Div - curl estimates with critical power weights

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1 Introduction

This note is based on [11]. But, the proof of the main result in [11] is not correct, though the result is true. Aim of this note is to give a proof of it, along the talk. It applies Bogovskiĭ formula instead of the Green function for the Neumann problem of Poisson equation, which was used in [11]. I learned this formula from Professor Hideo Kozono, and would like to thank him for teaching me it. It is should be emphasize that thanks to the formula, the result on 3-dimension in [11] is generalized to all dimension.

Div - curl estimate is a inequality of the form, which was firstly studied by Coifman-Linons-Meyer-Semmes [2]: for $p, q \in (n/(n+1), \infty)$ and $1/r = 1/p + 1/q < 1 + 1/n$,

$$\|(u \cdot \nabla)v\|_{H^r} \lesssim \|u\|_{H^q} \|\nabla v\|_{H^p}, \tag{1}$$

where u and v are vector valued functions; $u = \{u_j\}_{j=1}^n, v = \{v_j\}_{j=1}^n$, and div $u = \nabla \cdot u = 1$. Here

$$(u \cdot \nabla)v = \left(\sum_{j=1}^n u_j \partial_j v_1, \cdots, \sum_{j=1}^n u_j \partial_j v_n \right).$$

H^p, $(p \in (0, \infty))$ is the Hardy spaces: for any $\phi \in C_0^\infty(\mathbb{R}^n)$ with supp $\phi \subset\subset B(0,1)$ and $\displaystyle\int \varphi dx = 0$,

$$\|f\|_{H^p} := \|M_\phi[f]\|_{L^p}, \text{ where } M_\phi[f](x) := \sup_{t>0} |f * \phi_t(x)|,$$

where $g_t(x) := t^{-n} g(x/t)$. In the case $p = \infty$, define $H^\infty := L^\infty$.

As we mentioned in [11], we make use of a real interpolation spaces between weighted Hardy spaces.

Definition 1.1. *Let $p, q \in (0, \infty]$ and $\alpha \in \mathbb{R}$. Define Hardy spaces associated with Herz spaces $H_\alpha^{p,q}(\mathbb{R}^n)$ as*

$$H_\alpha^{p,q}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'; \|f\|_{H_\alpha^{p,q}} := \|M_\phi f\|_{L_\alpha^{p,q}} < \infty \right\},$$

where

$$\|f\|_{L_\alpha^{p,q}} := \left\| \{2^{\alpha k} \|f\|_{L^p(A_k)}\}_{k \in \mathbb{Z}} \right\|_{l^q}.$$

We explain notations. $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ denote the Schwartz spaces of rapidly decreasing smooth functions and tempered distributions on $\mathbb{R}^n$, respectively. For a measurable subset $E \subset \mathbb{R}^n$, $|E|$ and χ_E are the volume and the characteristic function of E, respectively. For any integers j, A_j denotes a annulus $\{x \in \mathbb{R}^n; 2^{j-1} \le |x| < 2^j\}$, and χ_j is the characteristic function of A_j. $B(x, r)$ is a ball in $\mathbb{R}^n$, centered at x of radius r. $\langle g\rangle_B := |B|^{-1}\int_B f dy$. Also, $A \lesssim B$ means $A \le cB$ with positive constant c, and $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

To give $(u\cdot\nabla)v$ a definition as a tempered distribution, we define Y by a space of all locally integrable functions f satisfying that there exist $c_f > 0$ and a seminorm $|\cdot|_{\mathcal{S}}$ of $\mathcal{S}$ so that $\displaystyle\int |f(x)\varphi(x)|dx \le c_f|\varphi|_{\mathcal{S}}$, for all $\varphi \in \mathcal{S}$.

The main result reads as follows.

Theorem 1.1. *For $n/(n+1) < p < \infty$, it holds*

$$\|(u\cdot\nabla)v\|_{H^{p,\infty}_{\alpha(p)}} \le c\|u\|_{L^\infty}\|\nabla v\|_{H^{p,n/(n+1)}_{\alpha(p)}},$$

for $u \in L^\infty(\mathbb{R}^n)^n$ with div $u = 0$ and $v \in \left(Y \cap W^{1,r}_{loc}(\mathbb{R}^n)\right)^n$ for some $r \in (1,\infty)$, where $\alpha(p) := n(1-1/p)+1$.

