Pointwise and weighted Calderón-Zygmund type estimates with applications to nonlinear PDEs

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The Weyl's lemma

Consider the simplest equation

$$\Delta u = 0 \qquad \text{in} \quad \mathbb{R}^n.$$

Weyl's lemma tells us that any distributional solution is smooth. Quantitatively,

$$\|\nabla^\alpha u\|_{L^\infty(B_r)} \leq \frac{C}{r^{|\alpha|+n}} \|u\|_{L^1(B_{2r})}.$$

That was the beginning of regularity theory!

The standard CZ theory

Next consider the equation

$$\Delta u = f \qquad \text{in} \quad \mathbb{R}^n.$$

Then CZ theory tells us that

$$f \in L^q \Longrightarrow \nabla^2 u \in L^q, \qquad 1 < q < \infty.$$

Note that this fails at the end-point cases $q = 1$ and $q = \infty$. Consequently, Sobolev embedding implies

$$\nabla u \in L^{\frac{nq}{n-q}}, \qquad q < n,$$

$$u \in L^{\frac{nq}{n-2q}}, \qquad q < n/2.$$

Pointwise representations

The bottom of the matter is

$$u(x) = c(n) \int_{\mathbb{R}^n} G(x, y) f(y) dy,$$

where

$$G(x, y = \begin{cases} |x - y|^{2-n} & \text{if} \quad n > 2 \\ -\log(|x - y|) & \text{if} \quad n = 2. \end{cases}$$

Then differentiating twice

$$\nabla^2 u(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy = [\mathbf{R}_i \mathbf{R}_j(f)]$$

where $K(x, y)$ is a singular integral kernel of CZ type. Hence the conclusion follows. Here R_j is the j-th Riesz transform.

Gradient estimates: Fractional integral approach

The pointwise representation says that

$$u(x) = \mathbf{I}_2 f(x), \qquad n > 2,$$

and

$$|\nabla u(x)| \leq c\, \mathbf{I}_1 |f|(x),$$

where $\mathbf{I}_\alpha$, $\alpha \in (0, n)$ is a fractional integral

$$\mathbf{I}_\alpha \mu(x) = c(n, \alpha) \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x - y|^{n-\alpha}}$$

$$= c \int_0^\infty \frac{\mu(B_t)(x)}{t^{n-\alpha}} \frac{dt}{t}.$$

Then

$$\mathbf{I}_\alpha : L^q \to L^{\frac{nq}{n-\alpha q}}, \qquad q > 1, \alpha q < n.$$

This gives the desired L^p control of u and ∇u.

Fractional integrals v.s. singular integrals

- **To bound u and ∇u we do not need to pass to $\nabla^2 u$ and thus CZ theory can be avoided. We work only with fractional integrals instead. This has an advantage when dealing with equations with bad coefficients over irregular domains.**
- **The theory of fractional integrals is different from the theory of singular integrals in that whereas the latter is based on cancellation properties of the kernel, the former only use the size of the kernel.**
- **In particular, the embedding**

$$\mathbf{I}_\alpha : L^q \to L^{\frac{nq}{n-\alpha q}}, \qquad \alpha q < n,$$

fails as $\alpha \to 0^+$.

A gradient estimate where singular integrals are needed

Now consider the equation

$$\Delta u = \operatorname{div} F \qquad \text{in} \quad \mathbb{R}^n.$$

We want to get the following bound

$$\|\nabla u\|_{L^q} \lesssim \|F\|_{L^q}, \qquad 1 < q < \infty,$$

i.e. the solution operator maps $\dot{W}^{-1,q}$ into $\dot{W}^{1,q}$.
Integrating by parts and differentiating the pointwise representation

$$\nabla u(x) = \int_{\mathbb{R}^n} \nabla_x \nabla_y G(x,y)\, F(y) dy = -[\mathbf{R_i R_j}]F.$$

Hence CZ theory applies and yields the above bound.

Warning

• **This L^q gradient estimate should not be expected to hold when the coefficients are not good or when the domain is irregular.**

• **Bad coefficient example and bad domain example will be discussed later on.**

Capacities

• **Sobolev capacity: Let $\alpha > 0$, $s > 1$, and let K be a compact set. Define**

$$\mathrm{Cap}(K, W^{\alpha,s}(\mathbb{R}^n)) := \inf\{\|u\|_{W^{\alpha,s}(\mathbb{R}^n)} : u \in S_K\},$$

where

$$S_K = \{u \in C_0^\infty(\mathbb{R}^n), u \geq 1 \text{ on } K\}.$$

• **Functions in $W^{\alpha,s}(\mathbb{R}^n)$ are generally not continuous. One can think of $\mathrm{Cap}(\cdot, W^{\alpha,s}(\mathbb{R}^n))$ is a device to measure the discontinuity of functions in $W^{\alpha,s}(\mathbb{R}^n)$, especially when $\alpha s \leq n$.**

• **Example (Lusin type theorem). If $f \in W^{\alpha,s}(\mathbb{R}^n)$, then f has a quasi-continuous representative $\tilde{f}$. That is, $f = \tilde{f}$ a.e. and for any $\epsilon > 0$ there exists an open set G such that $\mathrm{Cap}(G, W^{\alpha,s}(\mathbb{R}^n)) < \epsilon$ and $\tilde{f}$ is continuous in $\mathbb{R}^n \setminus G$.**

Capacities

- **Bessel capacity:**

$$\mathrm{Cap}_{\alpha,\, s}(K) = \inf \Big\{ \|f\|_{L^s}^s : f \geq 0, \mathbf{G}_\alpha f \geq 1 \text{ on } K \Big\},$$

where $\mathbf{G}_\alpha = \mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{\alpha}{2}}]$ **(Bessel kernel), and**

$$\mathbf{G}_\alpha f(x) = \int_{\mathbb{R}^n} \mathbf{G}_\alpha(x-y) f(y) dy.$$

- **Note that**

$$\mathbf{G}_\alpha(x) \leq A \begin{cases} \frac{1}{|x|^{n-\alpha}}, & 0 < |x| \leq 1 \\ e^{-a|x|}, & |x| > 1. \end{cases}$$

- **By Calderón-Zygmund theory**

$$W^{\alpha,s}(\mathbb{R}^n) = \{\mathbf{G}_\alpha f : f \in L^s(\mathbb{R}^n)\},$$

and thus

$$\mathrm{Cap}_{\alpha,s}(K) \simeq \mathrm{Cap}(K, W^{\alpha,s}(\mathbb{R}^n)).$$

Capacities

- **Riesz capacity:** $\alpha \in (0, n)$, $s > 1$,

$$\mathrm{cap}_{\alpha,\,s}(K) = \inf \Big\{ \|f\|_{L^s}^s : f \geq 0, \mathbf{I}_\alpha f \geq 1 \text{ on } K \Big\},$$

where recall that

$$\mathbf{I}_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

- $\mathrm{cap}_{\alpha,\,s}(\cdot)$ **is associated to the homogeneous Sobolev space** $\dot{W}^{\alpha,s}$**.**
- **Locally we also have the equivalence: For** $\alpha s < n$**,**

$$\mathrm{cap}_{\alpha,\,s}(K) \leq \mathrm{Cap}_{\alpha,\,s}(K) \leq C(R)\mathrm{cap}_{\alpha,\,s}(K), \qquad \forall K \subset B_R.$$

Capacities

- **For** $\alpha s > n$**,** $\mathrm{Cap}_{\alpha,s}(K) \geq c > 0$ **provided** K **is nonempty. But**
- **For** $\alpha s \geq n$**,** $\mathrm{cap}_{\alpha,s}(K) = 0$ **for any** K**.**
- **Capacity of a ball:** $\mathrm{cap}_{\alpha,s}(B_r) \simeq |B_r|^{1-\alpha s/n}$**,** $\alpha s < n$**.**
- **A lower estimate for general sets:**

$$\mathrm{cap}_{\alpha,s}(K) \gtrsim |K|^{1-\alpha s/n}, \quad \alpha s < n.$$

This follows from the Sobolev's inequality.

- **Relation to Hausdorff measure:**

$$\mathrm{cap}_{\alpha,s}(K) \leq c\mathcal{H}_{\infty}^{n-\alpha s}(K),$$

and moreover $\mathcal{H}^{n-\alpha s}(K) < \infty \Longrightarrow \mathrm{cap}_{\alpha,s}(K) = 0$.
On the other hand, $\mathrm{cap}_{\alpha,s}(K) = 0 \Longrightarrow \mathcal{H}^{t}(K) = 0$ **for all** $t > n - \alpha s$**.**

Capacities

Capacities play an important role in analysis and PDEs. For example, they are used to study:

• Pointwise behaviors of Sobolev functions (mentioned above).
• Removable singularities of solutions to PDEs. Example: Let E be a closed subset of Ω and $u \in Har(\Omega \setminus E) \cap L^{\infty}(\Omega \setminus E)$. If $\mathrm{cap}_{1,2}(E) = 0$ then $u \in Har(\Omega)$.
• Dirichlet problems on arbitrary domains (Wiener's criterion), etc.

We are interested in capacities mainly because of their relation to trace inequalities.

Capacities and trace inequalities

Theorem (Maz'ya-Adams-Dahlberg)

Let $\nu \in M^+(\mathbb{R}^n)$, $0 < \alpha < n$, *and* $1 < s < \infty$. *Then*

$$\int_{\mathbb{R}^n} (\mathbf{I}_\alpha f)^s d\nu \leq A_1 \int_{\mathbb{R}^n} f^s dx, \quad \forall f \in L^s(\mathbb{R}^n), f \geq 0.$$

$$\Updownarrow$$

$$\nu(K) \leq A_2 \,\mathrm{cap}_{\alpha,s}(K), \quad \forall K \subset \mathbb{R}^n.$$

• **If $\alpha \in \mathbb{N}$ then they are equivalent to the following weighted Poincaré-Sobolev's inequality:**

$$\int_{\mathbb{R}^n} |\varphi|^s d\nu \leq A_1 \int_{\mathbb{R}^n} |\nabla^\alpha \varphi|^s dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n).$$

Capacities and trace inequalities

- **For example, for $\alpha = 1$ we have**

$$\int_{\mathbb{R}^n} |\varphi|^s d\nu \leq A_1 \int_{\mathbb{R}^n} |\nabla\varphi|^s dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n).$$

$$\Updownarrow$$

$$\int_{\mathbb{R}^n} (\mathbf{I}_1 f)^s d\nu \leq A_1 \int_{\mathbb{R}^n} f^s dx, \quad \forall f \in L^s(\mathbb{R}^n), f \geq 0.$$

$$\Updownarrow$$

$$\nu(K) \leq A_2 \,\mathrm{cap}_{1,s}(K), \quad \forall K \subset \mathbb{R}^n.$$

- **For $\alpha = 2$, one has a similar result for $\mathbf{I}_2$ and Δ.**
- **Also, we have the inhomogeneous version**

$$\int_{\mathbb{R}^n} |\varphi|^s d\nu \leq C_1 \int_{\mathbb{R}^n} (|\nabla\varphi|^s + |\varphi|^s) dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n).$$

$$\Updownarrow$$

$$\nu(K) \leq C_2 \,\mathrm{Cap}_{1,s}(K), \quad \forall K \subset \mathbb{R}^n.$$

Capacities and trace inequalities

Balls versus sets:

- **Necessary condition:**

$$\nu(B_r) \le C\, \mathrm{cap}_{\alpha,s}(B_r) = C\, r^{n-\alpha s} \quad \forall B_r \subset \mathbb{R}^n.$$

- **Sufficient condition:** $\nu = g dx$ **and for some** $\epsilon > 0$

$$\int_{B_r} g^{1+\epsilon} dy \le C\, r^{n-(1+\epsilon)\alpha s} \quad \forall B_r \subset \mathbb{R}^n.$$

This is known as Fefferman-Phong condition (a Morrey space condition).

