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HNP.gap

Definition of $\text{Obs}(K/k)$ and $\text{Obs}_1(L/K/k)$ (Drakokhrust and Platonov [[PD85a](#), page 350], [[DP87](#), page 300]).

Let k be a number field, $L \supset K \supset k$ be a tower of finite extensions where L is normal over k . We call the group

$$\text{Obs}(K/k) = (N_{K/k}(\mathbb{A}_K^\times) \cap k^\times) / N_{K/k}(K^\times)$$

the total obstruction to the Hasse norm principle for K/k and

$$\text{Obs}_1(L/K/k) = (N_{K/k}(\mathbb{A}_K^\times) \cap k^\times) / ((N_{L/k}(\mathbb{A}_L^\times) \cap k^\times) N_{K/k}(K^\times))$$

the first obstruction to the Hasse norm principle for K/k corresponding to the tower $L \supset K \supset k$.

Theorem 1 (Drakokhrust and Platonov [[PD85a](#), page 350], [[PD85b](#), pages 789-790], [[DP87](#), Theorem 1]).

Let k be a number field, $L \supset K \supset k$ be a tower of finite extensions where L is normal over k . Let $G = \text{Gal}(L/k)$ and $H = \text{Gal}(L/K)$. Then

$$\text{Obs}_1(L/K/k) \simeq \text{Ker } \psi_1 / \varphi_1(\text{Ker } \psi_2)$$

where

$$\begin{array}{ccc} H/[H, H] & \xrightarrow{\psi_1} & G/[G, G] \\ \uparrow \varphi_1 & & \uparrow \varphi_2 \\ \bigoplus_{v \in V_k} \left(\bigoplus_{w|v} H_w/[H_w, H_w] \right) & \xrightarrow{\psi_2} & \bigoplus_{v \in V_k} G_v/[G_v, G_v] \end{array}$$

where ψ_1 , φ_1 and φ_2 are defined by the inclusions $H \subset G$, $H_w \subset H$ and $G_v \subset G$ respectively, and

$$\psi_2(h[H_w, H_w]) = x^{-1}hx[G_v, G_v]$$

for $h \in H_w = H \cap x^{-1}hx[G_v, G_v]$ ($x \in G$).

Let ψ_2^v be the restriction of ψ_2 to the subgroup $\bigoplus_{w|v} H_w/[H_w, H_w]$ with respect to $v \in V_k$ and ψ_2^{nr} (resp. ψ_2^{r}) be the restriction of ψ_2 to the unramified (resp. the ramified) places v of k .

Proposition 2 (Drakokhrust and Platonov [DP87]).

Let $k, L \supset K \supset k, G$ and H be as in Theorem 1.

- (i) ([DP87, Lemma 1]) Places $w_i \mid v$ of K are in one-to-one correspondence with the set of double cosets in the decomposition $G = \bigcup_{i=1}^{r_v} Hx_iG_v$ where $H_{w_i} = H \cap x_iG_vx_i^{-1}$;
- (ii) ([DP87, Lemma 2]) If $G_{v_1} \leq G_{v_2}$, then $\varphi_1(\text{Ker } \psi_2^{v_1}) \subset \varphi_1(\text{Ker } \psi_2^{v_2})$;
- (iii) ([DP87, Theorem 2]) $\varphi_1(\text{Ker } \psi_2^{\text{nr}}) = \Phi^G(H)/[H, H]$ where $\Phi^G(H) = \langle [h, x] \mid h \in H \cap xHx^{-1}, x \in G \rangle$;
- (iv) ([DP87, Lemma 8]) If $[K : k] = p^r$ ($r \geq 1$) and $\text{Obs}(K_p/k_p) = 1$ where $k_p = L^{G_p}$, $K_p = L^{H_p}$, G_p and H_p are p -Sylow subgroups of G and H respectively, then $\text{Obs}(K/k) = 1$.

Theorem 3 (Drakokhrust and Platonov [DP87, Theorem 3, Corollary 1]).

