Bilinear operators, commutators and smoothing

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Outline of the lectures

- Review of linear theory: classical results and tools
- Bilinear Calderón-Zygmund and related operators
- Commutators
- Compact bilinear operators
- The weighted case
- Beyond bilinear Calderón-Zygmund theory
- Smoothing of commutators
- Further connections

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The space of bounded mean oscillations

John-Nirenberg ('61): Given $f \in L^1_{loc}(\mathbb{R}^n)$, $f \in BMO$ if

$$\|f\|_{BMO} := \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q|\, dx = \fint_Q |f - f_Q| < \infty$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$. Equivalently, one could consider the supremum over all balls in $\mathbb{R}^n$.

1) BMO is a Banach space (identify f, g with $f - g =$const.),

2) It is invariant under translations and dilations,

3) It is a lattice, and

4) $L^\infty \subset BMO$.

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The L^p and Orlicz characterizations

For all $0 < p < \infty$,

$$\sup_Q \left(\fint_Q |f(x) - f_Q|^p \, dx \right)^{1/p} \leq C_{p,n} \|f\|_{BMO}.$$

In fact,

$$\|f\|_{BMO} \sim \sup_Q \left(\fint_Q |f(x) - f_Q|^p \, dx \right)^{1/p}, 1 < p < \infty.$$

Moreover,

$$\|f\|_{BMO} \sim \sup_Q \|f - f_Q\|_{\exp L, Q}.$$

(with $\varphi(t) = e^t - 1$, $\|g\|_{\varphi, Q} := \inf \left\{ \lambda > 0 : \fint_Q \varphi\left(\frac{|g(x)|}{\lambda}\right) \leq 1 \right\}$.)

Proof: John-Nirenberg's inequality.

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Muckenhoupt weights

- The A_p class: for $w \geq 0$, $1 < p < \infty$,

$$w \in A_p \Leftrightarrow [w]_{A_p} := \sup_Q \left(\fint_Q w \right) \left(\fint_Q w^{1-p'} \right)^{\frac{p}{p'}} < \infty.$$

- The Hardy-Littlewood maximal function:

$$M(f)(x) = \sup_{Q \ni x} \fint_Q f(y)\, dy$$

Theorem (Muckenhoupt, '72; Coifman-Fefferman, '74)

If $p \in (1, \infty)$, $w \in A_p$ if and only if $M : L^p(w) \to L^p(w)$ is bounded.

Proof: All A_p weights satisfy a reverse Hölder inequality.

A_p and BMO

- The A_1 class: for $w \geq 0$,

$$w \in A_1 \Leftrightarrow M(w)(x) \lesssim w(x).$$

- The A_∞ class: $A_\infty = \cup_{p \geq 1} A_p$.

Theorem

(a) $w \in A_\infty \Rightarrow \log(w) \in BMO$.
(b) $f \in BMO \Rightarrow e^{\delta f} \in A_\infty$, *for sufficiently small* $\delta > 0$.

The **canonical example** is the Hilbert transform:

$$H(f)(x) = \text{p.v.}\ \frac{1}{x} * f = \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{f(x-y)}{y}, f \in \mathcal{S}(\mathbb{R}).$$

Formally,

$$H(f)(x) = \int \underbrace{\kappa(x-y)}_{k(x,y)} f(y)\, dy, \kappa(x) = \frac{1}{x};$$

$$\widehat{H(f)}(\xi) = -i\text{sign}\,(\xi)\widehat{f}(\xi).$$

$k(x,y)$ is singular along the diagonal $\Delta := \{(x,y) \in \mathbb{R}^2 : x = y\}$,
yet $H : L^p \to L^p, 1 < p < \infty$ and $H : L^\infty \to BMO$.
The n-dimensional versions of H are the Riesz transforms.