- **Another equivalent condition: Kerman-Sawyer's testing condition**

$$\int_{B_r} (\mathbf{I}_\alpha \nu_{B_r})^{s'} dx \le C\, \nu(B_r) \quad \forall B_r \subset \mathbb{R}^n.$$

What does the condition $\nu(K) \leq C\,\mathrm{cap}_{\alpha,s}(K)$ tell us?

- **Since**

$$|K|^{1-\frac{\alpha s}{n}} \leq C\,\mathrm{cap}_{\alpha,\,s}(K)$$

we see that if $\nu = f \in L^{\frac{n}{\alpha s},\infty}(\Omega)$ **then** $\nu(K) \leq C\,\mathrm{cap}_{\alpha,s}(K)$ **and hence the trace inequality follows. Recall that for** $p > 1$**,**

$$f \in L^{p,\infty} \Leftrightarrow t^p|\{x \in \Omega : |f(x)| \geq t\}| \leq C \quad \forall t > 0.$$

$$\Leftrightarrow \int_K |f|dx \leq C|K|^{1-1/p}.$$

- **Strong type $\Longleftrightarrow$ weak type:**

$$\int_{\mathbb{R}^n} (\mathbf{I}_\alpha f)^s d\nu \leq C \int_{\mathbb{R}^n} f^s dx, \quad \forall f.$$

$$\Updownarrow$$

$$t^s \nu(\{\mathbf{I}_\alpha f \geq t\}) \leq C \int_{\mathbb{R}^n} f^s dx, \quad \forall f.$$

Capacities and trace inequalities

- **Capacitary weak type inequality:**

$$\mathrm{cap}_{\alpha,s}(\{\mathbf{I}_\alpha f \geq t\}) \leq \frac{1}{t^s} \int_{\mathbb{R}^n} f^s dx, \quad \forall f.$$

This is obvious from the definition of capacity.

- **Capacitary strong type inequality:**

$$\int_0^\infty \mathrm{cap}_{\alpha,s}(\{\mathbf{I}_\alpha f \geq t\})\, dt^s \leq C \int_{\mathbb{R}^n} f^s dx, \quad \forall f.$$

This is by no means obvious!

Capacities and trace inequalities

Theorem (Maz'ya-Verbitsky 1995)

Let $\nu \in M^+(\mathbb{R}^n)$, $0 < \alpha < n$, *and* $1 < s < \infty$. *Then*

$$\nu(K) \le A_3 \, \mathrm{cap}_{\alpha,s}(K), \quad \forall K \subset \mathbb{R}^n.$$

$$\Updownarrow$$

$$\int_K (\mathbf{I}_\alpha \nu)^{s'} dx \le A_4^{s'} \, \mathrm{cap}_{\alpha,\, s}(K), \quad \forall K \subset \mathbb{R}^n.$$

$$\Updownarrow$$

$$\mathbf{I}_\alpha[(\mathbf{I}_\alpha \nu)^{s'}](x) \le A_5^{\frac{1}{s-1}} \mathbf{I}_\alpha \nu(x) \quad \text{a.e. } x \in \mathbb{R}^n.$$

Remark:

- **The constants** A_i, $i = 1, \ldots, 5$, **are comparable.**
- **A similar result holds for** $\mathbf{G}_\alpha$ **and Bessel capacity** $\mathrm{Cap}_{\alpha,s}$**.**

Formulation by function spaces

• **Morrey space:** $\mathcal{L}^{p,\lambda}$**,** $p \geq 1$**,** $0 < \lambda \leq n$

$$\|f\|^p_{\mathcal{L}^{p,\lambda}} := \sup_{B_r} \frac{\int_{B_r} |f|^p dx}{r^{n-\lambda}}.$$

When $p = 1$**, we replace functions** f **with measures.**

• **Maz'ya space:** $\mathcal{M}^{p,\alpha,s}$**,** $p \geq 1$**,** $0 < \alpha s < n$

$$\|f\|^p_{\mathcal{M}^{p,\alpha,s}} := \sup_{K} \frac{\int_{K} |f|^p dx}{\mathrm{cap}_{\alpha,s}(K)}.$$

When $p = 1$**, we replace functions** f **with measures.**

Formulation by function spaces

• **Adams embedding:** $\mathbf{I}_\alpha : \mathcal{L}^{p,\lambda} \to \mathcal{L}^{\frac{\lambda p}{\lambda - \alpha p},\lambda}$ **holds for** $p > 1$ **and** $\alpha p < \lambda$. **But it fails for** $p = 1$. **Here** λ **acts like the dimension.**

• **Maz'ya-Verbitsky embedding:**

$$\mathbf{I}_\alpha : \mathcal{M}^{1,\alpha,s} \to \mathcal{M}^{s',\alpha,s}.$$

$$\mathbf{I}_\beta : \mathcal{M}^{p,\alpha,s} \to \mathcal{M}^{\frac{\alpha s p}{\alpha s - \beta p},\alpha,s}, \qquad p \geq 1, \beta p < \alpha s.$$

P.-Phan 2014.

• **Maz'ya versus Morrey:**

$$\mathcal{L}^{1+\epsilon,(1+\epsilon)\alpha s} \subset \mathcal{M}^{1,\alpha,s} \subset \mathcal{L}^{1,\alpha s}, \quad \epsilon > 0.$$

The first inclusion follows from Fefferman-Phong condition.

• $\nu \in \mathcal{M}^{1,\alpha,s}$ **and** $\mathbf{I}_\alpha \mu \leq C\, \mathbf{I}_\alpha \nu$ **a.e.** $\Longrightarrow \mu \in \mathcal{M}^{1,\alpha,s}$.

Capacities and trace inequalities

The Hardy-Littlewood maximal function M and standard CZO are bounded on $\mathcal{M}^{p,\alpha,s}$, $p > 1$ (Verbitsky).

Theorem (Verbitsky)

Let $f \in \mathcal{M}^{p,\alpha,s}$, where $p \geq 1$ and $\alpha s < n$. Suppose that for all weights $w \in A_1$,

$$\int_{\mathbb{R}^n} |g|^p w dx \leq K \int_{\mathbb{R}^n} |f|^p w dx,$$

where K depends only on n, p, and the A_1 constant of w. Then

$$\|g\|_{\mathcal{M}^{p,\alpha,s}} \leq C \|f\|_{\mathcal{M}^{p,\alpha,s}} .$$

- **A weight function $w \in A_1$ if $\exists A > 0$ s.t. $\mathbf{M}w \leq A\, w$ a.e.**
- **Application: Take $g = \mathbf{M}f$ or $g = Tf$, where $T = CZO$.**
- **The weighted estimate here is a substitute for pointwise estimate.**

Capacities and trace inequalities

The proof of the above theorem uses the following features of compact sets with positive capacity:

Lemma (Meyers 1970, Havin-Maz'ya 1972, Verbitsky 1985)

For any compact set $K \subset \mathbb{R}^n$ *with* $\mathrm{cap}_{\alpha,s}(K) > 0$, *there exists a measure* $\mu = \mu^K$ *(called capacitary measure of* K*) such that*

(i) $\mathrm{supp}(\mu) \subset K$, $\mu(K) = \mathrm{cap}_{\alpha,s}(K) = \|\mathbf{I}_\alpha \mu\|_{L^{s'}}^{s'}$.

(ii) $\mathbf{V}_{\alpha,s}\mu \geq 1$ *quasi-everywhere on* K. *Here* $\mathbf{V}_{\alpha,s}\mu = \mathbf{I}_\alpha(\mathbf{I}_\alpha\mu)^{\frac{1}{s-1}}$.

(iii) $\mathbf{V}_{\alpha,s}\mu \leq C(n,\alpha,s)$ *in* $\mathbb{R}^n$.

(iv) $\mathrm{cap}_{\alpha,s}\{\mathbf{V}_{\alpha,s}\mu \geq t\} \leq At^{-\min\{1,s-1\}}\mathrm{cap}_{\alpha,s}(K)$.

(v) $(\mathbf{V}_{\alpha,s}\mu)^\delta \in A_1$, *where* $0 < \delta < \frac{n}{n-\alpha}$ *if* $1 < s \leq 2 - \alpha/n$ *and* $0 < \delta < \frac{(s-1)n}{n-\alpha s}$ *if* $2 - \alpha/n < s < \infty$.

Morrey space version

Theorem

Let $f \in \mathcal{L}^{p,\lambda}$, where $p \geq 1$ and $0 < \lambda \leq n$. Suppose that for all weights $w \in A_1$,

$$\int_{\mathbb{R}^n} |g|^p w dx \leq K \int_{\mathbb{R}^n} |f|^p w dx,$$

where K depends only on n, p, and the A_1 constant of w. Then

$$\|g\|_{\mathcal{L}^{p,\lambda}} \leq C \|f\|_{\mathcal{L}^{p,\lambda}} .$$

• **Idea (Mengesha-P. 2010): Fix $0 < \epsilon < \lambda$ and apply the inequality with the weight**

$$w(x) = \min\{|x - z|^{-n+\lambda-\epsilon}, r^{-n+\lambda-\epsilon}\},$$

where $B_r(z)$ is the ball on which we want to control g.