Remark 1.1. *The same argument as the proof of Theorem 1.1, we can also show a weak type estimate:*

$$\|(u\cdot\nabla)v\|_{H^{(n/(n+1),\infty)}} \lesssim \|u\|_{L^\infty}\|\nabla v\|_{H^{n/(n+1)}}, \tag{2}$$

because $\alpha(n/(n+1)) = 0$. Here, $f \in H^{(p,q)} \iff M_\phi[f] \in L^{(p,q)}$, where $L^{(p,q)}$ is the Lorentz spaces. Similar estimates were established by Miyakawa [8]. This can be regarded as an endpoint case with $p = n/(n+1)$ and $q = \infty$ of [2]. The ingredient of the proof of (2) is the pointwise estimate:

$$\|N[v]\|_{L^{(n/(n+1),\infty)}} \lesssim \|\nabla v\|_{H^{n/(n+1)}}$$

instead of (4). This is achieved from the pointwise estimate (7) and a Fefferman-Stein's vector valued inequality (2) of Theorem 1 in [3].

Motivation of this research comes from the optimal L^2-energy decay for the incompressible Navier-Stokes equations. Wiegner [12] constructed global weak solutions u having

$$\|u(t)\|_{L^2} \lesssim t^{-(n+2)/4},$$

assuming that initial data $a \in L^2$ satisfying $\|e^{t\Delta}a\|_{L^2} \lesssim t^{-(n+2)/4}$. By Miyakawa-Schonbek [9], it is well-known that the decay order $(n+2)/4$ is optimal. $H^p_{\alpha(p)}$ is relevant to this order $(n+2)/4$, because one has that for $p \in (0,2]$,

$$\|e^{t\Delta}a\|_{L^2} \lesssim t^{-(n+2)/4}\|a\|_{H^p_{\alpha(p)}},$$

see [10] for the proof. The present author [10] investigated the L^2 decay of mild solutions by Kato [5] and constructed solutions whose decay order of L^2 energy is $\gamma < (n+2)/4$. One of reasons why the order γ in [10] did not reach to the optimal order $(n+2)/4$ is that div-curl estimate in [10] cannot allow us to deal with the critical exponent $\alpha = \alpha(p)$. As mentioned in Remark 7.3 in [7], the bilinear term $(u \cdot \nabla)v$ does not belong to $H^p_{\alpha(p)}$. This observation tells us that if we try to establish div-curl estimate with $\alpha = \alpha(p)$, we has to replace $H^p_{\alpha(p)}$ in the left hand side by some larger spaces. For the purpose, we use Hardy spaces associated to Herz spaces, as in [6] and [7]. Although, a critical div - curl lemma is proved in this article, the author does not know whether or not it is possible to construct global solutions having optimal L^2 decay from the similar argument as the previous paper [10].

2 Proof

Before we start the proof of Theorem 1.1, we point out the mistake in [**?**]. In [**?**], the Green function for the Neumann problem of Poisson equation:

$$-\Delta h = g \text{ in } B, \quad g = 0 \text{ on } \partial B$$

for $g \in C_0^\infty(B)$ with $\displaystyle\int gdx = 0$, is applied. I treated the solution h as a $C_0^2(B)$ function in [11]. Although $h \in C^2(\mathbb{R}^n)$, this is not true in general. To overcome this difficulty, as we mentioned above, we apply Bogovskiǐ formula. This is a representation, with a kernel function, of solutions to the divergence equation.

2.1 Bogovskiǐ formula

Let B be a ball in $\mathbb{R}^n$ and $g \in C_0^\infty(B)$ with $\displaystyle\int gdx = 0$. We refer Lemma III.3.1 in [4] for next lemma.

Lemma 2.1. *There exists a vector function* $\mathbf{K} = \{K_j\}_{j=1}^n$ *on* $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,y) : x = y\}$ *so that* $\mathbf{G}_B(x) := \displaystyle\int \mathbf{K}(x,y)g(y)dy \in C_0^\infty(B)^n$ *is a solution to the divergence equation* $\nabla \cdot \mathbf{G}_B = g$ *on* B *satisfying that for* $q \in (1,\infty)$

$$\|\mathbf{G}_B\|_{L^q} \lesssim |B|^{1/n}\|g\|_{L^q} \quad \text{and} \quad \|\nabla \cdot \mathbf{G}_B\|_{BMO} \lesssim \|g\|_{L^\infty}.$$

Remark 2.1. *The* $L^\infty - BMO$ *estimate above is deduced from the fact that the operator* $g \mapsto \mathbf{G}_B$ *is a Calderón-Zygmund operator.*

2.2 Vector valued restricted weak type inequality

Another ingredient for the proof of Theorem 1.1 is a vector-valued "restricted weak" type inequality for Hardy-Littlewood maximal operator; for $r \in (0, \infty)$

$$M_r f(x) := \sup_{B \ni x} \langle |f|^r \rangle_B^{1/r}, \ ,$$

where the supremum is taken over all ball B containing x. Define $Mf(x) := M_1 f(x)$. The following is a generalization of the result of Fefferman-Stein [3], and the proof is found in [11].