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Definition of Calderón-Zygmund kernel and operator

It captures the essential properties (of the kernel) of the Hilbert transform:

Definition

$K : \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \to \mathbb{C}$ is a Calderón-Zygmund (CZ) kernel if

$$|K(x,y)| \lesssim |x-y|^{-n},$$

$$|\nabla_{x,y} K(x,y)| \lesssim |x-y|^{-n-1}.$$

Definition

T is a (linear) Calderón-Zygmund operator if

(a) T is bounded on $L^2(\mathbb{R}^n)$;

(b) There exists a CZ kernel K such that for all $f \in \mathcal{D}$,

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Boundedness of Calderón-Zygmund operators

Theorem (Calderón-Zygmund, '50s)

Let T be a CZ operator. Then:

(1) $T : L^p \to L^p, 1 < p < \infty$;

(2) $T : L^1 \to L^{1,\infty}$;

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Theorem (Coifman-Fefferman, '74)

Let T be a CZ operator. Then:

(a) $T : L^p(w) \to L^p(w)$ *for all* $w \in A_p, 1 < p < \infty$;

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Pseudo-differential operators as CZ operators

$$T_\sigma(f)(x) := \int_{\mathbb{R}^n} \sigma(x,\xi)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi$$

- $\sigma(x,\xi) = m(x) : T_\sigma(f) = m \cdot f$
- $\sigma(x,\xi) = m(\xi) : T_\sigma(f) = m * f$

$$\sigma \in S^m_{\rho,\delta} \Leftrightarrow |\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \lesssim (1+|\xi|)^{m-\rho|\beta|+\delta|\alpha|},\ \forall x,\xi,\ \forall \alpha,\beta$$

Proposition (Mihlin, '50s; also Hörmander)

If $\sigma \in S^0_{1,0}$, then T_σ is a Calderón-Zygmund operator.

Proposition (Bourdaud, '80s)

If $\sigma \in S^0_{1,1}$, then T_σ has CZ kernel, but is not necessarily a CZ operator.

The Coifman-Rochberg-Weiss commutator

Let H be the Hilbert transform.
Define

$$[H, b](f)(x) = b(x)H(f)(x) - H(bf)(x)$$
$$= p.v. \int \frac{b(x) - b(y)}{x - y} f(y)\, dy$$

If $f \in L^p$ and $b \in L^q$, $1 < p < \infty$, $1 < q \leq \infty$, and $1/p + 1/q = 1/r$, we trivially have

$$\|[H, b](f)\|_{L^r} \lesssim \|b\|_{L^q} \|f\|_{L^p}.$$

Coifman-Rochberg-Weiss ('76): If $b \in BMO$, $f \in L^p$, $1 < p < \infty$,

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The result is also true if $H \rightsquigarrow T$ (CZ operator).

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The Cauchy integral trick

The idea of C-R-W was to represent $[T, b]$ as a Cauchy integral and use known weighted estimates for T.

Let

$$T_z(f) = e^{zb} T(e^{-zb}).$$

Then

$$[T, b](f)(x) = \frac{1}{2\pi i} \int_{|z|=\delta} \frac{T_z(f)(x)}{z^2} \, dz,$$

and to estimate the L^p-norm of $[T, b]$ with $p > 1$ it is enough to estimate

$$\sup_{|z|=\delta} \| T_z(f) \|_{L^p}$$

This can be done using weighted estimates since, for $|z| < \delta$ (small), $e^{(Re\, z)\, b}$ is an A_p weight.

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• Alvarez-Bagby- Kurtz-Pérez ('93): $L^p \to L^p$ boundedness of iterated commutators

$$[T,b]^k := [[[T,\underbrace{b],b],...,b}_{k \text{ times}}] \qquad k \geq 1,$$

provided a linear operator T satisfies weighted estimates.
• Despite the great generality of the Cauchy integral trick approach, the method can produce in some situations sharp estimates.

• Chung-Pereyra-Pérez (2011): If

$$\|T\|_{L^2(w)} \lesssim [w]_{A_2}^r,$$

then

$$\|[T,b]^k\|_{L^2(w)} \lesssim [w]_{A_2}^{r+k}.$$

• When $T = H$, the Hilbert transform on $\mathbb{R}$, or $T = R_j$, a Riesz transform on $\mathbb{R}^n$,

$$\|[T,b]^k\|_{L^2(w)} \lesssim [w]_{A_2}^{1+k};$$

this is sharp in terms of powers of the A_2-norm of the weight.

