha+n/q$$

or (ii):

$$2<q<\infty,\ m<-n(3/2-2/q),\ l>n/q \text{ and } l'>\alpha+n/2.$$

Then $S^m_{0,0}(l,l')\subset\mathcal{L}(K^\alpha_{p,q})$.

Proof. Theorem 3.2 has been already proved in the course of the proof of Theorem 3.1. When we consider non-homogeneous case, we do not need estimates of the case $j<0$ in the proof of Theorem 3.1. We check the case (i) with $0<q\le 1$ only. We write

$$\begin{aligned}f(x)&=f(x)\chi_{B(0,1)}(x)+\sum_{j\in\mathbb{N}}f(x)\chi_j(x)\\&=\sum_{j\ge0}f_j(x)\\&=\sum_{j\ge0}\lambda_ja_j(x),\end{aligned}$$

where $f_0(x)=f(x)\chi_{B(0,1)}(x),\ f_j(x)=f(x)\chi_j(x),\ (j\ge1),\ \lambda_j=|B_j|^{\alpha/n}\|f_j\|_{L^p}$ and $a_j(x)=\dfrac{f_j(x)}{|B_j|^{\alpha/n}\|f_j\|_{L^p}}$. Hence,

$$\begin{aligned}\|p(X,D)f\|^q_{K^\alpha_{p,q}}&=\|p(X,D)f\chi_{B(0,1)}\|^q_{L^p}+2^{\alpha q}\|p(X,D)f\chi_1\|^q_{L^p}+\sum_{k=2}^{\infty}2^{k\alpha q}\|p(x,D)f\chi_k\|^q_{L^p}\\&:=A+B+C.\end{aligned}$$

The term A is easily estimated as: $A \leq \sum_{j=0}^{\infty} |\lambda_j|^q \|p(X,D)a_j\chi_{B(0,1)}\|_{L^p}^q \leq \sum_{j=0}^{\infty} |\lambda_j|^q 2^{-jq} \lesssim \|f\|_{K_{p,q}^{\alpha}}^q$. Similarly, $B \lesssim \|f\|_{K_{p,q}^{\alpha}}^q$.

$$\begin{aligned}
C &= \sum_{k=2}^{\infty} 2^{k\alpha q} \|p(X,D)f\chi_k\|_{L^p}^q \\
&\leq \sum_{k=2}^{\infty} 2^{k\alpha q} \sum_{j=0}^{\infty} |\lambda_j|^q \|p(X,D)a_j\chi_k\|_{L^p}^q \\
&\lesssim \sum_{k=2}^{\infty} 2^{k\alpha q} \sum_{j=0}^{k-2} |\lambda_j|^q \|p(X,D)a_j\chi_k\|_{L^p}^q + \sum_{k=2}^{\infty} 2^{k\alpha q} \sum_{j=k-1}^{\infty} |\lambda_j|^q \|p(X,D)a_j\chi_k\|_{L^p}^q \\
&:= C_1 + C_2.
\end{aligned}$$

By the L^p-boundedness of $p(X,D)$, $C_2 \lesssim \sum_{k=2}^{\infty} \sum_{j=k-1}^{\infty} |\lambda_j|^q 2^{q(k-j)\alpha} \lesssim \|f\|_{K_{p,q}^{\alpha}}^q$. The estimates (3) gives $C_1 \lesssim \|f\|_{K_{p,q}^{\alpha}}^q$. Therefore we have $\|p(X,D)f\|_{K_{p,q}^{\alpha}} \lesssim \|f\|_{K_{p,q}^{\alpha}}$. □

Remark 3.1. *If $n(1-1/p) \lneq \alpha$, then we can take $-\alpha - n|1/p - 1/2|$ as the order of the symbol in Theorem 3.2.*

By using the following Proposition 3.1, the symbol $S_{0,0}^m(l,l')$ can be written as $S_{0,0}^m(L)$ in the statements of Theorem 3.1 and Theorem 3.2.