Proposition 2.1. *For $1 < r, p < \infty$ and $\alpha = n(1 - 1/p)$,*

$$\left\| \left(\sum_{l=1}^{\infty} (M f_l)^r \right)^{1/r} \right\|_{L_\alpha^{p,\infty}} \lesssim \left\| \left(\sum_{l=1}^{\infty} |f_l|^r \right)^{1/r} \right\|_{L_\alpha^{p,1}} .$$

This can be rewritten as the following form.

Corollary 2.1. *For $0 < r < 1 < p < \infty$ and $\alpha = n(1/r - 1/p)$,*

$$\left\| \sum_{l=1}^{\infty} M_r f_l \right\|_{L_\alpha^{p,\infty}} \lesssim \left\| \sum_{l=1}^{\infty} |f_l| \right\|_{L_\alpha^{p,r}} .$$

2.3 Complete of the proof of Theorem 1.1

The proof is almost same as that in [11] except for applying Lemma 2.1 instead of the Green function for the Neumann problem of Poisson equations.

Because

$$\|(u \cdot \nabla) v\|_{H_{\alpha(p)}^{p,\infty}} = \sum_{k=1}^{n} \left\| \sum_{j=1}^{n} u_j \partial_j v_k \right\|_{H_{\alpha(p)}^{p,\infty}} = \sum_{k=1}^{n} \left\| M_\phi \left(\sum_{j=1}^{n} u_j \partial_j v_k \right) \right\|_{L_{\alpha(p)}^{p,\infty}} ,$$

it is enough to show the inequality

$$\left\| M_\phi \left(\sum_{j=1}^{n} u_j \partial_j v \right) \right\|_{L_{\alpha(p)}^{p,\infty}} \lesssim \|u\|_{L^\infty} \|\nabla v\|_{H_{\alpha(p)}^{p,n/(n+1)}},$$

for all divergence free vector fields u and functions $v \in Y \cap W_{loc}^{1,r}$. Firstly, we give a definition of $\sum_{j=1}^{n} u_j \partial_j v$ as a tempered distribution as follows; for $\varphi \in \mathcal{S}$

$$\left\langle \sum_{j=1}^{n} u_j \partial_j v, \varphi \right\rangle := - \sum_{j=1}^{n} \int u_j(y) v(y) \partial_j \varphi(y) dy.$$

Our assumption ensures that the integral in the right hand side absolutely converges. Then, it follows

$$\sum_{j=1}^{n} u_j \partial_j v * \phi_t(x) = -C_\phi \|u\|_{L^\infty} \int v(y) \left[\sum_{j=1}^{n} \tilde{u}_j(y) \partial_{y_j} \phi_t(x-y) \right] dy,$$

where C_ϕ is a constant depending on ϕ, and $\tilde{u}_j(y) = \dfrac{u_j(y)}{C_\phi \|u\|_{L^\infty}}$. Owing to the divergence free condition on u, we see that for every $x \in \mathbb{R}^n$

$$\sum_{j=1}^{n} \tilde{u}_j(y) \partial_{y_j} \phi_t(x-y) = \sum_{j=1}^{n} \partial_{y_j} \left(\tilde{u}_j(y) \phi_t(x-y) \right) \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n_y). \tag{3}$$

Hence, we obtain the pointwise estimate

$$M_\phi \left(\sum_{j=1}^{n} u_j \partial_j v \right)(x) \le C_\phi \|u\|_{L^\infty} N[v](x),$$

where

$$N[v](x) := \sup_{t>0} \left| \int v(y) g(y) dy \right| \text{ and } g(y) = g(y;x,t) := \sum_{j=1}^{n} \tilde{u}_j(y) \partial_{y_j} \phi_t(x-y).$$

It is enough to prove that

$$\|N[v]\|_{L^{p,\infty}_{\alpha(p)}} \lesssim \|\nabla v\|_{H^{p,n/(n+1)}_{\alpha(p)}}. \tag{4}$$

To show this from a pointwise estimate, we make use of Bogovskiǐ formula Lemma 2.1 and the atomic decomposition in $H^{p,n/(n+1)}_{\alpha(p)}$ due to Miyachi [7].

