NUMBER OF RIGHT-ANGLE TRIANGLES (PAPER BY PACH–SHARIR)

Abstract.

1. Set up and notations

(1) $\mathcal{P} = \{p\}$ denotes a set of finite points in $\mathbb{R}^2$. We often write $N := \#\mathcal{P}$. Also, $\mathcal{L} = \{\ell\}$ denotes a set of finite lines in $\mathbb{R}^2$.
(2) Given $p_1, p_2, p_3 \in \mathbb{R}^2$, we denote a triangle spanned by these points by $\triangle(p_1, p_2, p_3)$. Note that $\triangle(p_1, p_2, p_3)$ may be degenerate.
(3) We are especially interested in a right-angled triangle and

$$\mathcal{T}(\mathcal{P}) := \{(p_1, p_2, p_3) \in (\mathcal{P})^3 : \triangle(p_1, p_2, p_3) \text{ is the right-angle triangle}\}$$

The main result of Pach–Sharir is as follows.

Theorem 1.1 (Pach–Sharir '92)**.** *For any* $\mathcal{P} \subset \mathbb{R}^2$,

e:PachSharir
$$\#\mathcal{T}(\mathcal{P}) \leq C(\#\mathcal{P})^2 \log \#\mathcal{P}. \tag{1.1}$$

We next aim to exhibit the actual estimate, proved by Pach–Sharir, that yields (1.1). The reason of doing this is because it may be interpreted as a certain (discrete) X-ray estimate. For this purpose, we need to introduce the discrete X-ray transform. Suppose we are given a finite points $\mathcal{P}$.

(1) For a line $\ell \in \mathbb{R}^2$, define

$$X[\mathcal{P}](\ell) := \#(\mathcal{P} \cap \ell).$$

(2) Let $\Theta = \Theta(\mathcal{P})$ be a set of directions that spanned by two points of $\mathcal{P}$: by denoting $\theta_{p,p'} := \frac{p-p'}{|p-p'|}$,

$$\Theta(\mathcal{P}) := \{\theta_{p,p'} : p \neq p' \in \mathcal{P}\}.$$

Note that

e:NumberDirections
$$\#\Theta(\mathcal{P}) \leq \binom{\#\mathcal{P}}{2} = \frac{1}{2}\#\mathcal{P}(\#\mathcal{P} - 1) \leq (\#\mathcal{P})^2. \tag{1.2}$$

(3) For $\theta \in \mathbb{S}^1$ and $p \in \mathbb{R}^2$, we set

$$\ell^\theta(p) := \{t\theta + p : t \in \mathbb{R}\} = \text{a line in direction } \theta \text{ and passing through } p.$$

We will consider a set of parallel lines in a fixed direction $\theta \in \Theta(\mathcal{P})$ whose centre runs over $\mathcal{P}$: for each $\theta \in \Theta(\mathcal{P})$,

$$\mathcal{L}^\theta = \mathcal{L}^\theta(\mathcal{P}) := \{\ell^\theta(p) : p \in \mathcal{P}\}.$$

Note that $\ell^\theta(p) = \ell^\theta(p')$ may happen even if $p \neq p'$. Thus,

$$I^\theta = I^\theta(\mathcal{P}) := \#\mathcal{L}^\theta(\mathcal{P}) \leq \#\mathcal{P}.$$

We will often label $\mathcal{L}^\theta(\mathcal{P})$ by

$$\mathcal{L}^\theta(\mathcal{P}) = \{\ell_1^\theta, \dots, \ell_{I^\theta}^\theta\} = \{\ell_i^\theta : i = 1, \dots, I^\theta\}.$$

Similarly, we will also consider a set of vertical lines:

$$\mathcal{L}^{\theta^\perp} = \mathcal{L}^{\theta^\perp}(\mathcal{P}) := \{\ell^{\theta^\perp}(p) : p \in \mathcal{P}\},$$

and label this set by

$$\mathcal{L}^{\theta^\perp}(\mathcal{P}) = \{\ell_1^{\theta^\perp}, \dots, \ell_{J^{\theta^\perp}}^{\theta^\perp}\} = \{\ell_j^{\theta^\perp} : j = 1, \dots, J^{\theta^\perp}\}, \quad J^{\theta^\perp} := \#\mathcal{L}^{\theta^\perp}(\mathcal{P}).$$

(4) Finally, for each $\ell_i^\theta \in \mathcal{L}^\theta(\mathcal{P})$ and $\ell_j^{\theta^\perp} \in \mathcal{L}^{\theta^\perp}(\mathcal{P})$, we denote

$$p_{ij}^\theta := \ell_i^\theta \cap \ell_j^{\theta^\perp},$$

which is a (unique) crossing point of two lines ℓ_i^θ and $\ell_j^{\theta^\perp}$.

