Orientations for gauge-theoretic moduli problems

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Plan of talk:

1. Gauge-theoretic moduli problem
   –general picture, motivation
2. Anti-self-dual instanton moduli space
   –Atiyah–Hitchin–Singer complex, Kuranishi model
3. Orientations
   –orientability and orientations of gauge-theoretic moduli spaces

- X: smooth manifold of real dimension n
- $P \to X$: principal G-bundle over X, G: Lie group

The gauge group $\mathcal{G}_P := \mathrm{Aut}(P)$ acts on $\mathcal{A}_P$, the space of all connections on P, by $u(A) := A - (d_A u)u^{-1}$, where $u \in \mathcal{G}_P$ and $A \in \mathcal{A}_P$, $\mathcal{G}_P$ is identified with $\Gamma(P \times_{Ad} G)$ and d_A is the covariant derivative on $\Gamma(P \times_{Ad} G)$ induced by A. We denote the quotient $\mathcal{A}_P/\mathcal{G}_P$ by $\mathcal{B}_P$.

Gauge-theoretic equations (e.g. anti-self-dual instanton equations) assign a vector bundle $\mathcal{E}$ over $\mathcal{B}_P$ and a section s of $\mathcal{E}$.

$$\begin{array}{ccc} & & \mathcal{E} \\ & & \downarrow \uparrow s \\ \mathcal{M}_P & \subset & \mathcal{B}_P. \end{array}$$

We call $\mathcal{M}_P := s^{-1}(0)$ a *gauge-theoretic moduli space*.

Gauge-theoretic moduli problem

Problem: construct a (virtual) fundamental cycle out of $\mathcal{M}_P$.

Applications: *intersection theory* on the (virtual) fundamental cycles produces deformation invariants such as Donaldson invariants, Gromov–Witten, Seiberg–Witten, Donaldson–Thomas ones and so on. Furthermore, the generating functions of these invariants typically have non-trivial properties such as modularity, which could indicate the origins of these theories perhaps.

Issues: smoothness, orientability, compactness of $\mathcal{M}_P$.

- For smoothness: use Freed–Uhlenbeck perturbation, virtual techniques by Behrend–Fantechi et al., or invoke derived stacks.
- For compactness: take up Uhlenbeck, Gieseker compactifications for vector bundles/sheaves, or stable map compactification for pseudo-holomorphic curves.

Anti-self-dual instantons

- X: closed, oriented, smooth four-manifold
- $P \to X$: principal G-bundle over X, G: Lie group

Fix a Riemannian metric g on X, and consider the Hodge star operator $*_g$ on $\Lambda^2_X := (\Lambda^2 T^*X)$. This satisfies $*_g^2 = 1$, so Λ^2_X decomposes as $\Lambda^2_X = \Lambda^+_X \oplus \Lambda^-_X$.

Definition: A connection on P is said to be an *anti-self-dual instanton*, or *ASD instanton* for short, if the curvature F_A of A satisfies $F_A^+ := \pi_+(F_A) = 0$, where $\pi_+ : \Gamma(\mathfrak{g}_P \otimes \Lambda^2_X) \to \Gamma(\mathfrak{g}_P \otimes \Lambda^+_X)$ is the projection and $\mathfrak{g}_P$ is the adjoint bundle of P.

Consider $\mathcal{M}^{ASD}_{P,g} := \{A \in \mathcal{A}_P \ : \ F_A^+ = 0\}/\mathcal{G}_P$, the *anti-self-dual instanton moduli space*. (The corresponding $(\mathcal{E}, s)$ in the earlier slide is given by $\mathcal{E} := \mathcal{A}_P \times_{\mathcal{G}_P} \Omega^+_X(\mathfrak{g}_P) \to \mathcal{B}_P$ and $s := F_A^+$.)

Atiyah–Hitchin–Singer complex: the infinitesimal deformation of an anti-self-dual instanton A is described by the following elliptic complex:

$$0 \to \Gamma(\mathfrak{g}_P \otimes \Lambda^0_X) \xrightarrow{d_A} \Gamma(\mathfrak{g}_P \otimes \Lambda^1_X) \xrightarrow{d_A^+} \Gamma(\mathfrak{g}_P \otimes \Lambda^+_X) \to 0,$$

where $d_A^+ := \pi_+ \circ d_A$.

We write its cohomology by $\mathbb{H}^i_A$ for $i = 0, 1, 2$.

Denote by $\Gamma_A := \{u \in \mathcal{G}_P \,:\, u(A) = A\}$ the stabilizer group of $\mathcal{G}_P$ at $[A] \in \mathcal{B}_P$.