Fix $x \in \mathbb{R}^n$ and $t \in (0,\infty)$. There exists $\varepsilon_0 > 0$ so that for all $\varepsilon \in (0,\varepsilon_0)$, supp $g_* \phi_\varepsilon \subset\subset B(x,t)$. Take $\eta_0 \in C_0^\infty(\mathbb{R}^n)$ such that $\eta(y) = \eta(y;x,t) := \eta_0 \left(\dfrac{y-x}{t} \right)$ satisfies that $0 \le \eta \le 1$ and for $\varepsilon \in (0,\varepsilon_0)$

$$\text{supp } g * \phi_\varepsilon \subset\subset \text{supp } \eta \subset\subset B(x,t), \text{ and } \eta \equiv 1 \text{ on supp } g * \phi_\varepsilon$$

Remark that $\|\eta\|_{L^p} = ct^{n/p}$ for all $p \in [1,\infty]$ with c independent of x, t and ε. Next we see that

$$\sigma_\varepsilon := \|\eta\|_{L^1}^{-1} \int g * \phi_\varepsilon dy \to 0 \text{ as } \varepsilon \searrow 0.$$

In fact, for a test function $\rho \in C_0^\infty(B(x,2t))$ with $\rho \equiv 1$ on $B(x,t)$, we have from (3)

$$\|\eta\|_{L^1} \sigma_\varepsilon = \langle g, \rho * \phi_\varepsilon \rangle \to \langle g, \rho \rangle = 0.$$

For $\varepsilon \in (0, \varepsilon_0)$, letting $g^\varepsilon := g * \phi_\varepsilon - \sigma_\varepsilon \eta \in C_0^\infty(B(x,t))$, we obtain $\displaystyle\int g^\varepsilon dy = 0$. From Lemma 2.1,

$$\mathbf{G}^\varepsilon(y) := \int \mathbf{K}(y,z) g^\varepsilon(z) dz \in C_0^\infty(B(x,t))$$

solves the Drichlet problem of Poisson equation:

$$\nabla \cdot \mathbf{G}^\varepsilon = g^\varepsilon \text{ in } B(x,t), \quad \mathbf{G}^\varepsilon = 0 \text{ on } \partial B(x,t).$$

Further, $\mathbf{G}^\varepsilon$ fulfills the following estimates: for all $q \in (1, \infty)$,

$$\|\mathbf{G}^\varepsilon\|_{L^q} \lesssim t^{-n+n/q} \quad \text{and} \quad \|\partial_j \mathbf{G}^\varepsilon\|_{BMO} \lesssim t^{-(n+1)}. \tag{5}$$

Indeed, these follows that $\|g^\varepsilon\|_{L^q} \lesssim t^{-1-n+n/q}$ and $\|g^\varepsilon\|_{L^\infty} \lesssim t^{-(n+1)}$, respectively.

Integration by parts yields

$$\int vg dy = -\lim_{\varepsilon \to 0} \int \nabla v \cdot \mathbf{G}^\varepsilon dy.$$

Therefore, one obtains

$$\left| \int vg dy \right| \le \sum_{k=1}^n \limsup_{0<\varepsilon<\varepsilon_0} \left| \int \partial_k v \mathbf{G}_k^\varepsilon dy \right|.$$

Since $\partial_k v \in H_{\alpha(p)}^{p,n/(n+1)}$, following Miyachi [7], it can be decomposed as

$$\partial_k v = \sum_{j=1}^\infty a_j^{(k)}$$

where supp $a_j^{(k)} \subset B_j = B(x_j, r_j)$, $a_j^{(k)} \in L^\infty$ and $\displaystyle\int x^\beta a_j^{(k)}(x) dx = 0$ for $|\beta| \le 1$, also

$$\left\| \sum_{j=1}^\infty \|a_j^{(k)}\|_{L^\infty} \chi_{B_j} \right\|_{L_{\alpha(p)}^{p,n/(n+1)}} \lesssim \|\partial_k v\|_{H_{\alpha(p)}^{p,n/(n+1)}}.$$

and one obtains

$$\left| \int vg dy \right| \le \sum_{k=1}^n \sum_{j=1}^\infty \limsup_{0<\varepsilon<\varepsilon_0} \left| \int a_j^{(k)} \mathbf{G}_k^\varepsilon dy \right|.$$