With these notations, the main estimate of Pach–Sharir may be stated as follows.

Theorem 1.2 (Pach–Sharir '92). *For any* $\mathcal{P} \subset \mathbb{R}^2$,

e:PachSharir-Xray

$$\sum_{\theta \in \Theta(\mathcal{P})} \sum_{i=1}^{I^\theta} \sum_{j=1}^{J^{\theta^\perp}} X[\mathcal{P}](\ell_i^\theta) X[\mathcal{P}](\ell_j^{\theta^\perp}) \mathbf{1}_{\mathcal{P}}(p_{ij}^\theta) \leq C(\#\mathcal{P})^2 \log(\#\mathcal{P}). \tag{1.3}$$

A continuous analogue to (1.3)

Let us try to catch a sense of (1.3). Use our familiar notation $Xf(\theta, v) := \int_{\mathbb{R}} f(t\theta + v)\, dt$. Then the continuous analogue to LHS of (1.3) is as follows:

$$\int_{\mathbb{S}^1} \int_{v_1 \in \langle\theta\rangle^\perp} \int_{v_2 \in \langle\theta\rangle} Xf(\theta, v_1) Xf(\theta^\perp, v_2) K(v_1, v_2; \theta)\, d\lambda_{\langle\theta\rangle^\perp}(v_1) d\lambda_{\langle\theta\rangle}(v_2) d\sigma(\theta),$$

where $K(v_1, v_2; \theta)$ is some integral kernel[a].

[a]maybe something like

$$K(v_1, v_2; \theta) = \mathbf{1}_{\operatorname{supp} f}(\ell^\theta(v_1) \cap \ell^{\theta^\perp}(v_2))???$$

We will see how (1.3) implies their main result (1.1) later.

2. Proof of Theorem of Pach–Sharir

Let us give a proof of (1.1). We take arbitrary $\mathcal{P} \subset \mathbb{R}^2$ and fix it below. We thus sometimes abbreviate the dependence of $\mathcal{P}$.

2.1. **Implication of** (1.3) $\Rightarrow$ (1.1).

In this subsection, we give an interpretation of the problem about the number of right-angle triangles in terms of the X-ray transform. A goal here is to show the following representation:

Claim 2.1. By using above notations,

e:NumberRightangle-Xray

$$\sharp\mathcal{T}(\mathcal{P}) = \sum_{\theta\in\Theta(\mathcal{P})} \sum_{i=1}^{I^\theta} \sum_{j=1}^{J^{\theta^\perp}} \big(X[\mathcal{P}](\ell_i^\theta) - 1\big)\big(X[\mathcal{P}](\ell_j^{\theta^\perp}) - 1\big)\mathbf{1}_{\mathcal{P}}(\ell_i^\theta \cap \ell_j^{\theta^\perp}). \tag{2.1}$$

Once one could see this claim, then it in particular follows that

berRightangle-Xray(Ineq)

$$\sharp\mathcal{T}(\mathcal{P}) \le \sum_{\theta\in\Theta(\mathcal{P})} \sum_{i=1}^{I^\theta} \sum_{j=1}^{J^{\theta^\perp}} X[\mathcal{P}](\ell_i^\theta) X[\mathcal{P}](\ell_j^{\theta^\perp})\mathbf{1}_{\mathcal{P}}(\ell_i^\theta \cap \ell_j^{\theta^\perp}). \tag{2.2}$$

Thus, their main result (1.1) would follow from their X-ray estimate (1.3).