Proposition 3.1. *For any $m \in \mathbb{R}$ and non-negative integers l, l', there exist non-negative integers P, Q such that $\|p\|_{m;l,l'} \lesssim |p|_{P+Q}^m$, $(p \in \mathscr{S})$.*

Proof. : We define $\langle D_y \rangle^2 = 1 + \sum_{i=1}^{n} D_{y_i}^2 = 1 - \triangle$. And let $N = [n+1-m/2]+1$, $M = N+1$, $P = (n+l)/2$ if $n+l$ is even, $N = [(n+l)/2]+1$ if $n+l$ is odd, $Q = n+l'$ if $n+l'$is even, $Q = [(n+l')/2]+1$ if $n+l'$ is odd. Then we have

$$\begin{aligned}
&\left|\mathscr{F}^{-1}\left[\theta_j(y)\theta_k(\eta)\hat{p}(y,\eta)\right](x,\xi)\right| = \left|\mathscr{F}^{-1}[\theta_j(y)\theta_k(\eta)] * p(x,\xi)\right| \\
&= \Big| \iint \Big(\frac{1}{(2\pi)^{2n}} \iint e^{i(yu+\eta v)} \langle D_y \rangle^{2M} \theta_j(y) \langle D_\eta \rangle^{2N} \theta_k(\eta) dy d\eta \Big) \\
&\times \langle u \rangle^{-2M} \langle v \rangle^{-2N} p(x-u, \xi - v) du dv \Big| \\
&\leq \iint \Big(\frac{1}{(2\pi)^{2n}} \iint \langle y \rangle^{-2P} \langle D_y \rangle^{2M} \theta_j(y) \langle \eta \rangle^{-2Q} \langle D_\eta \rangle^{2N} \theta_k(\eta) dy d\eta \Big) \\
&\times C_{P,Q,M,N} |p|_{P+Q}^m \langle \xi \rangle^m \langle v \rangle^{-m} \langle u \rangle^{-2M} \langle v \rangle^{-2N} du dv \\
&\leq C_{P,Q,M,N} |p|_{P+Q}^m 2^{-jl} 2^{-kl'} \langle \xi \rangle^m, \ (j,k \in \mathbb{Z}_+, \ x, \xi \in \mathbb{R}^n).
\end{aligned}$$

□

4 Interpolations

In this section, by using the interpolation theory for bilinear operators, we get rid of the condition of α : $n(1-1/p) \leq \alpha$ in Theorem 3.1. Furthermore, the duality argument allows us to take negative index α. First of all, we recall the defintions of interpolation for families of quasi-Banach spaces, [14]. Let $\triangle$ be the open unit disc in $\mathbb{C}$, and T the boundary of $\triangle$. We put a quasi-Banach space on for each $\theta \in T$: $(B(\theta), \|\cdot\|_{B(\theta)})$, and denote by $c(\theta)$ the constants in the quasi-triangle inequalities. We say that family $\{B(\theta)\}_{\theta\in T}$ is an interpolation family of quasi-Banach spaces if each $B(\theta)$ is cotinuously embedded in a Hausdorff topological vecter space $\mathcal{U}$, the function $\theta \to \|b\|_{B(\theta)}$ is measurable for each $b \in \bigcap_{\theta\in T} B(\theta)$, and $\log c(\theta) \in L^1(T)$; $\mathcal{U}$ is called the containing space of the given family $\{B(\theta)\}_{\theta\in T}$. We define

$$\beta = \Big\{ b \in \bigcap_{\theta\in T} B(\theta); \int_T \log^+ \|b\|_{B(\theta)} d\theta < \infty \Big\},$$

called the log-intersection space of the given family $\{B(\theta)\}_{\theta\in T}$. Let $\mathcal{G} = \mathcal{G}(\triangle, B(\cdot))$ be the space of all the β-valued analytic function of the form

$$g(z) = \sum_{j=1}^{m} \psi_j(z) b_j$$

for which $\|g\|_{\mathcal{G}} = \sup_{\theta\in T} \|g(\theta)\|_{B(\theta)} < \infty$, where $m \in \mathbb{N}$, $\psi_j \in N^+(\triangle)$, the positive Nevalinna class for $\triangle$, ([7]), and $b_j \in \beta$, $j = 1, ..., m$. For any $a \in \beta$ and $z \in \triangle$, we define

$$\|a\|_z = \inf \Big\{ \|g\|_{\mathcal{G}}; g \in \mathcal{G},\ g(z) = a \Big\}.$$