Compactness of commutators

Let $CMO = \overline{C_c^\infty}^{\|\cdot\|_{BMO}}$ (the closure of $C_c^\infty(\mathbb{R}^n)$ in the BMO norm)

- Uchiyama ('78): If $b \in CMO$, T is CZ operator and $1 < p < \infty$, then $[T, b] : L^p \to L^p$ is compact.

Relevant applications:

- Coifman, Lions, Meyer, Semmes ('93): Compensated compactness.
- Iwaniec-Sbordone ('98): Fredholm alternative for equations with CMO coefficients.
- Iwaniec-Sbordone ('92): Integrability of Jacobians.

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The argument relies on the **Fréchet-Kolmogorov characterization of precompactness in L^p**, namely

X is precompact in L^p if and only if

(1) $\sup_{f \in X} \|f\|_{L^p} < \infty$;

(2) $\lim_{A \to \infty} \|f\|_{L^p(\{|x|>A\})} = 0$ uniformly in $f \in X$;

(3) $\lim_{t \to 0} \|(f(t+\cdot) - f(\cdot)\|_{L^p} = 0$ uniformly in $f \in X$.

Bilinear operators with Calderón-Zygmund kernels

$$T(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y,z)f(y)g(z)\,dydz$$

- *Bilinear Calderón-Zygmund (BCZ) kernel*: on $\mathbb{R}^{3n}\setminus\{x=y=z\}$,

$$|\partial^\beta K(x,y,z)| \lesssim \Big(|x-y|+|y-z|+|z-x|\Big)^{-2n-|\beta|}, \quad |\beta|\le 1.$$

- *Bilinear Calderón-Zygmund operator (BCZO)*: T has a BCZ kernel **and** $T: L^{p_0}\times L^{q_0}\to L^{r_0}$, for *some* $1<p_0,q_0<\infty$, $\frac{1}{p_0}+\frac{1}{q_0}=\frac{1}{r_0}\le 1$.
- Grafakos-Torres (2002): If T is BCZO, then

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Examples of BCZO

Example (Bilinear Riesz transform)

$$R_1(f,g)(x) = \text{p.v.} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{x-y}{|(x-y, x-z)|^3} f(y)g(z)\, dydz.$$

Example (Translation invariant)

Let $\Omega \in L^1(\mathbf{R}^{2n})$ and Lipschitz of order $\epsilon \in (0,1)$, with $\int_{\mathbb{S}^{2n-1}} \Omega(u,v) d\sigma = 0$. Define

$$K_0(u,v) = |(u,v)|^{-2n} \Omega\Big(\frac{(u,v)}{|(u,v)|}\Big),$$

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Example (Bilinear Riesz transform)

$$R_1(f,g)(x) = \text{p.v.} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{x-y}{|(x-y, x-z)|^3} f(y)g(z)\, dydz.$$

Example (Translation invariant)

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$$T_\sigma(f,g)(x) = \int_{\mathbb{R}^{2n}} \sigma(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)}\, d\xi d\eta,$$

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• Based on BBMNT 2013, for $\sigma \in BS^m_{1,\delta}, 0 \le \delta < 1$, we need $m < 0$ to guarantee $T_\sigma : L^p \times L^q \to L^r$. For example, $BS^1_{1,\delta} : L^p \times L^q \not\to L^r$.

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$$2T(f,g) = \int_{\mathbb{S}^{2n-1}} \Omega(\theta,\omega) BH_{\theta,\omega}(f,g)(x)\, d\theta d\omega,$$

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Lacey-Thiele ('99), Grafakos-Li (2004): Proving that the bilinear Hilbert transform is uniformly bounded in $(\theta,\omega) \in \mathbb{S}^{2n-1}$ is highly non-trivial!