Trace inequalities with different exponents

Consider inequality of the the more general form

$$\int_{\mathbb{R}^n} |u|^q d\nu \leq C \int_{\mathbb{R}^n} |\nabla^\alpha u|^s dx, \quad \forall u \in C_0^\infty(\mathbb{R}^n).$$

- $q = s > 1$: **discussed above.**
- $q > s > 1$: **the characterization is**

$$\nu(K) \leq C\, \mathrm{cap}_{\alpha,s}(K)^{\frac{q}{s}}, \quad \forall K \subset \mathbb{R}^n.$$

$$\Updownarrow$$

$$\nu(B_r) \leq C\, \mathrm{cap}_{\alpha,s}(B_r)^{\frac{q}{s}} = C\, r^{(n-\alpha s)q/s}, \quad \forall \text{ balls } B_r \subset \mathbb{R}^n.$$

This is known as Adams' Theorem. This also holds for $q \geq s = 1$**.**

Trace inequalities with different exponents

- **$s > 1$, $s > q > 0$: a characterization is due to Cascante-Ortega-Verbitsky**

$$\mathbf{W}_{\alpha,s}\nu \in L^{\frac{q(s-1)}{s-q}}(d\nu),$$

where $\mathbf{W}_{\alpha,s}\nu$ is the Wolff's potential of ν

$$\mathbf{W}_{\alpha,s}\nu(x) = \int_0^\infty \left(\frac{\nu(B_r(x))}{r^{n-\alpha s}}\right)^{\frac{1}{s-1}} \frac{dt}{t}, \quad x \in \mathbb{R}^n.$$

- **Another characterization is due to Maz'ya-Netrusov:**

$$\int_0^\infty \left(\frac{t^{s/q}}{\Psi(t)}\right)^{\frac{q}{s-q}} \frac{dt}{t},$$

where $\Psi(t) = \inf\{\mathrm{cap}_{1,s}(A) : A \subset \mathbb{R}^n, \nu(A) \geq t\}$**.**

Trace inequalities with different exponents

- **The special case $s > 1$ and $q = 1$:**

$$\iff \nu \in (\dot{W}^{\alpha,s})^*$$

$$\Updownarrow$$

$$\mathbf{I}_\alpha \nu \in L^{s'}(dx)$$

$$\Updownarrow$$

$$\mathbf{W}_{\alpha,s}\nu \in L^1(d\nu)$$

- **The last equivalence is known as Wolff's inequality:**

$$\int_{\mathbb{R}^n} (\mathbf{I}_\alpha \nu(x))^{s'} dx \simeq \int_{\mathbb{R}^n} \mathbf{W}_{\alpha,s}\nu(x) d\nu(x).$$

Connection to Lane-Emden equation with measure data

Consider the equation

$$-\Delta u = u^q + \mu \text{ in } \mathbb{R}^n. \tag{1}$$

Here $u \geq 0$, $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, and $\mu \in M^+(\mathbb{R}^n)$. In integral form, this reads

$$u = \mathbf{I}_2(u^q) + \mathbf{I}_2\mu \qquad \text{a.e.}$$

Here we assume that $\mathbf{I}_2\mu < +\infty$ a.e.

Theorem (Baras-Pierre 1983, 1985)

Let $q > 1$ and $n > 2$. Then (1) has a nonnegative solution $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ if and only if

$$\int_{\mathbb{R}^n} \mathbf{I}_2(\phi)d\mu \leq (q-1)q^{-q'} \int_{\mathbb{R}^n} \phi^{q'} \mathbf{I}_2(\phi)^{1-q'} dx \quad \forall \phi \geq 0.$$

- **The proof makes use of duality and the linear nature of $\mathbf{I}_2$ (or Δ).**

Connection to Lane-Emden equation with measure data

Theorem (Adams-Pierre 1991)

Let $q > 1$ and $n > 2$.
(i) *If* (1) *has a nonnegative solution $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ then*

$$\mu(K) \leq C \,\mathrm{cap}_{2,q'}(K) \tag{2}$$

for all compact sets $K \subset \mathbb{R}^n$, with C independent of K.
(ii) *There exists a constant $C_0 = C_0(n, q)$ such that if* (2) *holds with $C \leq C_0$, then* (1) *has a nonnegative solution $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$.*

• **Note that** (2) $\Rightarrow$ **When μ is nonzero, the Hausdorff dimension of** $\mathrm{Supp}(\mu) \geq n - 2q'$. **That is μ has to be "soft" enough.**

Connection to Lane-Emden equation with measure data

Equations on bounded domains:

Theorem (Adams-Pierre 1991)

Suppose that $\mathrm{supp}\mu \Subset \Omega$, $q > 1$.

- *If the equation*

$$\begin{cases} -\Delta u = u^q + \mu \text{ in } \Omega, \\ u \geq 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{3}$$

has a solution then

$$\mu(K) \leq C\, \mathrm{Cap}_{2,q'}(K), \qquad \forall K \subset \Omega. \tag{4}$$

- *Conversely,* $\exists C_0 = C_0(n, q) > 0$ *such that if* (4) *holds with* $C \leq C_0$ *then* (3) *has a solution.*

Connection to Lane-Emden equation with measure data

The proof of these two theorems were based Baras-Pierre's result and the following characterization: For $s > 1$,

$$\nu(K) \leq C\,\mathrm{cap}_{2,s}(K)$$

$$\Updownarrow$$

$$\int_{\mathbb{R}^n} |\varphi|^s d\nu \leq C \int_{\mathbb{R}^n} |\Delta\varphi|^s dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n)$$

$$\Updownarrow$$

$$\int_{\mathbb{R}^n} |\varphi| d\nu \leq C \int_{\mathbb{R}^n} |\Delta\varphi|^s |\varphi|^{1-s} dx$$

$$\Updownarrow$$

$$\int_{\mathbb{R}^n} |\varphi|^q d\nu \leq C \int_{\mathbb{R}^n} |\Delta\varphi|^s |\varphi|^{q-s} dx, \qquad q \in [1, s].$$

A proof of $(1) \Rightarrow (2)$ (Verbitsky-Wheeden 1995):

$$(1) \Longrightarrow \mathbf{I}_2(u^q) \le u < +\infty \quad \text{a.e.}$$

$$\Updownarrow$$

$$\mathbf{I}_2[\mathbf{I}_2(u^q)^q] \le \mathbf{I}_2(u^q)$$

$$\Updownarrow$$

$$\int_{\mathbb{R}^n} (\mathbf{I}_2 f)^{q'} u^q dx \le C \int_{\mathbb{R}^n} f^{q'} dx, \quad \forall f.$$

$$\Updownarrow$$

$$\int_K u^q dx \le C \mathrm{cap}_{2,q'}(K), \quad \forall K.$$

$$\Downarrow (u \ge \mathbf{I}_2 \mu)$$

$$\int_K \mathbf{I}_2(\mu)^q dx \le C \mathrm{cap}_{2,q'}(K), \quad \forall K.$$

Connection to Lane-Emden equation with measure data

- **Intrinsic space of solutions:** $\mathcal{M}^{q,2,q'}$
- **Simple sufficient condition: Let** $\mu = f dx$.

$$f \in L^{\frac{n}{2q'},\infty}(\Omega) \qquad \text{(with small norm)}.$$

- **Fefferman-Phong sufficient condition: Let** $\mu = f dx$. **For some** $\epsilon > 0$

$$\int_{B_r} f^{1+\epsilon} dx \leq C\, r^{n-\frac{(1+\epsilon)2q}{(q-1)}}, \qquad \forall \text{ balls } B_r.$$

Here one checks only over balls, but a small bump on f **is needed.**

- **Liouville type theorem: If** $1 < q \leq \frac{n}{n-2}$ **(i.e.** $2q' \geq n$**) then** (1) **has no nonnegative global solution provided** $\mu \neq 0$.
- **On the other hand, if** $1 < q < \frac{n}{n-2}$ **then** (4) **is satisfied for some** $C > 0$ **provided** μ **is finite in** Ω. **In this case a solution exists in** Ω **provided** $\|\mu\|$ **is small enough.**

Removable Singularities for $-\Delta u = u^q$

Theorem (Adams-Pierre 1991)

Let $E \subset \Omega$ be compact. Then

$$\mathrm{Cap}_{2,\,q'}(E) = 0$$

is necessary and sufficient in order that:

$$\begin{cases} u \in L^q_{\mathrm{loc}}(\Omega \setminus E), \quad u \geq 0, \\ -\Delta u = u^q \quad \text{in} \quad \mathcal{D}'(\Omega \setminus E). \end{cases}$$

$$\Downarrow$$

$$\begin{cases} u \in L^q_{\mathrm{loc}}(\Omega), \quad u \geq 0, \\ -\Delta u = u^q \quad \text{in} \quad \mathcal{D}'(\Omega). \end{cases}$$

- **Remark: No information of u near E is needed.**

Removable Singularities for $-\Delta u = u^q$

Proof of the necessity part: By contradiction, suppose that $\mathrm{Cap}_{2,\,q'}(E) > 0$. Let μ^E be the capacitary measure for E. It is known that μ^E satisfies the capacitary condition. Thus there is a positive solution $u \in L^q_{\mathrm{loc}}$ to

$$-\Delta u = u^q + \epsilon\,\mu^E \quad \text{in} \quad \Omega,$$

provided ϵ is sufficiently small. As $\mathrm{supp}(\mu^E) \subset E$ and $\mu^E \neq 0$, we reach a contradiction!

Connection to Riccati type equation with measure data

Consider an equation with super-linear growth in the gradient

$$-\Delta u = |\nabla u|^q + \mu \text{ in } \mathbb{R}^n. \tag{5}$$

Here $u \in W^{1,q}_{\text{loc}}(\mathbb{R}^n)$ and $\mu \in M^+(\mathbb{R}^n)$.

Theorem (Hansson-May'za-Verbitsky 1999)

Let $q > 1$ and $n \geq 2$.
(i) *If* (5) *has a solution $u \in W^{1,q}_{\text{loc}}(\mathbb{R}^n)$ then*

$$\mu(K) \leq C \operatorname{cap}_{1,q'}(K) \tag{6}$$

for all compact sets $K \subset \mathbb{R}^n$, with C independent of K.
(ii) *There exists a constant $C_0 = C_0(n, q)$ such that if* (6) *holds with $C \leq C_0$, then* (5) *has a solution $u \in W^{1,q}_{\text{loc}}(\mathbb{R}^n)$.*

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Necessary conditions

Lemma (Hansson-Maz'ya-Verbitsky 1999)

Let $q > 1$ and $\mu \in M^+(\mathbb{R}^n)$. If (5) has a solution $u \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} \varphi^{q'} d\mu \leq (q'-1)^{q'-1} \int_{\mathbb{R}^n} |\nabla \varphi|^{q'} dx$$

and

$$\int_{\mathbb{R}^n} \varphi^{q'} |\nabla u|^q dx \leq (q')^{q'} \int_{\mathbb{R}^n} |\nabla \varphi|^{q'} dx$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n), \varphi \geq 0$.

- **Idea of the proof: Use $\varphi^{q'}$ as a test function for (5).**
- **Liouville exponent: $q_0 = \frac{n}{n-1}$.**
- **A priori estimate $\nabla u \in \mathcal{M}^{q,1,q'}$.**

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Nonlinear setting: The three model equations

Lane-Emden type:

$$-\Delta_p u = u^q + \mu, \qquad u \geq 0.$$

$$F_k[-u] = u^q + \mu, \qquad u \geq 0.$$

Riccati type:

$$-\Delta_p u = |\nabla u|^q + \mu.$$

Stationary Navier-Stokes:

$$\begin{cases} -\Delta U + U \cdot \nabla U + \nabla P &= F, \\ \mathrm{div}\, U &= 0. \end{cases}$$

$$U = (U_1, U_2, \ldots, U_n), \qquad F = (F_1, F_2, \ldots, F_n).$$

- **Here μ is a non-negative measure or even a signed distribution for Riccati type equations.**

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The p-Laplacian

$$\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u), \qquad p > 1.$$

• **In most cases it can be replaced by a more general quasilinear operator**

$$\mathcal{L}_p[u] := \mathrm{div}\mathcal{A}(x, \nabla u),$$

where $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n)$ satisfies certain growth and monotonicity conditions:

$$\mathcal{A}(x,\xi)\cdot\xi \approx |\xi|^p, \quad [\mathcal{A}(x,\xi_1) - \mathcal{A}(x,\xi_2)]\cdot(\xi_1 - \xi_2) > 0$$

for all x, ξ, and $\xi_1 \neq \xi_2$ in $\mathbb{R}^n$.