If N_z denotes the set of functions of β such that $\|a\|_z = 0$, the completion $B(z)$ of $(\beta/N_z, \|\cdot\|_z)$ will be called the interpolation space at z of the family $\{B(\theta)\}_{\theta\in T}$. We also denote $B(z)$ by $[B(\theta)]_z$.

Let $1 < p_0, p_1 < \infty$, $0 < q_0 < \infty$, $n(1 - 1/p_0) \le \alpha_0 < \infty$, $m_0 < -\alpha_0 - n|1/p_0 - 1/2|$, $m_1 = -n|1/p_1 - 1/2|$, and $0 < \theta < 1$. Then, we define $1/p(\theta) = (1-\theta)/p_0 + \theta/p_1$, $1/q(\theta) = (1-\theta)/q_0 + \theta/p_1$, $\alpha(\theta) = (1-\theta)\alpha_0$, and $m(\theta) = (1-\theta)m_0 + \theta m_1$. Let L be an integer sufficiently large. Following three equalities, which characterize the intermediate spaces obtained by the complex method of interpolation for the couples or families, are well known.

$$[S^{m_0}_{0,0}(L), S^{m_1}_{0,0}(L)]_\theta = S^{m(\theta)}_{0,0}(L) : \text{Páivárinta and Sommersaro, [24]},$$

$$[H\dot{K}^{\alpha_0}_{p_0,q_0}, H\dot{K}^{0}_{p_1,p_1}]_\theta = H\dot{K}^{\alpha(\theta)}_{p(\theta),q(\theta)} : \text{Hernández and Yang, [14]},$$

$$[\dot{K}^{\alpha_0}_{p_0,q_0}, \dot{K}^{0}_{p_1,p_1}]_\theta = \dot{K}^{\alpha(\theta)}_{p(\theta),q(\theta)} : \text{Hernández and Yang, [13]}.$$

Next we consider the following bilinear operator:

$$\begin{aligned} \mathscr{T} : & S^{m_0}_{0,0}(L) \times H\dot{K}^{\alpha_0}_{p_0,q_0} \to \dot{K}^{\alpha_0}_{p_0,q_0} \\ \text{or } & S^{m_1}_{0,0}(L) \times H\dot{K}^{0}_{p_1,p_1} \to \dot{K}^{0}_{p_1,p_1} ; (p,f) \longmapsto p(X,D)f. \end{aligned}$$

Theorem 4.1. *In the above situation, if L is sufficiently large, then*

$$\|\mathscr{T}(p,f)\|_{\dot{K}^{\alpha(\theta)}_{p(\theta),q(\theta)}} \lesssim |p|^{m(\theta)}_L \|f\|_{H\dot{K}^{\alpha(\theta)}_{p(\theta),q(\theta)}},$$

where $p \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$ *and* $f \in H\dot{K}^{\alpha(\theta)}_{p(\theta),q(\theta)}$.

Proof. : We follow the argument in Theorem 4.4.1 in [2]. For the sake of convenience, we write

$$A^0_\theta := S^{m(\theta)}_{0,0}(L),\ A^1_\theta := H\dot{K}^{\alpha(\theta)}_{p(\theta),q(\theta)},\ A^2_\theta := \dot{K}^{\alpha(\theta)}_{p(\theta),q(\theta)},$$

$$(B^0(\tau), B^1(\tau), B^2(\tau)) := \begin{cases} (S^{m_0}_{0,0}(L), H\dot{K}^{\alpha_0}_{p_0,q_0}, \dot{K}^{\alpha_0}_{p_0,q_0}), & \text{if } \tau \in T_0, \\ (S^{m_1}_{0,0}(L), H\dot{K}^{0}_{p_1,p_1}, \dot{K}^{0}_{p_1,p_1}), & \text{if } \tau \in T_1, \end{cases}$$