Let $k, L \supset K \supset k, G$ and H be as in Theorem 1. Let $H_i \leq G_i \leq G$ ($1 \leq i \leq m$), $H_i \leq H \cap G_i$, $k_i = L^{G_i}$ and $K_i = L^{H_i}$. If $\text{Obs}(K_i/k_i) = 1$ for any $1 \leq i \leq m$ and

$$\bigoplus_{i=1}^m \widehat{H}^{-3}(G_i, \mathbb{Z}) \xrightarrow{\text{cores}} \widehat{H}^{-3}(G, \mathbb{Z})$$

is surjective, then $\text{Obs}(K/k) = \text{Obs}_1(L/K/k)$. In particular, if $[K : k] = n$ is square-free, then $\text{Obs}(K/k) = \text{Obs}_1(L/K/k)$.

Theorem 4 (Drakokhrust [Dra89, Theorem 1], see also Opolka [Opo80, Satz 3]).

Let $k, L \supset K \supset k, G$ and H be as in Theorem 1. Let $\tilde{L} \supset L \supset k$ be a tower of Galois extensions with $\tilde{G} = \text{Gal}(\tilde{L}/k)$ and $\tilde{H} = \text{Gal}(\tilde{L}/K)$ which correspond to a central extension $1 \rightarrow A \rightarrow \tilde{G} \rightarrow G \rightarrow 1$ with $A \cap [\tilde{G}, \tilde{G}] \simeq M(G)$; the Schur multiplier of G (this is equivalent to the inflation $M(\tilde{G}) \rightarrow M(G)$ is zero map, see Beyl and Tappe [BT82, Proposition 2.13, page 85]). Then $\text{Obs}(K/k) = \text{Obs}_1(\tilde{L}/K/k)$. In particular, \tilde{G} is a Schur cover of G , i.e. $A \simeq M(G)$, then $\text{Obs}(K/k) = \text{Obs}_1(\tilde{L}/K/k)$.

FirstObstructionN

▸ `FirstObstructionN(G, H).ker`

returns the list $[l_1, [l_2, l_3]]$ where l_1 is the abelian invariant of the numerator of the first obstruction $\text{Ker } \psi_1 = \langle y_1, \dots, y_t \rangle$ with respect to G, H as in

Theorem 1, $l_2 = [e_1, \dots, e_m]$ is the abelian invariant of $H^{ab} = H/[H, H] = \langle x_1, \dots, x_m \rangle$ with $e_i = \text{order}(x_i)$ and $l_3 = [l_{3,1}, \dots, l_{3,t}]$, $l_{3,i} = [r_{i,1}, \dots, r_{i,m}]$ is the list with $y_i = x_1^{r_{i,1}} \cdots x_m^{r_{i,m}}$ for $H \leq G \leq S_n$.

▸ `FirstObstructionN(G).ker`

returns the same as `FirstObstructionN(G, H).ker` where $H = \text{Stab}_1(G)$ is the stabilizer of 1 in $G \leq S_n$.

FirstObstructionDnr

▸ `FirstObstructionDnr(G, H).Dnr`

returns the list $[l_1, [l_2, l_3]]$ where l_1 is the abelian invariant of the unramified part of the denominator of the first obstruction $\varphi_1(\text{Ker } \psi_2^{\text{nr}}) = \Phi^G(H)/[H, H] = \langle y_1, \dots, y_t \rangle$ with respect to G, H as in Proposition 2 (iii), $l_2 = [e_1, \dots, e_m]$ is the abelian invariant of $H^{ab} = H/[H, H] = \langle x_1, \dots, x_m \rangle$ with $e_i = \text{order}(x_i)$ and $l_3 = [l_{3,1}, \dots, l_{3,t}]$, $l_{3,i} = [r_{i,1}, \dots, r_{i,m}]$ is the list with $y_i = x_1^{r_{i,1}} \cdots x_m^{r_{i,m}}$ for $H \leq G \leq S_n$.