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The k-Hessian

Let $k = 1, 2, \ldots, n$. **It is defined that**

$F_k[u] =$ **the sum of** $k \times k$ **principal minors of the Hessian matrix** $\nabla^2 u$.

That is,

$$F_k[u] = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k},$$

where $\lambda_1, \ldots, \lambda_n$ **are eigenvalues of** $\nabla^2 u$. **In particular,**

$$F_1[u] = \Delta u, \qquad F_n[u] = \det(\nabla^2 u).$$

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Pointwise estimates for $-\Delta_p u = \mu$

Theorem (Kilpeläinen-Malý 1994)

Let $r = \mathrm{dist}(x, \partial\Omega)$. If $-\Delta_p u = \mu \in M^+(\Omega)$, $u \geq 0$ in Ω then

$$c_1 \mathbf{W}_{1,\,p}^{\frac{r}{3}} \mu(x) \leq u(x) \leq c_2 \mathbf{W}_{1,\,p}^{r} \mu(x) + \inf_{B_{r/3}(x)} u,$$

where

$$\mathbf{W}_{1,\,p}^{r} \mu(x) = \int_0^r \left[\frac{\mu(B_t(x))}{t^{n-p}} \right]^{\frac{1}{p-1}} \frac{dt}{t}.$$

For $\Omega = \mathbb{R}^n$ and $\inf_{\mathbb{R}^n} u = 0$ we have neater bounds

$$c_1 \mathbf{W}_{1,\,p} \mu(x) \leq u(x) \leq c_2 \mathbf{W}_{1,\,p} \mu(x).$$

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Characterization of Hölder continuity for $-\Delta_p u = \mu$

Corollary (Kilpeläinen-Malý 1994)

Let u be a solution of $-\Delta_p u = \mu \geq 0$ in Ω.
(i) *If there exists $\epsilon > 0$ such that $\mu(B_r(x)) \leq C\, r^{n-p+\epsilon}$ whenever $B_{2r}(x) \subset \Omega$, then $u \in C^{\gamma}_{\mathrm{loc}}(\Omega)$ for some $\gamma > 0$.*
(ii) *Conversely, if $u \in C^{\gamma}_{\mathrm{loc}}(\Omega)$ for some $\gamma > 0$ then*

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Pointwise estimates for $-\Delta_p u = \mu$

Global estimate on bounded domains with zero boundary data:

Theorem (P.-Verbitsky 2008)

Let μ be a finite signed measure in Ω. Suppose that u is a renormalized solution to

$$\begin{cases} -\Delta_p u = \mu \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Then

$$|u(x)| \leq C\, \mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)} |\mu|(x), \qquad \forall x \in \Omega.$$

- **One can replace Δ_p with $\mathcal{L}_p[\cdot] := \mathrm{div}\mathcal{A}(x, \nabla \cdot)$.**

Notion of renormalized solutions

For each integer $k > 0$ the truncation

$$T_k(u) := \max\{-k, \min\{k, u\}\}$$

belongs to $W_0^{1,p}(\Omega)$ and satisfies

$$-\mathrm{div}\, \mathcal{A}(x, \nabla T_k(u)) = \mu_k$$

in the sense of distributions in Ω for a finite measure μ_k in Ω. Moreover, if we extend both μ and μ_k by zero to $\mathbb{R}^n \setminus \Omega$ then

$$|\mu_k| \to |\mu|$$

weakly as measures in $\mathbb{R}^n$.

- **Fundamental solution:** $u(x) = c\,|x|^{\frac{p-n}{p-1}}$ **for** $p \neq n$**, and** $u(x) = -c\,\log|x|$ **for** $p = n$.

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Pointwise estimates for $F_k[-u] = \mu$

Recall that for $\alpha > 0$, $s > 1$

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k-**Hessian equations:** $\alpha = \frac{2k}{k+1}, \quad s = k+1.$

Theorem (Labutin 2002)

Let $r = \mathrm{dist}(x, \partial\Omega)$. *If* $F_k[-u] = \mu \in M^+(\Omega)$, $u \geq 0$ *in* Ω *then*

$$\mathbf{W}^{\frac{r}{3}}_{\frac{2k}{k+1},\, k+1}\mu(x) \lesssim u(x) \lesssim \mathbf{W}^r_{\frac{2k}{k+1},\, k+1}\mu(x) + \inf_{B_{r/3}(x)} u.$$

For $\Omega = \mathbb{R}^n$ *and* $\inf_{\mathbb{R}^n} u = 0$ *we have neater bounds*

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Notion of k-convexity (k-subharmonicity)

Definition

A function $v : \Omega \to [-\infty, \infty)$ is k-convex if v is USC and if whenever the graph of a quadratic polynomial q touches the graph of v from above at some point in Ω then $F_k[q] \geq 0$.

Remark: 1-convexity is subharmonicity and n-convexity is the convexity in the usual sense.

A function $v \in C^2(\Omega)$ **is** k**-convex in** Ω **if and only if** $F_j[v] \geq 0$ **in** Ω **for all** $j = 1, 2, \ldots, k$**.**

Trudinger-Wang 1999: If v **is** k**-convex in** Ω **then** $F_k[v]$ **can be understood as a Borel measure** $\mu_k[v]$ **in** Ω**. Moreover, if** $v \in C^2(\Omega)$ **then** $\mu_k[v] = F_k[v]$**.**

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Trudinger-Wang 1999: If v is k-convex in Ω then $F_k[v]$ can be understood as a Borel measure $\mu_k[v]$ in Ω. Moreover, if $v \in C^2(\Omega)$ then $\mu_k[v] = F_k[v]$.

Notion of k-convexity (k-subharmonicity)

Definition

A function $v : \Omega \to [-\infty, \infty)$ is k-convex if v is USC and if whenever the graph of a quadratic polynomial q touches the graph of v from above at some point in Ω then $F_k[q] \geq 0$.

Remark: 1-convexity is subharmonicity and n-convexity is the convexity in the usual sense.

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Relation to Riesz's potentials: Wolff type inequalities

Original Wolff's inequality: $1 < \alpha < n$, $s > 1$,

$$\int_{\mathbb{R}^n} (\mathbf{I}_\alpha \mu)^{\frac{s}{s-1}} dx \simeq \int_{\mathbb{R}^n} \mathbf{W}_{\alpha,\, s}\mu(x) d\mu(x).$$

A variant of Wolff's inequality (P-Verbitsky 2008): For $q > s - 1 > 0$, $1 < \alpha s < n$,

$$\|\mathbf{W}_{\alpha,s}\mu\|_{L^q}^q \simeq \|\mathbf{I}_{\alpha s}\mu\|_{L^{q/(s-1)}}^{q/(s-1)}$$

$$\simeq \|\mathbf{M}_{\alpha s}\mu\|_{L^{q/(s-1)}}^{q/(s-1)} \text{ (Muckenhoupt-Wheeden)}.$$

Here $\mathbf{M}_{\alpha s}$ is a fractional maximal function:

$$\mathbf{M}_{\alpha s}\mu(x) = \sup_{t>0} \frac{\mu(B_t(x))}{t^{n-\alpha s}}.$$

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Relation to Riesz's potentials: Wolff type Inequalities

Explicitly,

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Relation to Riesz's potentials: Wolff type Inequalities

- **Loosely speaking, $\mathbf{W}_{\alpha,s}\mu$ behaves like $(\mathbf{I}_{\alpha s}\mu)^{\frac{1}{s-1}}$.**

- **One can replace dx by $w(x)dx$ for any weight $w \in A_\infty$ (Muckenhoupt-Wheeden 1974).**

- $[\mathbf{W}_{\alpha,s}(\cdot)]^{s-1} : \mathcal{L} \to \mathcal{L}$, $[\mathbf{W}_{\alpha,s}(\cdot)]^{s-1} : \mathcal{M} \to \mathcal{M}$ **with explicit indices.**

- **This gives the precise mapping property of the solution operator.**

Application to quasilinear Lane-Emden type equations

Theorem (P.-Verbitsky, 2008)

Let $q > p-1$, $1 < p < n$, and $\mu \in M^+(\mathbb{R}^n)$. Then

$$\begin{cases} -\Delta_p u = u^q + \mu \text{ in } \mathbb{R}^n, \\ \inf_{\mathbb{R}^n} u = 0, \end{cases}$$

$$\Updownarrow$$

$$\mu(K) \leq C \,\mathrm{cap}_{p,\, \frac{q}{q-p+1}}(K).$$

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$$\mathbf{W}_{1,\,p}(\mathbf{W}_{1,\,p}\mu)^q \leq C\, \mathbf{W}_{1,\,p}\mu \qquad a.e.$$

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Relation to semilinear equations

- **Note that $\frac{q}{q-p+1} = \left(\frac{q}{p-1}\right)'$. Liouville exponent $q_0 = \frac{n(p-1)}{n-p}$.**
- **From the pointwise bounds by Wolff's potential, heuristically we are dealing with an integral equation of the form**

$$u \approx \mathbf{W}_{1,p}(u^q) + \mathbf{W}_{1,p}\mu.$$

- **Recall that**

$$\mu(K) \leq C\,\mathrm{cap}_{p,\,\frac{q}{q-p+1}}(K)$$
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$$u = \mathbf{I}_p(u^{q/(p-1)}) + \mathbf{I}_p\mu$$

As $\mathbf{I}_p = (-\Delta)^{-p/2}$, in some sense we have the equivalence

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Quasilinear Lane-Emden type equations

The proof of the implication

$$\begin{cases} -\Delta_p u = u^q + \mu \text{ in } \mathbb{R}^n, \\ \inf_{\mathbb{R}^n} u = 0. \end{cases} \Rightarrow \mu(K) + \int_K u^q dx \leq C \operatorname{cap}_{p, \frac{q}{q-p+1}}(K).$$

Proof: Let $\nu = u^q + \mu$. By the lower Wolff potential estimate

$$C\, \mathbf{W}_{1,\,p}\nu(x) \leq u(x) \quad \forall x \in \mathbb{R}^n.$$

From this we obtain

$$(\mathbf{W}_{1,\,p}\nu)^q dx \leq C\, u^q(x) dx \leq C\, d\nu.$$

$$\Downarrow$$

$$\int_{\mathbb{R}^n} (\mathbf{W}_{1,\,p}\nu)^q (\mathbf{M}_\nu g)^{\frac{q}{p-1}} dx \leq C \int_{\mathbb{R}^n} (\mathbf{M}_\nu g)^{\frac{q}{p-1}} d\nu$$

for all $g \in L_\nu^{\frac{q}{p-1}}$. Here $\mathbf{M}_\nu$ denotes the centered H-L maximal function associated to ν defined by

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continued...

$$\mathbf{M}_\nu f(x) = \sup_{r>0} \frac{\int_{B_r(x)} |f| \, d\nu}{\nu(B_r(x))}.$$

Since $\mathbf{M}_\nu$ is bounded on $L^s_\nu(\mathbb{R}^n)$, $s > 1$, we obtain

$$\int_{\mathbb{R}^n} (\mathbf{W}_{1,\,p}\nu)^q (\mathbf{M}_\nu g)^{\frac{q}{p-1}} dx \leq C \int_{\mathbb{R}^n} g^{\frac{q}{p-1}} d\nu.$$

From this inequality and the estimate

$$[\mathbf{W}_{1,\,p}\nu(x)]^q [\mathbf{M}_\nu g(x)]^{\frac{q}{p-1}} \geq C \, [\mathbf{W}_{1,\,p}(g d\nu)(x)]^q$$

we deduce

$$\int_{\mathbb{R}^n} [\mathbf{W}_{1,\,p}(g d\nu)(x)]^q dx \leq C \int_{\mathbb{R}^n} g^{\frac{q}{p-1}} d\nu.$$

continued...