Proof of (2.1). Fix a direction $\theta \in \Theta(\mathcal{P})$ and create a grid

$$\mathcal{L}^\theta \times \mathcal{L}^{\theta^\perp} = \{\ell_1^\theta, \dots, \ell_{I^\theta}^\theta\} \times \{\ell_1^{\theta^\perp}, \dots, \ell_{J^{\theta^\perp}}^{\theta^\perp}\}.$$

We then focus on right-angle triangles with an 'orientation' at θ or $\theta^\perp$, (equivalently those created from the grid $\mathcal{L}^\theta \times \mathcal{L}^{\theta^\perp}$). In order to give more precise definition, let us fist introduce a subset of $\mathcal{T}$ defined by

$$\mathcal{T}^\theta(p_{ij}^\theta) := \{\triangle(p_{ij}^\theta, p_{i'j}^\theta, p_{ij'}^\theta) : i' \in \{1, \dots, I^\theta\}\setminus\{i\},\ j' \in \{1, \dots, J^{\theta^\perp}\}\setminus\{j\} \text{ s.t. } p_{i'j}^\theta, p_{ij'}^\theta \in \mathcal{P}\},$$

for each $(i,j) \in \{1, \dots, I^\theta\} \times \{1, \dots, J^{\theta^\perp}\}$ such that $p_{ij}^\theta := \ell_i^\theta \cap \ell_j^{\theta^\perp} \in \mathcal{P}$. What does this subset mean? In one word, this is a set of all right-angle triangles in $\mathcal{T}$ whose 'orthogonal vertex' is at p_{ij}^θ; see my hand-written picture for more instinct! We then define

$$\mathcal{T}^\theta := \bigcup_{(i,j):p_{ij}^\theta\in\mathcal{P}} \mathcal{T}^\theta(p_{ij}^\theta).$$

This is a collection of all right-angle triangles whose shortest edge is oriented at either θ or $\theta^\perp$. Thus, $\mathcal{T}$, all right-angle triangles, may be decomposed into

$$\mathcal{T} = \bigcup_{\theta\in\Theta} \mathcal{T}^\theta = \bigcup_{\theta\in\Theta} \bigcup_{(i,j):p_{ij}^\theta\in\mathcal{P}} \mathcal{T}^\theta(p_{ij}^\theta).$$

As an important remark, we note that $\mathcal{T}^\theta(p_{ij}^\theta)$ and $\mathcal{T}^{\theta'}(p_{i'j'}^{\theta'})$ are 'independent' in the sense that

$$\mathcal{T}^\theta(p_{ij}^\theta) \cap \mathcal{T}^{\theta'}(p_{i'j'}^{\theta'}) = \emptyset$$

whenever $(\theta, i, j) \neq (\theta', i', j')$ (that is, either one of the following holds true: $\theta \neq \theta'$, $i \neq i'$, or $j \neq j'$). Therefore, we have that

$$\#\mathcal{T} = \sum_{\theta \in \Theta} \sum_{i=1}^{I^\theta} \sum_{j=1}^{J^{\theta^\perp}} \#\mathcal{T}^\theta(p_{ij}^\theta) \mathbf{1}_{\mathcal{P}}(p_{ij}^\theta).$$

Finally, for fixed (i, j) such that $p_{ij}^\theta \in \mathcal{P}$, we notice from the definition of $\mathcal{T}^\theta(p_{ij}^\theta)$ that

$$\begin{aligned} \#\mathcal{T}^\theta(p_{ij}^\theta) &= \#\{i' \in \{1, \ldots, I^\theta\} \setminus \{i\} : p_{i'j}^\theta \in \mathcal{P}\} \times \#\{j' \in \{1, \ldots, J^{\theta^\perp}\} \setminus \{j\} : p_{ij'}^\theta \in \mathcal{P}\} \\ &= \big(X[\mathcal{P}](\ell_j^{\theta^\perp}) - 1\big)\big(X[\mathcal{P}](\ell_i^\theta) - 1\big). \end{aligned}$$

This concludes the proof of (2.1). □

2.2. **Szemerédi–Trotter in X-ray language.**

Theorem 2.2 (Szemerédi–Trotter for lines)**.** *Let $\mathcal{L}$ be a finite collection of lines in $\mathbb{R}^2$, and $k \in \mathbb{N}$. Then*

e:ST-Line

$$\begin{aligned} \#\{k\text{-rich points of } \mathcal{L}\} &:= \#\{p \in \mathbb{R}^2 : \exists \ell_1, \ldots, \ell_k \in \mathcal{L} \text{ s.t. } x \in \ell_1 \cap \cdots \cap \ell_k\} \\ &\leq C \max\{\frac{(\#\mathcal{L})^2}{k^3}, \frac{\#\mathcal{L}}{k}\}. \end{aligned} \tag{2.3}$$

According to the well-known point-line duality, (2.3) is equivalent to the following:

Theorem 2.3 (Szemerédi–Trotter for points)**.** *Let $\mathcal{L}$ be a finite collection of lines in $\mathbb{R}^2$, $\mathcal{P}$ be a finite collection of points in $\mathbb{R}^2$, and $k \in \mathbb{N}$. Then*

e:ST-Point

$$\#\{\ell \in \mathcal{L} : \exists p_1, \ldots, p_k \in \mathcal{P} \cap \ell\} \leq C \max\{\frac{(\#\mathcal{P})^2}{k^3}, \frac{\#\mathcal{P}}{k}\}. \tag{2.4}$$

The inequality (2.4) may be described in terms of the X-ray transform as follows:

Corollary 2.4 (Szemerédi–Trotter in terms of X-ray transform)**.** *Let $\mathcal{L}$ be a finite collection of lines in $\mathbb{R}^2$, $\mathcal{P}$ be a finite collection of points in $\mathbb{R}^2$, and $k \in \mathbb{N}$. Then*

e:ST-X-ray

$$\#\{\ell \in \mathcal{L} : X[\mathcal{P}](\ell) \geq k\} \leq C \max\{\frac{(\#\mathcal{P})^2}{k^3}, \frac{\#\mathcal{P}}{k}\}. \tag{2.5}$$

Proof. Clearly, $\exists p_1, \ldots, p_k \in \mathcal{P} \cap \ell$ is equivalent to $X[\mathcal{P}](\ell) \geq k$. □

Szemerédi–Trotter = weak-type estimate of the X-ray transform

The threshold of k in (2.5) is given by $k = \sqrt{\#\mathcal{P}}$.

(1) In the case of $k \geq \sqrt{\#\mathcal{P}}$, (2.5) becomes

$$\#\{\ell \in \mathcal{L} : X[\mathcal{P}](\ell) \geq k\} \leq C\frac{\#\mathcal{P}}{k}.$$

This may be manifestly read as

$$\|X[\mathbf{1}_{\mathcal{P}}]\|_{L^{1,\infty}(\mathcal{L})} \text{“} \leq \text{”} C\|\mathbf{1}_{\mathcal{P}}\|_{L^1}.$$

(2) In the case of $k \leq \sqrt{\#\mathcal{P}}$, (2.5) becomes

$$\#\{\ell \in \mathcal{L} : X[\mathcal{P}](\ell) \geq k\} \leq C\frac{(\#\mathcal{P})^2}{k^3}.$$

This may be manifestly read as

e:ST-L3/2-weakL3 (2.6)
$$\|X[\mathbf{1}_{\mathcal{P}}]\|^3_{L^{3,\infty}(\mathcal{L})} \text{“} \leq \text{”} C\|\mathbf{1}_{\mathcal{P}}\|^3_{L^{\frac{3}{2}}}.$$

In particular, (2.6) suggests a strong type estimate of $X[\mathcal{P}]$ by loosing some logarithmic factor. This is indeed the case as follows:

Corollary 2.5 (Strong $L^{\frac{3}{2}}$-L^3 bound of the X-ray transform)**.** *Let $\mathcal{L}$ be a finite collection of lines in $\mathbb{R}^2$, $\mathcal{P}$ be a finite collection of points in $\mathbb{R}^2$, and $N := \#\mathcal{P}$. Then*

e:ST-StrongX-ray (2.7)
$$\|\mathbf{1}_{\{X[\mathcal{P}] \leq \sqrt{N}\}} X[\mathcal{P}]\|^3_{L^3(\mathcal{L})} \leq C \log N \|\mathbf{1}_{\mathcal{P}}\|^3_{L^{\frac{3}{2}}} = C(\#\mathcal{P})^2 \log \#\mathcal{P}.$$

Proof. This is perhaps standard argument to upgrade some weak-type estimate to the strong one by allowing logarithmic loss.

$$\begin{aligned}
\|\mathbf{1}_{\{X[\mathcal{P}] \leq \sqrt{N}\}} X[\mathcal{P}]\|^3_{L^3(\mathcal{L})} &= \sum_{\ell \in \mathcal{L} : X[\mathcal{P}](\ell) \leq \sqrt{N}} X[\mathcal{P}](\ell)^3 \\
&= \sum_{k=0}^{\sqrt{N}} \sum_{\ell \in \mathcal{L} : X[\mathcal{P}](\ell) = k} X[\mathcal{P}](\ell)^3 \\
&= \sum_{k=0}^{\sqrt{N}} k^3 L_{=k},
\end{aligned}$$