▸ `FirstObstructionDnr(G).Dnr`

returns the same as `FirstObstructionDnr(G, H).Dnr` where $H = \text{Stab}_1(G)$ is the stabilizer of 1 in $G \leq S_n$.

FirstObstructionDr

▸ `FirstObstructionDr(G, G_v, H).Dr`

returns the list $[l_1, [l_2, l_3]]$ where l_1 is the abelian invariant of the ramified part of the denominator of the first obstruction $\varphi_1(\text{Ker } \psi_2^v) = \langle y_1, \dots, y_t \rangle$ with respect to G, G_v, H as in Theorem 1, $l_2 = [e_1, \dots, e_m]$ is the abelian invariant of $H^{ab} = H/[H, H] = \langle x_1, \dots, x_m \rangle$ with $e_i = \text{order}(x_i)$ and $l_3 = [l_{3,1}, \dots, l_{3,t}]$, $l_{3,i} = [r_{i,1}, \dots, r_{i,m}]$ is the list with $y_i = x_1^{r_{i,1}} \cdots x_m^{r_{i,m}}$ for $H \leq G \leq S_n$.

▸ `FirstObstructionDr(G, G_v).Dr`

returns the same as `FirstObstructionDr(G, G_v, H).Dr` where $H = \text{Stab}_1(G)$ is the stabilizer of 1 in $G \leq S_n$.

SchurCoverG

▸ `SchurCoverG(G).SchurCover`

returns one of the Schur covers \tilde{G} of G in a central extension $1 \rightarrow A \rightarrow \tilde{G} \xrightarrow{\pi} G \rightarrow 1$ with $A \simeq M(G)$; Schur multiplier of G (see

Karpilovsky [Kap87, page 16]). The Schur covers \tilde{G} are stem extensions, i.e. $A \leq Z(\tilde{G}) \cap [\tilde{G}, \tilde{G}]$, of the maximal size.

▸ `SchurCoverG(G).epi`

returns the surjective map π in a central extension $1 \rightarrow A \rightarrow \tilde{G} \xrightarrow{\pi} G \rightarrow 1$ with $A \simeq M(G)$; Schur multiplier of G (see Karpilovsky [Kap87, page 16]).

The Schur covers \tilde{G} are stem extensions, i.e. $A \leq Z(\tilde{G}) \cap [\tilde{G}, \tilde{G}]$, of the maximal size.

These functions are based on the built-in function `EpimorphismSchurCover` in GAP.

MinimalStemExtensions

▸ `MinimalStemExtensions(G)[j].MinimalStemExtension`

(resp. `MinimalStemExtensions(G)[j].epi`) returns the j -th minimal stem extension $\bar{G} = \tilde{G}/A'$, i.e. $\bar{A} \leq Z(\bar{G}) \cap [\bar{G}, \bar{G}]$, of G provided by the Schur cover \tilde{G} of G via `SchurCoverG(G).SchurCover` where A' is the j -th maximal subgroups of $A = M(G)$ (resp. the surjective map $\bar{\pi}$) in the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & A = M(G) & \longrightarrow & \tilde{G} & \xrightarrow{\pi} & G \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & \bar{A} = A/A' & \longrightarrow & \bar{G} = \tilde{G}/A' & \xrightarrow{\bar{\pi}} & G \longrightarrow 1 \end{array}$$

(see Robinson [Rob96, Exercises 11.4]).

This function is based on the built-in function `EpimorphismSchurCover` in GAP.

StemExtensions

▸ `StemExtensions(G)[j].StemExtension`

(resp. `StemExtensions(G)[j].epi`) returns the j -th stem extension $\bar{G} = \tilde{G}/A'$, i.e. $\bar{A} \leq Z(\bar{G}) \cap [\bar{G}, \bar{G}]$, of G provided by the Schur cover \tilde{G} of G via `SchurCoverG(G).SchurCover` where A' is the j -th maximal subgroups of $A = M(G)$ (resp. the surjective map $\bar{\pi}$) in the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & A = M(G) & \longrightarrow & \tilde{G} & \xrightarrow{\pi} & G \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & \bar{A} = A/A' & \longrightarrow & \bar{G} = \tilde{G}/A' & \xrightarrow{\bar{\pi}} & G \longrightarrow 1 \end{array}$$

(see Robinson [[Rob96](#), Exercises 11.4]).