Thus by Wolff's inequality one gets

$$\int_{\mathbb{R}^n} [\mathbf{I}_p(g d\nu)(x)]^{\frac{q}{p-1}} dx \leq C \int_{\mathbb{R}^n} g^{\frac{q}{p-1}} d\nu.$$

for all $g \in L_\nu^{\frac{q}{p-1}}$**,** $g \geq 0$**. Note that** $\mathbf{I}_p$ **is linear and thus by duality**

$$\int_{\mathbb{R}^n} [\mathbf{I}_p(f)(x)]^{\frac{q}{q-p+1}} d\nu \leq C \int_{\mathbb{R}^n} f^{\frac{q}{q-p+1}} dx.$$

$$\Downarrow$$

$$\nu(K) \leq C\, \mathrm{cap}_{p, \frac{q}{q-p+1}}(K).$$

Here is another proof:

Let $\nu = u^q + \mu$. By the lower Wolff potential estimate

$$C\,\mathbf{W}_{1,\,p}\nu(x) \le u(x) \quad \forall x \in \mathbb{R}^n.$$

From this we obtain, for every ball $B \subset \mathbb{R}^n$,

$$(\mathbf{W}_{1,\,p}\nu_B)^q dx \le C\,u^q(x)dx \le C\,d\nu.$$

$$\Downarrow$$

$$\int_B (\mathbf{W}_{1,\,p}\nu_B)^q dx \le C\,\nu(B).$$

$$\Downarrow \text{(localized Wolff)}$$

$$\int_B (\mathbf{I}_p\nu_B)^{\frac{q}{p-1}} dx \le C\,\nu(B) \quad \textbf{(Kerman-Sawyer)}.$$

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Equations on bounded domains

Theorem (P.-Verbitsky, 2008)

Let $q > p-1$. Suppose that $\mathrm{supp}\mu \Subset \Omega$.

$$\begin{cases} -\Delta_p u = u^q + \mu \text{ in } \Omega, \\ u \geq 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

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where $R = \mathrm{diam}(\Omega)$.

- **No restriction on Ω is needed here.**

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Some sufficient conditions

- **Simple sufficient condition:**

$$\mu = f \in L^{\frac{n(q-p+1)}{pq},\infty}(\Omega).$$

This answers a question posed by Bidaut-Veron 2002.

- **Fefferman-Phong sufficient condition:**

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Removable Singularities for $-\Delta_p u = u^q$

Theorem (P.-Verbitsky, 2008)

Let $E \subset \Omega$ be compact. Then

$$\mathrm{Cap}_{p, \frac{q}{q-p+1}}(E) = 0$$

is necessary and sufficient in order that:

$$\begin{cases} u \in L^q_{\mathrm{loc}}(\Omega \setminus E), \quad u \geq 0, \\ -\Delta_p u = u^q \quad \text{in} \quad \mathcal{D}'(\Omega \setminus E). \end{cases}$$

$$\Downarrow$$

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Remark: No information of u near E is needed.

Hessian equations of Lane-Emden type

Theorem (P.-Verbitsky, 2008)

Let $q > k$. Suppose $\mathrm{supp}\mu \Subset \Omega$, *where* Ω *is uniformly* $(k-1)$*-convex.*

$$\begin{cases} F_k[-u] = u^q + \mu \text{ in } \Omega, \\ u \geq 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \iff \mu(K) \leq C\,\mathrm{Cap}_{2k,\frac{q}{q-k}}(K).$$

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$$\mathbf{I}_{2k}^{2R}(\mathbf{I}_{2k}^{2R}\mu)^{\frac{q}{2k}} \leq C\,\mathbf{I}_{2k}^{2R}\mu \qquad a.e.,$$

where $R = \mathrm{diam}(\Omega)$.

- **$(k-1)$-convexity of Ω: $H_j(\partial\Omega) > 0, j = 1, \ldots, k-1$; H_j denotes the j-mean curvature of the boundary $\partial\Omega$.**
- **Similar result in $\mathbb{R}^n$. Liouville exponent $q_0 = \frac{nk}{n-2k}$.**

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Lane-Emden type equations with two weights

$$-\Delta_p u = \sigma u^q + \mu, \qquad u \geq 0.$$

$$F_k[-u] = \sigma u^q + \mu, \qquad u \geq 0.$$

Here σ and μ are nonnegative measures. For simplicity we will discuss these equations on the whole $\mathbb{R}^n$.

Lane-Emden type equations with two weights

Theorem (Kalton-Verbitsky 1999, P.-Verbitsky 2009)

$$-\Delta_p u = \sigma u^q + \mu, \qquad u \geq 0.$$

$$\Updownarrow$$

$$\mathbf{W}_{1,\,p}[(\mathbf{W}_{1,\,p}\mu)^q d\sigma](x) \leq C\,\mathbf{W}_{1,\,p}\mu(x).$$

$$\Updownarrow$$

$$\int_B [\mathbf{W}_{1,\,p}\mu_B(y)]^q d\sigma(y) \leq C\mu(B), \quad \forall \text{balls } B.$$

The last line is Kerman-Sawyer type condition.

Lane-Emden type equations with two weights

Moreover, these conditions are also equivalent to the following pair of conditions:

$$\int_{\mathbb{R}^n} [\mathbf{W}_{1,\,p}(g d\mu)(y)]^q d\sigma(y) \le C \int_{\mathbb{R}^n} g^{\frac{q}{p-1}} d\mu, \quad \forall g \ge 0$$

and for all $x \in \mathbb{R}^n$ **and** $r > 0$**,**

$$\int_0^r \left(\frac{\sigma(B_t(x))}{t^{n-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \cdot \left[\int_r^\infty \left(\frac{\mu(B_t(x))}{t^{n-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \right]^{\frac{q}{p-1}-1} \le C.$$

The last one is referred to as infinitesimal inequality. It can be written as

$$\mathbf{W}^r_{1,\,p}\sigma \cdot [\mathbf{W}_{1,\,p}\mu - \mathbf{W}^r_{1,\,p}\mu]^{\frac{q}{p-1}-1} \le C.$$

Lane-Emden type equations with two weights

An a priori estimate for solution:

$$\int_0^r \left(\frac{\sigma(B_t(x))}{t^{n-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \cdot \left[\int_r^\infty \left(\frac{\int_{B_t(x)} u^q d\sigma}{t^{n-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \right]^{\frac{q}{p-1}-1} \leq C.$$

Corollary

Suppose that $0 \in \Omega$ and that $u \geq 0$ is a solution to the differential inequality

$$-\Delta_p u \geq \frac{u^q}{|x|^p}, \quad q > p-1,$$

or

$$F_k[-u] \geq \frac{u^q}{|x|^{2k}}, \quad q > k.$$

Then $u \equiv 0$.

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Pointwise gradient estimates for $-\Delta_p u = \mu$

Theorem (Duzaar-Mingione 2010, Kuusi-Mingione 2013)

Let $p > 2 - 1/n$ and suppose that u solves $-\Delta_p u = \mu$ in Ω. Then for any ball $B_R(x) \subset \Omega$

$$|\nabla u(x)| \leq C\,[\mathbf{I}_1^R|\mu|(x)]^{\frac{1}{p-1}} + C \fint_{B_R(x)} |\nabla u(y)|dy,$$

where

$$\mathbf{I}_1^R\mu(x) = \int_0^R \frac{\mu(B_t(x))}{t^{n-1}}\frac{dt}{t}.$$

A history: An earlier result in the case $p \geq 2$ reads

Theorem (Duzaar-Mingione (2009?) 2011)

Let $p \geq 2$ and suppose that u solves $-\Delta_p u = \mu$ in Ω. Then for any ball $B_R(x) \subset \Omega$

$$|\nabla u(x)| \leq C\,\mathbf{W}^R_{\frac{1}{p},p}|\mu|(x) + C \fint_{B_R(x)} |\nabla u(y)|dy,$$

where

$$\mathbf{W}^R_{\frac{1}{p},p}\mu(x) = \int_0^R \left[\frac{\mu(B_t(x))}{t^{n-1}}\right]^{\frac{1}{p-1}} \frac{dt}{t}.$$

- For $p > 2$

$$\left[\mathbf{I}^R_1|\mu|(x)\right]^{\frac{1}{p-1}} \leq C\mathbf{W}^{2R}_{\frac{1}{p},p}|\mu|(x).$$

Pointwise gradient estimates for $-\Delta_p u = \mu$

A history: An earlier result in the case $p \geq 2$ reads

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Some consequences:

Corollary (Duzaar-Mingione, Kuusi-Mingione)

Let $p > 2 - 1/n$ *and suppose that* $u \in W^{1,p}(\Omega)$ *solves* $-\Delta_p u = \mu$ *in* Ω.
(i) *Lipschitz continuity criterion:*

$$\mathbf{I}_1^R|\mu| \in L^\infty_{\mathrm{loc}}(\Omega) \text{ for some } R > 0 \Longrightarrow \nabla u \in L^\infty_{\mathrm{loc}}(\Omega).$$

(i) *Criterion for the continuity of* ∇u*: If* $\lim_{R\to 0} \mathbf{I}_1^R|\mu|(x) = 0$ *locally uniformly in* x *then* ∇u *is continuous in* Ω*. In particular, this holds if* $\mu \in L^{n,1}(\Omega)$*, i.e.,*

$$\int_0^\infty |\{x \in \Omega : |\mu(x)| \geq t\}|^{\frac{1}{n}} dt < +\infty.$$

- **These criteria are independent of p.**

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Pointwise gradient estimates for $-\Delta_p u = \mu$

Equations with general structure: $-\mathrm{div}\mathcal{A}(x, \nabla u) = \mu$.

• **Growth and ellipticity conditions: for some** $p > 2 - 1/n$,

$$|\mathcal{A}(x,\xi)| \leq A|\xi|^{p-1}, \qquad \nabla_\xi |\mathcal{A}(x,\xi)| \leq A|\xi|^{p-2},$$
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for all $x \in \mathbb{R}^n$**, and** $\xi, \lambda \in \mathbb{R}^n \setminus \{0\}$.

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$$|\mathcal{A}(x_1,\xi) - \mathcal{A}(x_2,\xi)| \leq C|x_1 - x_2|^\alpha |\xi|^{p-1}.$$

This Hölder condition can also be replaced by a Dini condition.