where T_0 and T_1 are subsets of T so that

$$\frac{1}{p(\theta)} = \int_{T_0} \frac{1}{p_0} P_\theta(\tau) d\tau + \int_{T_1} \frac{1}{p_1} P_\theta(\tau) d\tau,$$

$$\frac{1}{q(\theta)} = \int_{T_0} \frac{1}{q_0} P_\theta(\tau) d\tau + \int_{T_1} \frac{1}{p_1} P_\theta(\tau) d\tau,$$

$$\alpha(\theta) = \int_{T_0} \alpha_0 P_\theta(\tau) d\tau + \int_{T_1} \alpha_1 P_\theta(\tau) d\tau,$$

and

$$m(\theta) = \int_{T_0} m_0 P_\theta(\tau) d\tau + \int_{T_1} m_1 P_\theta(\tau) d\tau$$

where $P_\theta(\tau)$ is the Poisson kernel for evaluation at θ.

Let us be reminded that the space $B(\theta)$ defines β, $\mathcal{G}$ and N_z as was explained above. We use the notations β^k, $\mathcal{G}^k$ and N_z^k, if β, $\mathcal{G}$ and N_z is defined by $B(\theta) = A_\theta^k$, $(k = 0, 1, 2)$.

It is not hard to see that $N_\theta^k = \{0\}$ for each k. Let $\varepsilon > 0$, $a_0 \in \beta^0$ and $a_1 \in \beta^1$. Then there exist $f_0 \in \mathcal{G}^0$ and $f_1 \in \mathcal{G}^1$: $f_0 = \sum_{i=1}^{k_0} \varphi_i b_i$, $f_1 = \sum_{j=1}^{k_1} \psi_j c_j$ such that $f_0(\theta) = a_0$, $f_1(\theta) = a_1$, $\|f_0\|_{\mathcal{G}^0} \le \|a_0\|_\theta + \varepsilon$ and $\|f_1\|_{\mathcal{G}^1} \le \|a_1\|_\theta + \varepsilon$, where $\varphi_i, \psi_j \in N^+(\triangle)$ and $b_i \in \beta^0, c_j \in \beta^1$. Let C_0 and C_1 be constants in the inequalities $\|\mathscr{T}(p,f)\|_{\dot{K}^{\alpha_0}_{p_0,q_0}} \le C_0 |p|_L^{m_0} \|f\|_{H\dot{K}^{\alpha_0}_{p_0,q_0}}$, $\|\mathscr{T}(p,f)\|_{\dot{K}^{\alpha_1}_{p_1,p_1}} \le C_1 |p|_L^{m_1} \|f\|_{H\dot{K}^{\alpha_1}_{p_1,p_1}}$, and we set

$$g(z) = (C_0 + C_1)^{-1} \sum_{i=1}^{k_0} \sum_{j=1}^{k_1} \varphi_i(z) \psi_j(z) \mathscr{T}(b_i, c_j).$$

Then $g(\theta) = (C_0 + C_1)^{-1} \mathscr{T}(a_0, a_1)$, $g \in \mathcal{G}^2$ and

$$\begin{aligned}
\|g\|_{\mathcal{G}^2} &= \sup_{\tau \in T} \|g(\tau)\|_{B^2(\tau)} \\
&= \sup_{\tau \in T} \frac{1}{C_0 + C_1} \|\mathscr{T}(\sum_{i=1}^{k_0} \varphi_i(\tau) b_i, \sum_{j=1}^{k_1} \psi_j(\tau) c_j)\|_{B^2(\tau)} \\
&\le \sup_{\tau \in T} \|\sum_{i=1}^{k_0} \varphi_i(\tau) b_i\|_{B^0(\tau)} \sup_{\tau \in T} \|\sum_{j=1}^{k_1} \psi_j(\tau) c_j\|_{B^1(\tau)} \\
&= \|f_0\|_{\mathcal{G}^0} \|f_1\|_{\mathcal{G}^1}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\|\mathscr{T}(a_0, a_1)\|_\theta &\le (C_0 + C_1) \|g\|_{\mathcal{G}^2} \\
&\le (C_0 + C_1) \|f_0\|_{\mathcal{G}^0} \|f_1\|_{\mathcal{G}^1} \\
&\le (C_0 + C_1)(\|a_0\|_\theta + \varepsilon)(\|a_1\|_\theta + \varepsilon),
\end{aligned}$$