Definition: A connection A of P is called *irreducible* if Γ_A coincides with the centre of G and *reducible* otherwise.

Kuranishi model

Theorem (Atiyah–Hitchin–Singer) Let A be an anti-self-dual instanton. Then there exists an open neighbourhood U of 0 in $\mathbb{H}^1_A$ and a differentiable map $\kappa : U \to \mathbb{H}^2_A$ with $\kappa(0) = 0$ and the first derivative of κ vanishing at 0, which is Γ_A-equivariant if A is reducible, such that the moduli space $\mathcal{M}^{ASD}_{P,g}$ around $[A]$ is locally modeled on $\kappa^{-1}(0)/\Gamma_A$

Remark: One needs an appropriate Sobolev space setting to prove the above, for example in order to use an implicit function theorem in the infinite-dimensional setting.

Note that $\mathbb{H}^0_A = 0$, if A is irreducible. Also, if $\mathbb{H}^2_A = 0$, then $\mathbb{H}^1_A$ is the *tangent space* at $[A] \in \mathcal{M}^{ASD}_{P,g}$ for A irreducible, so $\mathbb{H}^2_A$ is the *obstruction space* to deforming the equivalence classes $[A]$ of irreducible connections in $\mathcal{M}^{ASD}_{P,g}$.

In the analytic setting, if $G = SU(2)$ or $SO(3)$, then $\mathbb{H}^2_A = 0$ for a generic choice of Riemannian metrics g on X. In addition, if b^+_X, the dimension of maximal positive subspace for the intersection form on $H_2(X, \mathbb{Z})$, is positive, then there are no reducible connections other than the trivial one again for a generic metric g. Hence we obtain:

Theorem (Atiyah–Hitchin–Singer, Freed–Uhlenbeck, Donaldson–Kronheimer) Let X be a closed, oriented, simply-connected, smooth four-manifold, and let $P \to X$ be a principal G-bundle over X. Take the structure group G of P to be $SU(2)$ or $SO(3)$, and assume that $b^+_X > 0$. Then $\mathcal{M}^{ASD}_{P,g}$ is a smooth manifold of the expected dimension for a generic choice of metrics g on the underlying four-manifold.

Remark: One can use *topological stacks* and *derived manifolds* when the above assumptions fail.

Orientations

Finite-dimensional model: let X be a smooth n-manifold.

a) X is *orientable* if the determinant bundle $L := \Lambda^n TX$ of TX is trivial.

b) An *orientation* is a choice of trivialization of L.

ASD instantons case: consider the family of elliptic operators parametrized by $\mathcal{A}_P$ given by:

$$\delta_A := (d_A^*, d_A^+) : \Gamma(\mathfrak{g}_P \otimes \Lambda_X^1) \to \Gamma(\mathfrak{g}_P \otimes (\Lambda^0 \oplus \Lambda_X^+)).$$

By the Fredholm property of elliptic operators, it has a well-defined determinant:

$$\mathcal{L}^A := \det(\operatorname{ind} \delta_A) := \det(\ker \delta_A) \otimes \det(\operatorname{coker} \delta_A)^*.$$

This defines a line bundle on $\mathcal{A}_P$, which descends to $\mathcal{L} \to \mathcal{B}_P$.

If there are no reducible connections (so $\mathbb{H}^0_A = 0$) and $\mathbb{H}^2_A = 0$ for all $[A] \in \mathcal{M}^{ASD}_{P,g}$, then $\iota^*(\mathcal{L})$ is isomorphic to the determinant line bundle of the tangent bundle of $\mathcal{M}^{ASD}_{P,g}$, where $\iota : \mathcal{M}^{ASD}_{P,g} \hookrightarrow \mathcal{B}_P$ is the natural inclusion.

Theorem (i) Donaldson, ii) Donaldson–Kronheimer) Let X be a closed, oriented, smooth 4-manifold, and let $P \to X$ be a principal G-bundle over X. Assume that either i) the structure group G of P is $U(m)$ or $SU(m)$; or ii) X is simply-connected and G is a simlpy-connected, simple Lie group. Then

a) $\mathcal{L} \to \mathcal{B}_P$ is trivial, hence the smooth part of $\mathcal{M}^{ASD}_P$ is orientable; and
b) a canonical orientation can be determined by choosing an orientation on $H^1(X)$ and $H^+(X)$.