From (5), we immediately see that

$$\left| \int a_j^{(k)} \mathbf{G}_k^\varepsilon dy \right| \le \|a_j^{(k)}\|_{L^\infty} |B(x,t)|^{1-1/q} \|\mathbf{G}_k^\varepsilon\|_{L^q} \lesssim \|a_j^{(k)}\|_{L^\infty}.$$

When $x \notin 4B_j$, if $Ct < |x - x_j|$ with $C > 8/3$, then it holds $B_j \cap B(x,t) = \emptyset$ and $\int a_j^{(k)} \mathbf{G}_k^\varepsilon dy = 0$. On the other hand, if $Ct \geq |x - x_j|$, then we can derive the decay estimate

$$\limsup_{0<\varepsilon<\varepsilon_0} \left| \int a_j^{(k)} \mathbf{G}_k^\varepsilon dy \right| \lesssim \|a_j^{(k)}\|_{L^\infty} \left(\frac{r_j}{|x - x_j|} \right)^{n+1}. \tag{6}$$

We may assume $x \neq x_j$. Using the moment condition on $a_j^{(k)}$ twice, one has

$$\begin{aligned}
\int a_j^{(k)}(y) \mathbf{G}_k^\varepsilon(y) dy &= \int a_j^{(k)}(y) \left(\mathbf{G}_k^\varepsilon(y) - \mathbf{G}_k^\varepsilon(x_j) \right) dy \\
&= \sum_{s=1}^n \int_0^1 \int a_j^{(k)}(y)(y - x_j)_s (\partial_s \mathbf{G}_k^\varepsilon)(\theta y + (1-\theta)x_j) dy d\theta \\
&= \sum_{s=1}^n \int_0^1 \int a_j^{(k)}(y)(y - x_j)_s \left[(\partial_s \mathbf{G}_k^\varepsilon)(\theta y + (1-\theta)x_j) - \langle \partial_s \mathbf{G}_k^\varepsilon \rangle_{B(x_j, \theta r_j)} \right] dy d\theta.
\end{aligned}$$

From this representation, the decay estimate (6) is derived as follows from (5);

$$\begin{aligned}
\left| \int a_j^{(k)}(y) \mathbf{G}_k^\varepsilon(y) dy \right| &\lesssim r_j \|a_j^{(k)}\|_{L^\infty} \sum_{s=1}^n \int_0^1 \theta^{-n} \int_{B(x_j, \theta r_j)} \left| \partial_s \mathbf{G}_k^\varepsilon(y) - \langle \partial_s \mathbf{G}_k^\varepsilon \rangle_{B(x_j, \theta r_j)} \right| dy d\theta \\
&\lesssim r_j^{n+1} \|a_j^{(k)}\|_{L^\infty} \sum_{s=1}^n \|\partial_s \mathbf{G}_k^\varepsilon\|_{BMO} \\
&\lesssim \left(\frac{r_j}{t} \right)^{n+1} \|a_j^{(k)}\|_{L^\infty} \\
&\lesssim \left(\frac{r_j}{|x - x_j|} \right)^{n+1} \|a_j^{(k)}\|_{L^\infty}.
\end{aligned}$$

As mentioned in [7], because $\left(\frac{1}{1 + |x - x_j|/r_j} \right)^{n+1} \approx M_{n/(n+1)}(\chi_{B_j})(x)$, as a consequence it follows that for all $x \in \mathbb{R}^n$,

$$N[v](x) = \sup_{t>0} \left| \int v(y) g(y; x, t) dy \right| \lesssim \sum_{k=1}^n \sum_{j=1}^\infty \|a_j^{(k)}\|_{L^\infty} M_{n/(n+1)}(\chi_{B_j})(x). \tag{7}$$

Now, we apply Corollary 2.1 with $r = n/(n+1)$ and obtain

$$\|N[v]\|_{L^{p,\infty}_{\alpha(p)}} \lesssim \sum_{k=1}^n \left\| \sum_{j=1}^\infty \|a_j^{(k)}\|_{L^\infty} \chi_{B_j} \right\|_{L^{p,n/(n+1)}_{\alpha(p)}} \lesssim \sum_{k=1}^n \|\partial_k v\|_{H^{p,n/(n+1)}_{\alpha(p)}} = \|\nabla v\|_{H^{p,n/(n+1)}_{\alpha(p)}}.$$

Here we have used $n(1 - 1/p) + n((n+1)/n - 1) = n(1 - 1/p) + 1 = \alpha(p)$. The proof is completed.

Remark 2.2. *In [1], the pointwise estimate (6) with $n + 1 - \varepsilon$ replaced by $n + 1$ was proved.*

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