This function is based on the built-in function `EpimorphismSchurCover` in GAP.

Resolution of G

▸ `ResolutionNormalSeries(LowerCentralSeries(G), $n+1$)`

returns a free resolution RG of G when G is nilpotent.

▸ `ResolutionNormalSeries(DerivedSeries(G), $n+1$)`

returns a free resolution RG of G when G is solvable.

▸ `ResolutionFiniteGroup(G , $n+1$)`

returns a free resolution RG of G when G is finite.

These functions are the built-in functions of [HAP](#) in GAP.

ResHnZ

▸ `ResHnZ(RG , RH , n).HnGZ`

returns the abelian invariants of $H^n(G, \mathbb{Z})$ with respect to Smith normal form, for free resolution RG of G .

▸ `ResHnZ(RG , RH , n).HnHZ`

returns the abelian invariants of $H^n(H, \mathbb{Z})$ with respect to Smith normal form, for free resolution RH of H .

▸ `ResHnZ(RG , RH , n).Res`

returns the list $L = [l_1, \dots, l_s]$ where

$$H^n(G, \mathbb{Z}) = \langle x_1, \dots, x_s \rangle \xrightarrow{\text{res}} H^n(H, \mathbb{Z}) = \langle y_1, \dots, y_t \rangle,$$

$\text{res}(x_i) = \prod_{j=1}^t y_j^{l_{i,j}}$ and $l_i = [l_{i,1}, \dots, l_{i,t}]$ for free resolutions RG and RH of G and H respectively.

▸ `ResHnZ(RG , RH , n).Ker`

returns the list $L = [l_1, [l_2, l_3]]$ where l_1 is the abelian invariant of

$\text{Ker}\{H^n(G, \mathbb{Z}) \xrightarrow{\text{res}} H^n(H, \mathbb{Z})\} = \langle y_1, \dots, y_t \rangle$, $l_2 = [d_1, \dots, d_s]$ is the abelian invariant of $H^n(G, \mathbb{Z}) = \langle x_1, \dots, x_s \rangle$ with $d_i = \text{ord}(x_i)$ and

$l_3 = [l_{3,1}, \dots, l_{3,t}]$, $l_{3,j} = [r_{j,1}, \dots, r_{j,s}]$ is the list with $y_j = x_1^{r_{j,1}} \cdots x_s^{r_{j,s}}$ for free resolutions RG and RH of G and H respectively.

▸ `ResHnZ(RG , RH , n).Coker`

returns the list $L = [l_1, [l_2, l_3]]$ where $l_1 = [e_1, \dots, e_t]$ is the abelian invariant

of $\text{Coker}\{H^n(G, \mathbb{Z}) \xrightarrow{\text{res}} H^n(H, \mathbb{Z})\} = \langle \overline{y}_1, \dots, \overline{y}_t \rangle$ with $e_j = \text{ord}(\overline{y}_j)$,

$l_2 = [d_1, \dots, d_s]$ is the abelian invariant of $H^n(H, \mathbb{Z}) = \langle x_1, \dots, x_s \rangle$ with $d_i = \text{ord}(x_i)$ and $l_3 = [l_{3,1}, \dots, l_{3,t}]$, $l_{3,j} = [r_{j,1}, \dots, r_{j,s}]$ is the list with

$\overline{y_j} = \overline{x_1}^{r_{j,1}} \cdots \overline{x_s}^{r_{j,s}}$ for free resolutions RG and RH of G and H respectively.