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Bilinear commutators

1. $[T, b]_1(f, g) = T(bf, g) - bT(f, g)$
2. $[T, b]_2(f, g) = T(f, bg) - bT(f, g)$
3. $[T, \mathbf{b}](f, g) = [[T, b_1]_1, b_2]_2(f, g)$, where $\mathbf{b} = (b_1, b_2)$

Formally, if T has kernel K, then

$$C_1 = \int\int K(x, y, z)(b(y) - b(x))f(y)g(z)\,dydz,$$

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Boundedness of commutators

Theorem

Let T BCZO and $b \in BMO$, $\mathbf{b} = (b_1, b_2) \in BMO \times BMO$. Then, for all $1/p + 1/q = 1/r$, $1 < p, q < \infty$:

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Are $C_1 - C_3$ "better" if $b, \mathbf{b}$ are assumed smoother?

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Compact bilinear operators

Definition (Calderón, '64)

Let X, Y, Z be normed spaces and $T : X \times Y \to Z$ bilinear operator.
T is *compact* if $\{T(x, y) : \|x\|, \|y\| \leq 1\}$ is precompact in Z.
T is *separately compact* if $T_x : Y \to Z$ and $T_y : X \to Z$ are compact for all $x \in X$, $y \in Y$.

• If T is bilinear compact, then T is separately compact. The converse is false.

• If Z is Banach, the space of compact bilinear operators is a closed linear subspace of the space of $X \times Y \to Z$ bounded operators.

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Simple examples

Example (Bounded, not separately compact)

$T : C[0,1] \times C[0,1] \to C[0,1]$ (with $\|\cdot\|_{L^\infty}$):

$$T(f,g) = f \cdot g;$$

T is not separately compact (hence not compact): $T_{f=1} = Id$ is not compact (Riesz's theorem).

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$S : C[0,1] \times C[0,1] \to C[0,1]$ (with $\|\cdot\|_{L^\infty}$):

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Digression: continuous vs separately continuous operators

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Let X, Y, Z be normed spaces and $T : X \times Y \to Z$ a bilinear op. T is *continuous* if $\|T\| < \infty$, where

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Equivalently, $T : X \times Y \to Z$ is continuous, with the product space endowed by $\|(x,y)\| := \|x\|_X + \|y\|_Y, \forall (x,y) \in X \times Y$. T is *separately continuous* if $T_x : Y \to Z$ and $T_y : X \to Z$ are continuous for all $x \in X$, $y \in Y$.

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Question: Is it possible to understand bilinear compactness via separate compactness? NO! Even if we assume ALL spaces X, Y, Z Banach, there are simple examples that show separate compactness DOES NOT IMPLY compactness.

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Commutators as compact bilinear operators

• Fernandez-da Silva (2010): The notion of compactness in multilinear setting was previously considered only in the context of interpolation.

Theorem (B.-Torres, 2013)

Let T be a BCZO, $b \in CMO$, $1/p + 1/q = 1/r$, $1 < p, q < \infty$ and $1 \leq r < \infty$. Then

$$[T, b]_1, [T, b]_2 : L^p \times L^q \to L^r \text{ are compact.}$$

Similarly, if $\mathbf{b} \in CMO \times CMO$, then

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For $0 \leq a < 2n$,

$$T_a(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_a(x,y,z)f(y)g(z)\,dydz;$$

$$|\partial^\beta K_a(x,y,z)| \lesssim (|x-y|+|y-z|+|z-x|)^{-2n+a-|\beta|},\ |\beta| \leq 1.$$

- $a = 0$: BCZ kernel

Example

$$K_a(x,y,z) = (|x-y|+|x-z|)^{-2n+a}, 0 < a < 2n;$$

T_a is the bilinear Riesz potential operator.

Fractional versions of BCZO

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Let T_a *be the bilinear fractional operator with kernel* K_a, *and* $b \in CMO$. *Then*

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$$BI_a(f,g)(x) = \int_{\mathbb{R}^n} \frac{f(x-y)g(x+y)}{|y|^{n-a}}\,dy \ldots$$

"close" to $BHT = BH_{-1,1}$ (in dimension $n = 1$).