• **Example:** $\mathrm{div}\mathcal{A}(x,\nabla u) = \mathrm{div}\Big(A(x)\nabla u \cdot \nabla u\Big)^{\frac{p-2}{2}} A(x)\nabla u, \ A \in C^\alpha$.

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Why the restriction $p > 2 - 1/n$?

- **Generally, solutions of $-\Delta_p u = \mu$ satisfy $\nabla u \in L^q_{\rm loc}$ for any $0 < q < \frac{n(p-1)}{n-1}$. Thus when $p > 2 - 1/n$, ∇u is locally integrable.**
- **$W^{1,1}$ comparison estimate (Duzaar-Mingione): With $u \in W^{1,p}_{\rm loc}(\Omega)$ being a solution of $-\Delta_p u = \mu$, let $w \in u + W^{1,p}_0(B_r)$ with $\Delta_p w = 0$ in B_r where $B_r \Subset \Omega$. Then there is a constant $C = C(n,p)$ such that**

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Comments on the proof:

- **Some sort of "interpolation" between $W^{1,1}$ and $C^{1,\alpha}$ estimates.**

- **Making use of the above $W^{1,1}$ comparison estimate.**

- **Making use of $C^{1,\alpha}$ estimate for homogeneous equations in a mean oscillation decay form: Let $w \in W^{1,p}(\Omega)$ be a solution of $\Delta_p u = 0$. Then there exist $\alpha \in (0,1)$ and $C \geq 1$ such that**

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holds for all concentric balls $B_\rho \subset B_R \subset \Omega$. Here $(\overline{\nabla w})_{B_\rho}$ is the average of ∇w over the ball B_ρ.

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Weighted gradient estimates for $-\mathrm{div}\mathcal{A}(x, \nabla u) = \mu$

When $\mathcal{A}(x,\xi)$ is no longer Hölder (or Dini) continuous in x, the pointwise estimate fails. In that case it can be replaced by a weighted estimate. The condition on $\mathcal{A}(x,\xi)$ in the x-variable is then relaxed to VMO or small BMO.

Definition

We say that $\mathcal{A}(x,\xi)$ satisfies a (δ, R_0)-BMO condition for some $\delta, R_0 > 0$ if

$$[\mathcal{A}]_{R_0} := \sup_{y\in\mathbb{R}^n,\, 0<r\leq R_0} \fint_{B_r(y)} \Upsilon(B_r(y))(x)dx \leq \delta,$$

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Weighted gradient estimates for $-\mathrm{div}\mathcal{A}(x, \nabla u) = \mu$

Muckenhoupt-Wheeden type bounds:

Theorem (P. 2014)

Let $2 - 1/n < p \leq n$, $0 < q < \infty$, and let w be an A_∞ weight. Assume that $\mathcal{A}(x,\xi)$ satisfies a (δ, R_0)-BMO condition for some small $\delta > 0$. Assume also that $\partial\Omega$ is sufficiently flat in the sense of Reifenberg. Then for any renormalized solution u to the boundary value problem

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla u) &= \mu \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{cases}$$

we have

$$\int_\Omega |\nabla u|^q w(x) dx \leq C \int_\Omega \mathbf{M}_1(\mu)^{\frac{q}{p-1}} w(x) dx.$$

- **Here $\mathbf{M}_1$ is the fractional maximal function of order 1.**
- **The solution-gradient operator maps $\mathcal{L} \to \mathcal{L}$ and $\mathcal{M} \to \mathcal{M}$, etc.**

Reifenberg flat domains

Definition

We say that Ω is a (δ, R_0)-Reifenberg flat domain if for every $x \in \partial\Omega$ and every $r \in (0, R_0]$, there exists a hyperplane $L(x,r)$ such that

$$D[\partial\Omega \cap B_r(x), L(x,r) \cap B_r(x)] \leq \delta\, r.$$

- Essentially, this says that the boundary is sufficiently flat at every boundary point and at every scale. Locally, it is trapped between two hyperplanes separated by a small positive distance proportional to the scale.
- Reifenberg 1960. Appears in minimal surfaces and free boundary problems.
- Examples: C^1 domains, Lipschitz domains with small Lipschitz constant, or even some fractal domains.

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Here is a closer look:

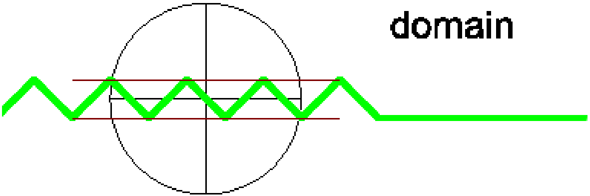

Figure: A closer look at RF domain

Gradient estimate for $\Delta_p u = \mathrm{div} F$ (distributional data)

A nonlinear singular operator: The p-harmonic transform.
Let $\mathbf{f} \in L^p(\Omega, \mathbb{R}^n)$. Consider the problem

$$\begin{cases} \Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u) = \mathrm{div}|\mathbf{f}|^{p-2}\mathbf{f}, \\ u \in W_0^{1,p}(\Omega). \end{cases} \tag{7}$$

The energy estimate (take u as a test function and IBP):

$$\int_\Omega |\nabla u|^p dx \leq \int_\Omega |\mathbf{f}|^p dx$$

The p-harmonic transform is defined by

$$\mathcal{H}_p : L^p(\Omega, \mathbb{R}^n) \to L^p(\Omega, \mathbb{R}^n)$$

$$\mathcal{H}_p(\mathbf{f}) = \nabla u.$$

The Harmonic transform

The case $p = 2$ and $\Omega = \mathbb{R}^n$: By means of Fourier transform we find

$$\begin{aligned} \mathcal{H}_2(\mathbf{f}) &= -[\mathbf{R}_{ij}]\mathbf{f} \\ &= -c(n)\,\mathrm{p.v.}\int_{\mathbb{R}^n} \frac{< x-y, \mathbf{f}(y) > (x-y)}{|x-y|^{n+2}} dy. \end{aligned}$$

Here $[\mathbf{R}_{ij}]$ is the matrix of second order Riesz transforms:

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Calderón-Zygmund 1952:

$$||\mathcal{H}_2(\mathbf{f})||_{L^q(\mathbb{R}^n)} \leq C_q \, ||\mathbf{f}||_{L^q(\mathbb{R}^n)}, \qquad \forall q > 2-1 = 1.$$

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Bad coefficient example

Meyer's example: The function $u(x) = \frac{x_1}{\sqrt{|x|}}$ **solves the equation**

$$\mathbf{div}(A(x)\nabla u) = 0 \qquad \text{in} \quad \mathbb{R}^2,$$

where

$$A(x) = \frac{1}{4|x|^2}\begin{bmatrix} 4x_1^2 + x_2^2 & 3x_1x_2 \\ 3x_1x_2 & x_1^2 + 4x_2^2 \end{bmatrix} \qquad x = (x_1, x_2).$$

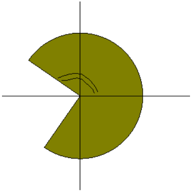
$$\nabla u \notin L^q(B_1) \qquad \forall q \geq 4.$$

Here $A(x)$ **is bounded but discontinuous at the origin!**

Bad domain example

Let $\frac{\pi}{2} < \theta_0 < \pi$ and consider the (non-convex) domain:

$$\Omega_{\theta_0} = \{(r, \theta) : 0 < r < 1 \text{ and } -\theta_0 < \theta < \theta_0\}.$$



For $\lambda = \frac{\pi}{2\theta_0} < 1$, let $u(r, \theta) = r^{\lambda} \cos(\lambda\theta)$. Then $\Delta u = 0$ in Ω_{θ_0} and $u = 0$ when $\theta = \pm\theta_0$.
Near the origin, we have $|\nabla u| = \lambda r^{\lambda - 1} = \lambda r^{\frac{\pi}{2\theta_0} - 1}$.
Thus for any $q > 4$ we can find a θ_0 (near π) such that

$$|\nabla u| \notin L^q(B_\epsilon(0) \cap \Omega_{\theta_0}), \qquad \epsilon > 0.$$

The p-harmonic transform: Basic question

Question: Is $\mathcal{H}_p$ bounded on L^q for $q > p-1$? Yes, when $q > p$. The case $p-1 < q < p$ is widely open. Difficulty: No duality available!

Theorem (Iwaniec 1983, Kinnunen-Zhou 1999, 2001)

Let $\Omega = \mathbb{R}^n$ or Ω be bounded with $C^{1,\alpha}$ boundary. Suppose that $p < q < \infty$. Then one has

$$\mathcal{H}_p : L^q(\Omega, \mathbb{R}^n) \to L^q(\Omega, \mathbb{R}^n)$$

with

$$||\mathcal{H}_p(\mathbf{f})||_{L^q} \leq C||\mathbf{f}||_{L^q}.$$

- **Byun 2004, Byun-Wang 2007: C^1 or even Reifenberg flat domains and small BMO coefficients. Caffarelli-Peral 1998: interior bounds by a perturbation technique.**
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The p-harmonic transform: Basic question

A result slightly below p:

Theorem (Iwaniec-Sbordone 1994)

Let Ω be a bounded regular domain. There exists small $\epsilon > 0$ such that for all $p - \epsilon < q < p$ one has

$$\mathcal{H}_p : L^q(\Omega, \mathbb{R}^n) \to L^q(\Omega, \mathbb{R}^n)$$

$$||\mathcal{H}_p(\mathbf{f})||_{L^q} \leq C||\mathbf{f}||_{L^q}.$$

- **Conjecture (Iwaniec 1983):** $\mathcal{H}_p$ **is bounded on** L^q **for all** $p - 1 < q < p$**.**

The p-harmonic transform: A result slightly below p

- **P. 2011, 2014: OK if we assume in addition that the solution is p-superharmonic, i.e., $\mathrm{div}\,|\mathbf{f}|^{p-2}\mathbf{f} \leq 0$ (for $\Omega = \mathbb{R}^n$ and $p > 2 - 1/n$).**
- **Adimurthi-P. 2016: Lorentz and Morrey estimates over domains whose complement is uniformly p-thick w.r.t $\mathrm{cap}_{1,p}$ (a very mild restriction on Ω).**

Definition (Uniform p-thickness)

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $1 < p < n$. We say that the complement $\Omega^c := \mathbb{R}^n \setminus \Omega$ is uniformly p-thick with constants $r_0, b > 0$, if the inequality

$$\mathrm{cap}_{1,p}(\overline{B_r(x)} \cap \Omega^c) \geq b\,\mathrm{cap}_{1,p}(\overline{B_r(x)})) = cb\, r^{n-p}$$

holds for any $x \in \partial\Omega$ and $r \in (0, r_0]$.

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The p-harmonic transform: Weighted version

Theorem (P. 2011, Mengesha-P. 2012, 2016, Adimurthi-P. 2016)

Suppose that Ω is a bounded sufficiently flat domain (in the sense of Reifenberg). Let u be a solution to

$$\begin{cases} \Delta_p u &= \operatorname{div} |\mathbf{f}|^{p-2}\mathbf{f} \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{cases}$$

Then one has the estimate

$$||\nabla u||_{L^q(\Omega,w)} \leq C\, ||\mathbf{f}||_{L^q(\Omega,w)}, \qquad \forall q \geq p,$$

provided the weight w is in the Muckenhoupt class $A_{q/p}$.