which implies the conclusion $\|\mathscr{T}(p,f)\|_{A_\theta^2} \lesssim |p|_L^{m(\theta)} \|f\|_{A_\theta^1}$. □

In particular, we consider the case $q_1 = 2$ in Theorem 4.1. By elementary calculation, we obtain

Corollary 4.1. *Let $1 < p < \infty$, $0 < q < \infty$, $0 < \alpha$, $1 - 1/p - \alpha/n < \min\big(1/p, 1/q, 1/2\big)$ and L be an integer sufficiently large. If $m < -\alpha - n|1/p - 1/2|$, then $S^m_{0,0}(L) \subset \mathcal{L}(H\dot{K}^\alpha_{p,q}, \dot{K}^\alpha_{p,q})$.*

Proof. The condition $1 - 1/p - \alpha/n < \min\big(1/p, 1/q, 1/2\big)$ guarantees that there exists $0 < \theta < 1$, $1 < p_0 < \infty$ and $0 < q_0 < \infty$ such that

$$1/p = (1-\theta)/p_0 + \theta/2, \ 1/q = (1-\theta)/q_0 + \theta/2$$

and

$$n(1 - 1/p_0) \leq \alpha/(1-\theta).$$

This and Theorem 4.1 complete the proof of Corollary 4.1. □

Remark 4.1. *If $0 < p, q \leq 2$ and $1 < p$, then the condition $1 - 1/p - \alpha/n < \min\big(1/p, 1/q, 1/2\big)$ is always satisfied. The range of α in Corollary 4.1 is wider than that of Theorem 4.1.*

Remark 4.2. *We remark that the conclusion of Corollary 4.1 holds if the index L is larger than at least, when $1 < p \leq 2$,*

$$[\frac{3n}{4}] + [\frac{3n}{4} + \frac{1}{2}[\frac{\alpha + n/p}{1 - min(1/q, 1/2)} - \frac{n^2}{2(\alpha + n/p)}] + 1] + 4$$

and when $2 < p < \infty$,

$$[\frac{3n}{4}] + [\frac{3n}{4} + \frac{1}{2}[\frac{\alpha}{1 - \min(1/p, 1/q)}] + 1] + 4.$$

The duality argument gives us the boundedness of $S^m_{0,0}(L)$ on the Herz spaces with $\alpha < 0$,

Corollary 4.2. *Let $1 < p, q < \infty$, $0 < \alpha < n(1 - 1/p)$, $1 - 1/p - \alpha/n < \min\big(1/p, 1/q, 1/2\big)$, and L be an integer sufficiently large. If $m < -\alpha - n|1/p - 1/2|$ then $S^m_{0,0}(L) \subset \mathcal{L}(\dot{K}^{-\alpha}_{p',q'})$, where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$.*

Remark 4.3. *Let p be in $\mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$ and p^* be the adjoint operator of p. Note that the inequality $|p^*|^m_L \leq |p|^m_{L+2([(n+1)/2]+1)}$ holds. Hence, the index L in Corollary 4.2 must be larger than $L_0 + 2([(n+1)/2] + 1)$ where L_0 is the minimal requirement for L in Corollary 4.1.*

Remark 4.4. *The author believes that the complex interpolation theorem for Herz-type Hardy spaces with $p \leq 1$ holds. We will be able to obtain the boundedness of pseudo-differential operators of class $S^m_{0,0}$ on the Herz-type Hardy spaces with $q \leq 1$, if the interpolation theorem holds.*

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