Theorem (BDMT, 2015)

Let $0 < a < n$, $1 < p, q, r < \infty$, $\frac{1}{p} + \frac{1}{q} < 1$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{a}{n}$, and $b \in CMO$. Then,

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Commutators of BPSDOs and compactness

Corollary (to BT 2013)

Let $\sigma \in BS^0_{1,\delta}, 0 \leq \delta < 1$, $b \in CMO$, $1/p + 1/q = 1/r$, $1 < p, q < \infty$ *and* $1 \leq r < \infty$. *Then*

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Let $\sigma \in BS^0_{1,\delta}, 0 \le \delta < 1$, $b \in CMO$, $1/p + 1/q = 1/r$, $1 < p, q < \infty$ *and* $1 \le r < \infty$. *Then*

$$[T_\sigma, b]_1, [T_\sigma, b]_2 : L^p \times L^q \to L^r \text{ are compact.}$$

Similarly, if $\mathbf{b} \in CMO \times CMO$, *then*

$$[T_\sigma, \mathbf{b}] \text{ is compact.}$$

$T \in BCZO, b \in CMO \Rightarrow [T, b]_1 : L^p \times L^q \to L^r$ compact

• **The proof relies on the Fréchet-Kolmogorov-Riesz theorem** characterizing the pre-compactness of a set in L^r.

• **By density arguments**: enough to check that for $b, f, g \in C_c^\infty$:

(a) For all $\epsilon > 0$, there exists an $A > 0$ ($A = A(\epsilon)$, but independent of f and g):

$$\left(\int_{|x|>A} \left|[T, b]_1(f, g)(x)\right|^r dx\right)^{1/r} \lesssim \epsilon \|f\|_{L^p} \|g\|_{L^q}.$$

(b) For $\epsilon \in (0, 1)$ there exists a sufficiently small t_0 ($t_0 = t_0(\epsilon)$, but independent of f and g) such that for all $0 < |t| < t_0$:

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- Let $|x| > A$ (sufficiently large).

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|[T,b]_1(f,g)(x)| &\leq \int\int_{y\in \text{supp } b} |K(x,y,z)||b(y)||f(y)||g(z)|\, dydz \\
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&\lesssim \int_{y\in \text{supp } b} \frac{|f(y)|}{|x-y|^n} \int \frac{|g(z)|}{(|x-y|+|x-z|)^n}\, dz\, dy \\
&\lesssim 2^n|x|^{-n} \int_{y\in \text{supp } b} |f(y)| \left(\int (|x-y|+|x-z|)^{-nq'}\, dz\right)^{1/q'} dy\, \|g\|_{L^q} \\
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- More involved: a further decomposition is required.
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  1. a variant of the maximal truncated bilinear singular integral
  2. the smoothness estimate of the kernel
  3. the L^p boundedness of the Hardy-Littlewood maximal function
- The second commutator is handled similarly.
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If $p \in (1, \infty)$, $w \in A_p$ if and only if $M : L^p(w) \to L^p(w)$ is bounded.

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Characterization of $A_{\mathbf{p}}$ and inclusions

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Compactness of commutators: the weighted case

Theorem (BDMT, 2015)

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• The difficulty in the work of Clop-Cruz 2013 is circumventing the non-translation of weighted measure wdx. The A_p condition is crucial.

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Beyond BCZ theory: commuting $BS^1_{1,0}$ and Lip^1

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$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \lesssim (1+|\xi|+|\eta|)^{1+\delta|\alpha|-(|\beta|+|\gamma|)}.$$

Example

$\sigma_1(\xi,\eta) = (1+|\xi|^2+|\eta|^2)^{1/2}$-bilinear counterpart of the symbol of $J^1 := (I-\Delta)^{1/2}$.

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Smoothing of $[BS^1_{1,0}, Lip^1]$

Given $\sigma \in BS^1_{1,0}$, the associated BPSDO:

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Theorem (B.-Oh, 2014)

Let $\sigma \in BS^1_{1,0}$ and $a \in Lip^1$. Then, $[T_\sigma, a]_j, j = 1, 2$, are BCZOs. In particular, $[T_\sigma, a]_j : L^p \times L^q \to L^r$ for Hölder triples (p, q, r) and appropriate end-point results.

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Let $\sigma \in BS^1_{1,0}$, $a \in Lip^1$ and $b \in CMO$. Then $[[T_\sigma, a]_j, b]_k, j, k = 1, 2$, are compact bilinear operators $L^p \times L^q \to L^r$.