The p-harmonic transform: Weighted version

•**Recall that for $s > 1$, the Muckenhoupt class A_s consists of nonnegative functions $w \in L^1_{\rm loc}(\mathbb{R}^n)$ such that**

$$\sup_B \left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{\frac{-1}{s-1}}\right)^{s-1} < +\infty.$$

Note that $A_r \subset A_s$ for $r \leq s$. The function $w(x) = |x|^a \in A_s$ iff $-n < a < n(s-1)$.

• **This estimate gives a nonlinear version of weighted norm inequalities for singular integrals: Hunt-Muckenhoupt-Wheeden 1973 (1D), Coifman-Fefferman 1974.**

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• **The sharp result should be an estimate for $A_{q/(p-1)}$ weights. Note that $A_{q/p} \subset A_{q/(p-1)}$. But from the point of view of extrapolation theory, such a result would imply Iwaniec's conjecture.**

• **For the linear equation**

$$\begin{cases} \mathrm{div} A(x)\nabla u &= \mathrm{div}\,\mathbf{f} \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{cases}$$

where A has small BMO seminorm, one can take $w \in A_q$ instead of $A_{q/2}$ (Adimurthi-Mengesha-P. 2016, Bulicek-Diening-Schwarzacher 2016, Dong-Kim 2016).

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Some features in the theory

• **Some sort of "interpolation" between $W^{1,p}$ (energy) estimate and $W^{1,\infty}$ (or $C^{1,\alpha}$) estimate using the H-L maximal function (or the F-S sharp maximal function).**

• **Making use of the following $W^{1,p}$ (energy) comparison estimate: With $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ being a solution of $\Delta_p u = \operatorname{div} |\mathbf{f}|^{p-2}\mathbf{f}$, let $w \in u + W^{1,p}_0(B_R)$ with $\Delta_p w = 0$ in B_R where $B_R \Subset \Omega$. Then**

$$\fint_{B_R} |\nabla u - \nabla w|^p \, dx \leq C \fint_{B_R} |\mathbf{f}|^p \, dx$$

if $p \geq 2$ and

$$\fint_{B_R} |\nabla u - \nabla w|^p \, dx \leq C \left(\fint_{B_R} |\mathbf{f}|^p \, dx\right)^{p-1} \left(\fint_{B_R} |\nabla u|^p \, dx\right)^{2-p}$$

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Some features in the theory

• **For the end-point case $q = p$, we need the following $W^{1,p-\delta}$ comparison estimate: With u and w as above, then for any sufficiently small $\delta > 0$ we have**

$$\fint_{B_R} |\nabla u - \nabla w|^{p-\delta}\, dx \le C \fint_{B_R} |\mathbf{f}|^{p-\delta}\, dx + C\, \delta^{\frac{p-\delta}{p-1}} \fint_{B_R} |\nabla u|^{p-\delta}\, dx$$

if $p \ge 2$ and

$$\fint_{B_R} |\nabla u - \nabla w|^{p-\delta}\, dx \le C \Big(\fint_{B_R} |\mathbf{f}|^{p-\delta}\, dx \Big)^{p-1} \Big(\fint_{B_R} |\nabla u|^{p-\delta}\, dx \Big)^{2-p} + C\, \delta^{p-\delta} \fint_{B_R} |\nabla u|^{p-\delta}\, dx$$

if $1 < p < 2$.

• **Adimurthi-P. 2015 by Hodge decomposition (Iwaniec) or Lipschitz truncation method (Lewis).**

Some features in the theory

• **The theory works for all $p > 1$.**

• **A control by the H-L maximal function: for $q > 0$ and $w \in A_\infty$ we have**

$$\int_\Omega |\nabla u|^q w dx \leq C \int_\Omega \left[\mathbf{M}(|\mathbf{f}|^p)^{\frac{1}{p}}\right]^q w dx.$$

• **For $q = p$ one needs to lower the power p to $p - \delta$ (Adimurthi-P. 2016).**

Quasilinear Riccati type equations: measure data

Using the gradient pointwise bound of Duzaar-Mingione-Kuusi, one has

Theorem

(i) *Let* $q > p - 1$ *and suppose that* $\mathrm{supp}\mu \Subset \Omega$, $\mu \geq 0$. *Then*

$$\begin{cases} -\Delta_p u = |\nabla u|^q + \mu \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases} \Longrightarrow \mu(K) \leq C\,\mathrm{Cap}_{1, \frac{q}{q-p+1}}(K).$$

(ii) *Conversely, let* $q > p - 1$, $p > 2 - 1/n$. *If* $\partial\Omega \in C^{1,\alpha}$ *and if* μ *is a finite signed measure such that*

$$|\mu|(K) \leq C\,\mathrm{Cap}_{1, \frac{q}{q-p+1}}(K) \text{ with a small } C > 0,$$

then the above equation has a solution.

Quasilinear Riccati type equations: measure data

• **An equivalent condition:** $\mathbf{I}_1^{2R}[\mathbf{I}_1^{2R}(|\mu|)^{\frac{q}{p-1}}] \leq C\, \mathbf{I}_1^{2R}|\mu|$ **a.e., where** $R = \mathrm{diam}(\Omega)$.

• **The proof of the existence result is based on an application of Schauder Fixed Point Theorem on the set**

$$S_M := \left\{ u \in W_0^{1,1}(\Omega) : |\nabla u| \leq M \left(\mathbf{I}_1^{2R}|\mu|\right)^{\frac{1}{p-1}} \text{ a.e.} \right\}$$

for some $M > 0$. **In the case** $p > 2$, **one can also replace** $\left(\mathbf{I}_1^{2R}|\mu|\right)^{\frac{1}{p-1}}$ **with** $\mathbf{W}_{1/p,p}^{2R}|\mu|$.

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Removable singularities for $-\Delta_p u = |\nabla u|^q$

Theorem

Let $K \subset \Omega$ be a compact set and let $q > p - 1$, $p > 2 - 1/n$. Then the condition

$$\mathrm{Cap}_{1,\,\frac{q}{q-p+1}}(K) = 0$$

is necessary and sufficient so that any solution u to the problem

$$\begin{cases} u \in W^{1,q}_{\mathrm{loc}}(\Omega \setminus K), \\ -\Delta_p u = |\nabla u|^q \quad \text{in } \mathcal{D}'(\Omega \setminus K), \\ u \text{ is } p\text{-superharmonic in } \Omega, \end{cases} \tag{8}$$

is also a solution to a similar problem with Ω in place of $\Omega \setminus K$.

Equations with signed distributional data

Observation: The last existence result is sharp when $\mu \geq 0$ but it is not for sign changing (oscillatory) data.

Example: Suppose that $1 < p < n$ and $q > p$ and let $s = \frac{q}{q-p+1}$. Then $0 < s < n$. Fix $\epsilon > 0$ such that $\epsilon + s < n$ and define

$$\sigma(x) = |x|^{-\epsilon - s} \sin(|x|^{-\epsilon}).$$

Then σ^+ (hence σ) does not belong to the space $\mathcal{M}^{1,s}(B_1(0))$. Thus the theorem could not be applied to the equation

$$\begin{cases} -\Delta_p u &= |\nabla u|^q + \lambda\, \sigma \quad \text{in } B_1(0), \\ u &= 0 \quad \text{on } \partial B_1(0), \end{cases}$$

for any real number $\lambda \neq 0$.

The case $q \geq p$ with Morrey oscillatory data

Theorem (Mengesha-P. 2016)

Let $q \geq p > 1$ and assume that $\partial\Omega$ is sufficiently flat. Let $\sigma = \operatorname{div} \mathbf{f}$. Then there exists a constant $c_0 = c_0(n, p, q, \Omega)$ such that if

$$\|\mathbf{f}\|_{\mathcal{L}^{\frac{q(1+\epsilon)}{p-1}, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)} \leq c_0, \tag{9}$$

then the Riccati type equation

$$\begin{cases} -\Delta_p u &= |\nabla u|^q + \sigma \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

admits a solution $u \in W_0^{1,\, q(1+\epsilon)}(\Omega)$ with $|\nabla u| \in \mathcal{L}^{q(1+\epsilon),\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)$.

The case $q \geq p$ with Morrey oscillatory data

• **Condition (9) holds if $|\mathbf{f}| \in L^{\frac{n(q-p+1)}{p-1},\infty}$ with small norm.**

• **Let's go back to the example $\sigma(x) = |x|^{-\epsilon-s}\sin(|x|^{-\epsilon})$. It is not hard to show that one can write**

$$\sigma(x) = \operatorname{div}\mathbf{f}(x) \qquad \text{where } |\mathbf{f}| \in L^{\frac{n(q-p+1)}{p-1},\infty}.$$

Thus the last theorem is applicable to this datum.

The case $q \geq p$ with Morrey oscillatory data

• **In the case $q = p$, one can replace $\mathcal{L}^{\frac{q(1+\epsilon)}{p-1}, \frac{q(1+\epsilon)}{q-p+1}}$ with $\mathcal{M}^{\frac{q}{p-1}, 1, \frac{q}{q-p+1}}$ (Adimurthi-P., in preparation). This case was also studied by many authors Jaye-Maz'ya-Verbitsky 2012, 2013, Ferone-Murat 2000, 2014, etc. This case has a connection to the Schrödinger type equation $-\Delta_p v = \sigma v^{p-1}, v \geq 0$.**

• **In the linear case $p = 2$, the existence result holds for all $q > 1$. Moreover, the Morrey space $\mathcal{L}^{q(1+\epsilon), \frac{q(1+\epsilon)}{q-1}}$ can be replaced by the Maz'ya space $\mathcal{M}^{q,1,q'}$ (Adimurthi-Mengesha-P., in preparation).**

• **Comparison to measure data: $\mathcal{M}^{1,1,\frac{q}{q-p+1}} \subset \mathrm{div}(\mathcal{M}^{\frac{q}{p-1},1,\frac{q}{q-p+1}})$.**

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A necessary condition

Theorem

Let $p > 1$, $q > p-1$. Suppose that σ is a distribution in a bounded domain Ω such that the Riccati type equation

$$-\Delta_p u = |\nabla u|^q + \sigma \quad \text{in } \mathcal{D}'(\Omega)$$

admits a solution $u \in W^{1,q}(\Omega)$ with $|\nabla u| \in \mathcal{M}^{q,1,\frac{q}{q-p+1}}(\Omega)$. Then there exists a vector field $\mathbf{f}$ on Ω such that

$$\sigma = \operatorname{div} \mathbf{f} \quad \text{and} \quad |\mathbf{f}| \in \mathcal{M}^{\frac{q}{p-1},1,\frac{q}{q-p+1}}(\Omega).$$

- **A similar result also holds if we replace $\mathcal{M}^{q,1,\frac{q}{q-p+1}}$ with $\mathcal{L}^{q(1+\epsilon),\frac{q(1+\epsilon)}{q-p+1}}$.**