where

$$L_{=k} := \#\{\ell \in \mathcal{L} : X[\mathcal{P}](\ell) = k\}.$$

By introducing

$$L_{\geq k} := \#\{\ell \in \mathcal{L} : X[\mathcal{P}](\ell) \geq k\},$$

we readily see that

$$L_{=k} = L_{\geq k} - L_{\geq k+1}.$$

Thus,

$$\begin{aligned}\|\mathbf{1}_{\{X[\mathcal{P}]\leq\sqrt{N}\}}X[\mathcal{P}]\|^3_{L^3(\mathcal{L})} &= \sum_{k=0}^{\sqrt{N}} k^3(L_{\geq k}-L_{\geq k+1})\\ &= 0+\sum_{k=1}^{\sqrt{N}} k^3 L_{\geq k} - \sum_{k=1}^{\sqrt{N}}(k-1)^3 L_{\geq k} - (\sqrt{N})^3 L_{\geq\sqrt{N}+1}\\ &= \sum_{k=1}^{\sqrt{N}}(3k^2-3k+1)L_{\geq k} - (\sqrt{N})^3 L_{\geq\sqrt{N}+1}\\ &\leq 4\sum_{k=1}^{\sqrt{N}} k^2 L_{\geq k}.\end{aligned}$$

We now then apply Szemerédi–Trotter in terms of the X-ray transform (2.5) to conclude that

$$\|\mathbf{1}_{\{X[\mathcal{P}]\leq\sqrt{N}\}}X[\mathcal{P}]\|^3_{L^3(\mathcal{L})} \leq C\sum_{k=1}^{\sqrt{N}} k^2\frac{(\#\mathcal{P})^2}{k^3} = C(\#\mathcal{P})^2\log(\#\mathcal{P}).$$

□

2.3. **Conclude the proof of** (1.3)**.** Given above preparation, we are now at the stage of doing something trivial, that is Cauchy–Schwarz. First, we separate three cases

e:BilineearXray

$$\begin{aligned}&\sum_{\theta\in\Theta}\sum_{i=1}^{I^\theta}\sum_{j=1}^{J^{\theta^\perp}} X[\mathcal{P}](\ell_i^\theta)X[\mathcal{P}](\ell_j^{\theta^\perp})\mathbf{1}_{\mathcal{P}}(p_{ij}^\theta)\\ &=\sum_{\theta\in\Theta}\sum_{i:X[\mathcal{P}](\ell_i^\theta)\leq\sqrt{N}} X[\mathcal{P}](\ell_i^\theta)\sum_{j:X[\mathcal{P}](\ell_j^{\theta^\perp})\leq\sqrt{N}} X[\mathcal{P}](\ell_j^{\theta^\perp})\mathbf{1}_{\mathcal{P}}(p_{ij}^\theta)\\ &\quad+\text{term involving}\sum_{i:X[\mathcal{P}](\ell_i^\theta)>\sqrt{N}}+\text{term involving}\sum_{j:X[\mathcal{P}](\ell_j^{\theta^\perp})>\sqrt{N}}.\end{aligned}\tag{2.8}$$

As we will see in the end, the main contribution comes from the first term. So, we will focus on how to deal with the first term. We fix $\theta\in\Theta$ and estimate

e:CS1

$$\begin{aligned}&\sum_{i:X[\mathcal{P}](\ell_i^\theta)\leq\sqrt{N}}\ \sum_{j:X[\mathcal{P}](\ell_j^{\theta^\perp})\leq\sqrt{N}} X[\mathcal{P}](\ell_i^\theta)X[\mathcal{P}](\ell_j^{\theta^\perp})\mathbf{1}_{\mathcal{P}}(p_{ij}^\theta)\\ &\leq\big(\sum_{i,j}\mathbf{1}_{\{i:X[\mathcal{P}](\ell_i^\theta)\leq\sqrt{N}\}}X[\mathcal{P}](\ell_i^\theta)^2\mathbf{1}_{\mathcal{P}}(p_{ij}^\theta)\big)^{\frac12}\big(\sum_{i,j}\mathbf{1}_{\{j:X[\mathcal{P}](\ell_j^{\theta^\perp})\leq\sqrt{N}\}}X[\mathcal{P}](\ell_j^{\theta^\perp})^2\mathbf{1}_{\mathcal{P}}(p_{ij}^\theta)\big)^{\frac12}\\ &=\big(\sum_{i:X[\mathcal{P}](\ell_i^\theta)\leq\sqrt{N}}X[\mathcal{P}](\ell_i^\theta)^2\sum_j\mathbf{1}_{\mathcal{P}}(p_{ij}^\theta)\big)^{\frac12}\big(\sum_{j:X[\mathcal{P}](\ell_j^{\theta^\perp})\leq\sqrt{N}}X[\mathcal{P}](\ell_j^{\theta^\perp})^2\sum_i\mathbf{1}_{\mathcal{P}}(p_{ij}^\theta)\big)^{\frac12}.\end{aligned}\tag{2.9}$$