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With σ_1 and a as above:

$$\|T_{\sigma_1}(af,g) - aT_{\sigma_1}(f,g)\|_{L^r} \lesssim \|a\|_{Lip^1}\|f\|_{L^p}\|g\|_{L^q}.$$

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- A bilinear counterpart of the Kato-Ponce commutator estimate

$$\|[J^s, f](g)\|_{L^r} \lesssim \|\nabla f\|_{L^\infty} \|J^{s-1} g\|_{L^r} + \|J^s f\|_{L^r} \|g\|_{L^\infty}.$$

Sketch of proof B.-Oh 2014

Uses the bilinear $T1$ theorem.

Theorem (Grafakos-Torres, 2001; Hart, 2014)

Let T be an operator with BCZ kernel. Then, T is BCZO iff
(1) $T(1,1), T^{*1}(1,1), T^{*2}(1,1) \in BMO$
(2) T satisfies the bilinear weak-boundedness property.

- If K is the kernel of T_σ, the kernel of $[T_\sigma, a]_1$ is

$$K_1(x,y,z) = (a(y) - a(x))K(x,y,z).$$

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Checking the *BMO* conditions for $[T_\sigma, a]$

- Reduction to $BS^0_{1,0}$: For $f, g \in \mathcal{S}$,

$$T_\sigma(f,g) = \sum_{j=1}^{n} (T_j^1(D_j f, g) + T_j^2(f, D_j g)).$$

The symbols of T_j^1, T_j^2 are in $BS^0_{1,0}$.

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- Easy calculations show

$$([T,a]_1)^{*1} = -[T^{*1},a]_1,$$

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The bilinear weak-boundedness property

• Denote $S = [T_\sigma, a]_1$. Show that for all $x_0 \in \mathbb{R}^n$, $t > 0$ and all *normalized bump functions* ϕ_j:

$$|\langle S(\phi_1^{x_0,t}, \phi_2^{x_0,t}), \phi_3^{x_0,t}\rangle| \lesssim t^n.$$

• A normalized bump: $\phi \in C_c^\infty$, supp $(\phi) \in \{|x| < 1\}$, $|\partial^\alpha \phi(x)| \lesssim 1$, $\forall \alpha$

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Beyond bilinear Hörmander classes

Let $\psi \neq 0$, $\psi \in \mathcal{S}(\mathbb{R}^{2n})$ with

$$\operatorname{supp}\psi \subset \{(\xi,\eta) : 1 < |\xi| + |\eta| < 2\}$$

$(g_k)_{k\geq 0} \subset C^\infty(\mathbb{R}^n)$ are so that

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$\phi \in \mathcal{S}(\mathbb{R}^n) : \operatorname{supp}\widehat{\phi} \subset \{\tau \in \mathbb{R}^n : |\tau| < r\}$; $\phi_{2^{-k}}(x) := 2^{nk}\phi(2^k x)$

Definition

$\sigma_{\phi,\psi,g}(x,\xi,\eta) := \sum_{k=0}^{\infty}(g_k * \phi_{2^{-k}})(x)\psi(2^{-k}\xi, 2^{-k}\eta)$

$$\sigma_{\phi,\psi,g} \notin \cup_{\delta\in[0,1)} BS^m_{1,\delta}$$

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A new class of symbols

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Some remarks

1. $BS^0_{1,1}$ yields BCZ kernels but NOT BCZOs (failure of bilinear $T1$ theorem)

2. The class $\tilde{BS}^0_{1,1} = \{\sigma \in BS^0_{1,1} : \sigma^{*1}, \sigma^{*2} \in BS^0_{1,1}\}$ is known to be the *largest* one to produce BCZOs.

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4. The proof of B.-Oh 2014 DOES NOT apply: reducing to $BS^0_{1,1}$ does not help since $BS^0_{1,1}$ is not closed under transposition!

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Commutators of the Bilinear Hilbert Transform

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•...produces (some) boundedness of $[BHT, b]_1$ with $b \in BMO$ via the "Cauchy integral trick" of Coifman-Rochberg-Weiss.
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