Some open problems

• **Pointwise gradient estimate for $-\Delta_p u = \mu$ in the singular case $1 < p \leq 2 - 1/n$.**

• **Estimate on the size of the singular sets of $-\Delta_p u = u^q$ and $F_k[-u] = u^q$. Conjecture: $\mathrm{cap}_{p,\frac{q}{q-p+1}}(S(u)) = 0$ for the first equation and $\mathrm{cap}_{2k,\frac{q}{q-k}}(S(u)) = 0$ for the second equation. Pacard 1992, 1993, and Adams 2012 considered the case $p = 2$ (or $k = 1$).**

• **$W^{1,q}$ estimate for $-\Delta_p u = \mathrm{div}\, |\mathbf{f}|^{p-2}\mathbf{f}$ when $p - 1 < q < p$ (Iwaniec's conjecture), or $W^{1,p}$ weighted estimate for $A_{p'}$ weights.**

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- **Estimate on the size of the singular sets of $-\Delta_p u = u^q$ and $F_k[-u] = u^q$. Conjecture: $\mathrm{cap}_{p,\frac{q}{q-p+1}}(S(u)) = 0$ for the first equation and $\mathrm{cap}_{2k,\frac{q}{q-k}}(S(u)) = 0$ for the second equation. Pacard 1992, 1993, and Adams 2012 considered the case $p = 2$ (or $k = 1$).**

- **$W^{1,q}$ estimate for $-\Delta_p u = \mathrm{div}\,|\mathbf{f}|^{p-2}\mathbf{f}$ when $p - 1 < q < p$ (Iwaniec's conjecture), or $W^{1,p}$ weighted estimate for $A_{p'}$ weights.**

- **Existence results for $-\Delta_p u = |\nabla u|^q + \mathrm{div}\,\mathbf{f}$ in the case $q > p - 1$ and $|\mathbf{f}| \in \mathcal{M}^{\frac{q}{p-1},1,\frac{q}{q-p+1}}$. This is known in the case $p = 2$ or $q = p$.**

Stationary Navier-Stokes equations

First, the Cauchy problem for non-stationary N-S equations:

$$u_t + (u \cdot \nabla)u + \nabla p = \Delta u, \quad \mathrm{div}\, u = 0, \quad u(x,0) = u_0(x).$$

$$u = u(x,t) = (u_1, u_2, \ldots u_n).$$

Time-global existence with *small* initial data:

- **T. Kato:** $u_0 \in L^n$.
- **T. Kato, Cannone, Federbush, Y. Meyer, M. Taylor:**

$$u_0 \in L^{n,\infty}, \qquad u_0 \in \mathcal{L}^{p,p}, 1 \le p \le n.$$

- **Koch-Tataru:** $u_0 \in BMO^{-1}$.
- **Bourgain-Pavlović: Ill-posedness in** $\dot{B}^{-1}_{\infty,\infty}$.
- **Yoneda 2010: Ill-posedness in a space smaller than** $\dot{B}^{-1}_{\infty,q}$ **for any** $q > 2$.

Stationary Navier-Stokes equations

For $1 \leq p < \infty, 0 < \lambda \leq n$**,**

$$\|f\|_{\mathcal{L}^{p,\lambda}} = \sup_{B_R} \left(R^{\lambda - n} \int_{B_R} |f|^p dx \right)^{\frac{1}{p}}.$$

$$\|f\|_{BMO^{-1}} = \sup_{B_R} \left(\frac{1}{|B_R|} \int_{B_R} \int_0^{R^2} |e^{t\Delta} f(y)|^2 dt dy \right)^{\frac{1}{2}}.$$

$$\|f\|_{B^{-1}_{\infty,\infty}} = \sup_{t>0} t^{\frac{1}{2}} \left\| e^{t\Delta} f(\cdot) \right\|_{L^\infty}.$$

One has the continuous emdeddings: $1 \leq p \leq n$

$$L^n \subset L^{n,\infty} \subset \mathcal{L}^{p,p} \subset BMO^{-1} \subset \dot{B}^{-1}_{\infty,\infty}.$$

Critical spaces:

$$\|f\|_E = \|\lambda f(\lambda \cdot)\|_E, \qquad \forall \lambda > 0.$$

Stationary Navier-Stokes equations

Stationary Navier-Stokes:

$$-\Delta U + U \cdot \nabla U + \nabla P = F, \qquad \mathrm{div}\, U = 0.$$

$$U = (U_1, U_2, \ldots, U_n), \qquad F = (F_1, F_2, \ldots, F_n).$$

It is invariant under the scaling

$$(U, P, F) \mapsto (U_\lambda, P_\lambda, F_\lambda),$$

where

$$U_\lambda = \lambda U(\lambda \cdot), \quad P_\lambda = \lambda^2 P(\lambda \cdot), \quad F_\lambda = \lambda^3 F(\lambda \cdot) \quad \forall \lambda > 0.$$

Stationary Navier-Stokes equations

Integral form:

$$U = \Delta^{-1}\mathbb{P}\nabla \cdot (U \otimes U) - \Delta^{-1}\mathbb{P}F, \tag{10}$$

where $\mathbb{P} := Id - \nabla\Delta^{-1}\nabla\cdot$ **is the Laray projection onto the divergence-free vector fields.**

A simple observation:

$$|\Delta^{-1}\nabla \cdot A| \leq C\,\mathbf{I}_1|A|.$$

Thus equation (10) **can be treated similarly to the integral equation**

$$u = \mathbf{I}_1(u^2) + f.$$

The role of $\mathcal{L}^{2,2}$

$\mathcal{L}^{2,2}$ is the largest Banach space $E \subset L^2_{\mathrm{loc}}(\mathbb{R}^n)$ that is invariant under translation and that $\|\lambda U(\lambda \cdot)\|_E = \|U\|_E$.
Thus one is tempted to look for solutions in $\mathcal{L}^{2,2}$ under the smallness condition

$$\left\|(-\Delta)^{-1} F\right\|_{\mathcal{L}^{2,2}} \leq \epsilon.$$

However, it seems impossible to prove such existence results under this condition as for $U \in \mathcal{L}^{2,2}$ the matrix $U \otimes U$ would belong to $\mathcal{L}^{1,2}$, but unfortunately the first order Riesz potentials of functions in $\mathcal{L}^{1,2}$ may not even belong to $L^2_{\mathrm{loc}}(\mathbb{R}^n)$.

- **Kozono-Yamazaki 1995: Existence in the space $\mathcal{L}^{2+\epsilon,\,2+\epsilon}$, $\epsilon > 0$.**

The space $\mathcal{V}^{2,2}$

We define

$$\mathcal{V}^{2,2}(\mathbb{R}^n) := \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^n) : \|u\|_{\mathcal{V}^{2,2}(\mathbb{R}^n)} < +\infty\},$$

where

$$\|u\|_{\mathcal{V}^{2,2}(\mathbb{R}^n)} = \sup_{K \subset \mathbb{R}^n} \left[\frac{\int_K u^2 dx}{\mathrm{cap}_{1,2}(K)}\right]^{\frac{1}{2}}.$$

That is, $\mathcal{V}^{2,2}(\mathbb{R}^n) = \mathcal{M}^{2,1,2}(\mathbb{R}^n)$**.**
Embeddings:

$$\mathcal{L}^{2+\epsilon,2+\epsilon} \subset \mathcal{V}^{2,2} \subset \mathcal{L}^{2,2}, \qquad \forall \epsilon > 0.$$

Main result

Theorem (Phan-P., 2013)

There exists a sufficiently small number $\delta_0 > 0$ *such that if* $\left\|(-\Delta)^{-1}F\right\|_{\mathcal{V}^{2,2}} < \delta_0$, *then the equation* (10) *has unique solution* U *satisfying*

$$||U||_{\mathcal{V}^{2,2}} \leq C \left\|(-\Delta)^{-1}F\right\|_{\mathcal{V}^{2,2}}.$$

Stationary Navier-Stokes equations

Bilinear estimates: Let

$$B(U, V) = \Delta^{-1}\mathbb{P}\nabla \cdot (U \otimes V).$$

One has

$$B : \mathcal{V}^{2,2} \times \mathcal{V}^{2,2} \to \mathcal{V}^{2,2}$$

with

$$\|B(U, V)\|_{\mathcal{V}^{2,2}} \leq C \|U\|_{\mathcal{V}^{2,2}} \|V\|_{\mathcal{V}^{2,2}} .$$

Stability results

Let $U \in \mathcal{V}^{2,2}$ be the solution of (10) with external force F satisfying

$$\left\|(-\Delta)^{-1} F\right\|_{\mathcal{V}^{2,2}} < \delta_0.$$

Consider the Cauchy problem

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \Delta u + F, & \text{in } \mathbb{R}^n \times [0, \infty), \\ \nabla \cdot u = 0, & \text{in } \mathbb{R}^n \times [0, \infty), \\ u(0) = u_0, & \text{in } \mathbb{R}^n, \end{cases} \tag{11}$$

where $u_0 \in \mathcal{V}^{2,2}$ with $\mathrm{div} u_0 = 0$.

Goal: To show that for u_0 near U there exists a unique time-global solution $u(t)$ of (11) such that as time $t \to \infty$ we have $u(t) \to U$ in some sense.

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Stability results

Theorem (Phan-P., 2013)

Let $\sigma_0 \in (1/2, 1)$. There exists a number $0 < \delta_1 \leq \delta_0$ such that for $||(-\Delta)^{-1}F||_{\mathcal{V}^{2,2}} < \delta_1$, the following results hold:
There is a positive number $\epsilon_0 > 0$ such that for every u_0 satisfying $||u_0 - U||_{\mathcal{V}^{2,2}} < \epsilon_0$, there exists uniquely a time-global solution $u(x,t)$ of (11) *with the initial condition being understood as*

$$\sup_{t>0} t^{\alpha/2} ||(-\Delta)^{\frac{\alpha}{2}}[u(\cdot,t) - u_0]||_{\mathcal{V}^{2,2}} \leq C||u_0 - U||_{\mathcal{V}^{2,2}}$$

for all $\alpha \in [-1, 0]$. Moreover, for every $\sigma \in [0, \sigma_0]$, the solution u enjoys the time-decay estimate

$$||(-\Delta)^{\frac{\sigma}{2}}[u(\cdot,t) - U]||_{\mathcal{V}^{2,2}} \leq C\, t^{\frac{-\sigma}{2}} ||u_0 - U||_{\mathcal{V}^{2,2}}. \tag{12}$$

Stability results

Remarks:

- **When $\sigma = 0$, the estimate (12) provides the Lyapunov stability of the stationary solution U. Moreover, it also implies that the solution u remains in $\mathcal{V}^{2,2}$ at all time.**
- **When $\sigma \in (0, \sigma_0]$, we have the asymptotic stability**

$$\lim_{t \to +\infty} \|(-\Delta)^{\frac{\sigma}{2}}[u(\cdot, t) - U]\|_{\mathcal{V}^{2,2}} = 0.$$

- **Kozono-Yamazaki 1995: Stability in smaller Morrey spaces.**

THANK YOU FOR YOUR ATTENTION!