In general, suppose we are given $E_0, E_1 \to X$ real vector bundles of the same rank over a compact manifold X, and $D : \Gamma(E_0) \to \Gamma(E_1)$, a linear elliptic operator. We write $E_\bullet := (E_0, E_1, D)$.

Let $A \in \mathcal{A}_P$. This induces a connection on $\mathfrak{g}_P \to X$. Then consider the elliptic linear operator twisted by A:

$$D^A : \Gamma(\mathfrak{g}_P \otimes E_0) \to \Gamma(\mathfrak{g}_P \otimes E_1).$$

As D^A is elliptic on a compact manifold, we have that

$$\det(D^A) = \det(\ker D^A) \otimes \det(\operatorname{coker} D^A)^*$$

is a one-dimensional vector space.

This defines a line bundle on $\mathcal{A}_P$, which descends to a line bundle $L_P^{E_\bullet} \to \mathcal{B}_P$, the *determinant line bundle* of $\mathcal{B}_P$.

We call $O_P^{E_\bullet} := (L_P^{E_\bullet} \setminus 0(\mathcal{B}_P))/(0, \infty)$ the *orientation bundle* of $\mathcal{B}_P$, where $0(\mathcal{B}_P)$ is the zero section. This is a principal $\mathbb{Z}_2$-bundle.

Definition:

a) $(\mathcal{B}_P, E_\bullet)$ is *orientable* if $O_P^{E_\bullet}$ is isomorphic to the trivial principal $\mathbb{Z}_2$-bundle $\mathcal{B}_P \times \mathbb{Z}_2$.

b) An *orientation* on $(\mathcal{B}_P, E_\bullet)$ is an isomorphism $\mathcal{B}_P \times \mathbb{Z}_2 \xrightarrow{\cong} O_P^{E_\bullet}$ of principal $\mathbb{Z}_2$-bundles.

Problems:

a) Under what condition on $X, G, P, E_\bullet$, is $(\mathcal{B}_P, E_\bullet)$ orientable?

b) If it is orientable, can we construct a natural orientation, i.e. is there a way of choosing an orientation which is independent of extra data (e.g. Riemannian metrics) on X?

[Joyce–T–Upmeier] solves these problems for various gauge-theoretic moduli spaces. One important technique is the excision theorem by Markus Upmeier, *A categorified excision principle for elliptic symbol families*, preprint, 2019.

Methods for orientations

- Excision theorems for the orientation bundles.
- Orientations from complex structures on $E_\bullet = (E_0, E_1, D)$ or G, namely, if $E_0, E_1 \to X$ are complex vector bundles, and the symbol of D is complex linear, then we have a canonical trivialization $O_P^{E_\bullet} \xrightarrow{\cong} \mathcal{B}_P \times \mathbb{Z}_2$ coming from the complex structures.
- Relating orientations of moduli spaces for Lie subgroups $H \subset G$ such as $U(m_1) \times U(m_2) \subset U(m_1 + m_2)$, $U(m) \hookrightarrow SU(m+1)$, $U(m) \hookrightarrow Sp(m)$ and so on.
- Stabilization, e.g. consider the direct limit $\mathcal{B}_{P\oplus\mathbb{C}^\infty} := \varinjlim_{k\to\infty} \mathcal{B}_{P\oplus\mathbb{C}^k}$ for a principal $U(m)$-bundle via *gluing maps* $\mathcal{B}_{P\oplus\mathbb{C}^k} \to \mathcal{B}_{P\oplus\mathbb{C}^{k+1}}$, and the direct limit of principal $\mathbb{Z}_2$-bundles $O^{E_\bullet}_{P\oplus\mathbb{C}^\infty} \to \mathcal{B}_{P\oplus\mathbb{C}^\infty}$.
- etc. see [Joyce–T–Upmeier].

For the anti-self-dual instanton moduli spaces, we obtain:

Theorem (Joyce–T–Upmeier)

Let X be a closed, oriented, smooth four-manifold, and let $P \to X$ be a principal G-bundle over X, where G is a connected Lie group.

1) Choose an orientation on $H^0(X) \oplus H^1(X) \oplus H^+(X)$ and on $\mathfrak{g}$, and a $Spin^c$-structure $\mathfrak{s}$ on X. Then we can construct a canonical orientation on $\mathcal{M}_P^{ASD}$ for all principal G-bundles $P \to X$.
2) If G is simply-connected, or if $G = U(m)$, then the orientation on $\mathcal{M}_P^{ASD}$ in the above 1) is independent of the choice of $Spin^c$-structure $\mathfrak{s}$.