KerResH3Z

KerResH3Z was improved to [ver.2019.10.04](#) from [ver.2020.03.19](#)

▸ `KerResH3Z(G,H)`

returns the list $L = [l_1, [l_2, l_3]]$ where l_1 is the abelian invariant of $\text{Ker}\{H^3(G, \mathbb{Z}) \xrightarrow{\text{res}} \bigoplus_{i=1}^{m'} H^3(G_i, \mathbb{Z})\} = \langle y_1, \dots, y_t \rangle$ where $H_i \leq G_i \leq G$, $H_i \leq H \cap G_i$, $[G_i : H_i] = n$ and the action of G_i on $\mathbb{Z}[G_i/H_i]$ may be regarded as nTm ($n \leq 15$) which is not in [HKY22, Table 1] or [HKY23, Table 1], $l_2 = [d_1, \dots, d_s]$ is the abelian invariant of $H^3(G, \mathbb{Z}) = \langle x_1, \dots, x_s \rangle$ with $d_{i'} = \text{ord}(x_{i'})$ and $l_3 = [l_{3,1}, \dots, l_{3,t}]$, $l_{3,j} = [r_{j,1}, \dots, r_{j,s}]$ is the list with $y_j = x_1^{r_{j,1}} \cdots x_s^{r_{j,s}}$ for groups G and H (cf. Theorem 3).

KerResH3Z(*G*,*H*:iterator) was added to [ver.2019.10.04](#) from [ver.2020.03.19](#)

▸ `KerResH3Z(G,H:iterator)`

returns the same as `KerResH3Z(G, H)` but using the built-in function `IteratorByFunctions` of GAP in order to run fast (by applying the new function `ChooseGilterator(G, H)` to choose suitable $G_i \leq G$ instead of the old one `ChooseGi(G, H)`).

ConjugacyClassesSubgroupsNGHOrbitRep

ConjugacyClassesSubgroupsNGHOrbitRep was added to [ver.2019.10.04](#) from [ver.2020.03.19](#)

▸ `ConjugacyClassesSubgroupsNGHOrbitRep(ConjugacyClassesSubgroups(G),H)`

returns the list $L = [l_1, \dots, l_t]$ where t is the number of subgroups of G up to conjugacy, $l_r = [l_{r,1}, \dots, l_{r,u_r}]$ ($1 \leq r \leq t$), $l_{r,s}$ ($1 \leq s \leq u_r$) is a representative of the orbit $\text{Orb}_{N_G(H) \backslash G / N_G(G_{v_r,s})}(G_{v_r,s})$ of $G_{v_r,s} \leq G$ under the conjugate action of G which corresponds to the double coset $N_G(H) \backslash G / N_G(G_{v_r,s})$ with $\text{Orb}_{G/N_G(G_{v_r})}(G_{v_r}) = \bigcup_{s=1}^{u_r} \text{Orb}_{N_G(H) \backslash G / N_G(G_{v_r,s})}(G_{v_r,s})$ corresponding to r -th subgroup $G_{v_r} \leq G$ up to conjugacy.

MinConjugacyClassesSubgroups

MinConjugacyClassesSubgroups was added to [ver.2019.10.04](#) from [ver.2020.03.19](#)

▸ `MinConjugacyClassesSubgroups(L)`

returns the minimal elements of the list $l = \{H_i^G\}$ where

$H_i^G = \{x^{-1}H_ix \mid x \in G\}$ for some subgroups $H_i \leq G$ which satisfy the condition that if $H_i^G, H_r^G \in l$ and $H_i \leq H_r$, then $H_j^G \in l$ for any $H_i \leq H_j \leq H_r$.

IsInvariantUnderAutG

IsInvariantUnderAutG was added to [ver.2019.10.04](#) from [ver.2020.03.19](#)

▸ IsInvariantUnderAutG(l)

returns true if the list $l = \{H_i^G\}$ is closed under the action of all the automorphisms $\text{Aut}(G)$ of G where $H_i^G = \{x^{-1}H_ix \mid x \in G\}$ ($H_i \leq G$). If not, this returns false.

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