Notice that

$$\sum_j\mathbf{1}_{\mathcal{P}}(p_{ij}^\theta)=\#\{j\in\{1,\ldots,J^{\theta^\perp}\}:\ell_i^\theta\cap\ell_j^{\theta^\perp}\in\mathcal{P}\}=X[\mathcal{P}](\ell_i^\theta),$$

and similarly

$$\sum_i \mathbf{1}_{\mathcal{P}}(p^\theta_{ij}) = X[\mathcal{P}](\ell_j^{\theta^\perp}).$$

Therefore,

$$\begin{aligned}
&\sum_{i:X[\mathcal{P}](\ell_i^\theta)\le\sqrt{N}} \ \sum_{j:X[\mathcal{P}](\ell_j^{\theta^\perp})\le\sqrt{N}} X[\mathcal{P}](\ell_i^\theta)X[\mathcal{P}](\ell_j^{\theta^\perp})\mathbf{1}_{\mathcal{P}}(p^\theta_{ij}) \\
&\le \Big(\sum_{i:X[\mathcal{P}](\ell_i^\theta)\le\sqrt{N}} X[\mathcal{P}](\ell_i^\theta)^3\Big)^{\frac{1}{2}} \Big(\sum_{j:X[\mathcal{P}](\ell_j^{\theta^\perp})} X[\mathcal{P}](\ell_j^{\theta^\perp})^3\Big)^{\frac{1}{2}}.
\end{aligned}$$

By taking a summation in θ and applying CS again,

e:CS2

$$\begin{aligned}
&\text{First term} \\
&= \sum_{\theta\in\Theta} \sum_{i:X[\mathcal{P}](\ell_i^\theta)\le\sqrt{N}} \ \sum_{j:X[\mathcal{P}](\ell_j^{\theta^\perp})\le\sqrt{N}} X[\mathcal{P}](\ell_i^\theta)X[\mathcal{P}](\ell_j^{\theta^\perp})\mathbf{1}_{\mathcal{P}}(p^\theta_{ij}) \\
&\le \sum_{\theta\in\Theta} \Big(\sum_{i:X[\mathcal{P}](\ell_i^\theta)\le\sqrt{N}} X[\mathcal{P}](\ell_i^\theta)^3\Big)^{\frac{1}{2}} \Big(\sum_{j:X[\mathcal{P}](\ell_j^{\theta^\perp})\le\sqrt{N}} X[\mathcal{P}](\ell_j^{\theta^\perp})^3\Big)^{\frac{1}{2}} \\
&\le \Big(\sum_{\theta} \sum_{i:X[\mathcal{P}](\ell_i^\theta)\le\sqrt{N}} X[\mathcal{P}](\ell_i^\theta)^3\Big)^{\frac{1}{2}} \Big(\sum_{\theta} \sum_{j:X[\mathcal{P}](\ell_j^{\theta^\perp})\le\sqrt{N}} X[\mathcal{P}](\ell_j^{\theta^\perp})^3\Big)^{\frac{1}{2}} \qquad (2.10) \\
&= \sum_{\theta} \sum_{i:X[\mathcal{P}](\ell_i^\theta)\le\sqrt{N}} X[\mathcal{P}](\ell_i^\theta)^3 = \|\mathbf{1}_{\{X[\mathcal{P}](\ell)\le\sqrt{N}\}} X[\mathcal{P}]\|^3_{L^3(\mathcal{L})},
\end{aligned}$$

where $\mathcal{L}$ denotes all lines spanned by two points of $\mathcal{P}$. We conclude the desired estimate for the first term of (2.8) from $L^{\frac{3}{2}}$–$L^3(\mathcal{L})$ boundedness of $X[\mathcal{P}]$ (2.7).

We are left to handle other terms in (2.8). However, these terms will be bounded by $\#\mathcal{P}\log\#\mathcal{P}$, and thus it is an error term; see original paper by Pach–Sharir for details.