Part 1) is new, both the orientability of $\mathcal{B}_P$ and the use of $Spin^c$-structures in constructing canonical orientations.

Structure of proof:

- Find a CW complex $Y \subset X$ of dimension 2, and a trivialization $P|_{X \setminus Y} \to (X \setminus Y) \times G$. Then choose an open neighbourhood U of Y in X such that U retract onto Y, an open subset $V \subset X$ with $\bar{V} \subset X \setminus Y$ and $U \cup V = X$, and connection $\hat{A}$ on P, which is trivial over $V \subset X \setminus Y$.
- X does not necessarily have an almost complex structure, but by using a *Spin*^{c}-structure, one can introduce an almost complex structure on the above U.
- Applying an excision theorem to these U, V etc., one obtains a choice for orientations at $[\hat{A}]$, using the almost complex structure on $E_\bullet|_U$. Then take paths from $[\hat{A}]$ to other connections $[A]$ in the space of connections on P in order to make choices at $[A]$. Prove that this orientation is independent of choices in the construction.

[Joyce–T–Upmeier] describes/solves the orientation problems also for flat connections on Riemann surfaces, flat connections and Casson invariants on 3-manifolds, Seiberg–Witten invariants, Vafa–Witten and Kapustin–Witten equations on 4-manifolds, Haydys–Witten equations on 5-manifolds, and Donaldson–Thomas instantons on compact symplectic 6-manifolds.

For the orientation problems for G_2-instantons, $Spin(7)$-instantons and $DT4$ moduli spaces, see

- D. Joyce and M. Upmeier, *Canonical orientations for moduli spaces of G_2-instantons with gauge group $SU(m)$ or $U(m)$*, arXiv:1811.02405, 2018.
- Y. Cao and D. Joyce, *Orientability of moduli spaces of $Spin(7)$-instantons*, arXiv:1811.09658, 2018.
- Y. Cao, J. Gross and D. Joyce, *On orientations for moduli spaces of coherent sheaves on Calabi–Yau manifolds*, in preparation, 2019.

Variations of the ASD instanton moduli space

I. Kapustin–Witten equations on closed four-manifolds

Let X be a closed, oriented, smooth four-manifold, and let $P \to X$ be a principal G-bundle over X, where G is a connected Lie group. For $(A, \mathfrak{a}) \in \mathcal{A}_P \times \Gamma(\mathfrak{g}_P \otimes \Lambda^1_X)$, we consider the following equations:

$$d_A^* \mathfrak{a} = 0, \quad d_A^- \mathfrak{a} = 0, \text{and}$$
$$F_A^+ + \pi_+([\mathfrak{a}, \mathfrak{a}]) = 0,$$

where $d_A^- := \pi_-(d_A)$, and $\pi_- : \Gamma(\mathfrak{g}_P \otimes \Lambda^2_X) \to \Gamma(\mathfrak{g}_P \otimes \Lambda^-)$ is the projection.

Remark: Kapustin and Witten introduced a one parameter family of equations. The above is obtained by specifying the parameter, and corresponds to Simpson's equations (i.e. (poly-)stable Higgs bundles) when the underlying manifold is a Kähler surface.

The gauge group $\mathcal{G}_P = \mathrm{Aut}\,(P)$ acts on pairs $(A, \mathfrak{a}) \in \mathcal{A}_P \times \Gamma(\mathfrak{g}_P \otimes \Lambda^1_X)$. We say that $(A, \mathfrak{a})$ is *irreducible* if the stabilizer group in $\mathcal{G}_P$ is trivial.

Denote by $\mathcal{M}_P^{KW}$ the moduli space of gauge equivalence classes of irreducible solutions $(A, \mathfrak{a})$ to the Kapustin–Witten equations.

The elliptic operator $E_\bullet = (E_0, E_1, D)$ in this case is

$$D : \Gamma(\Lambda^1_X \oplus \Lambda^1_X) \to \Gamma(\Lambda^0_X \oplus \Lambda^+_X \oplus \Lambda^-_X \oplus \Lambda^0_X),$$

where

$$D := \begin{pmatrix} d^* & d^+ & 0 & 0 \\ 0 & 0 & d^- & d^* \end{pmatrix}^T .$$

So $E_\bullet$ is the direct sum $E_\bullet = E_\bullet^+ \oplus E_\bullet^-$, where $E_\bullet^+$ is as in the anti-self-dual instantons case, and $E_\bullet^-$ is $E_\bullet^+$ for the opposite orientation on X. Thus $O_P^{E_\bullet} \cong O_P^{E_\bullet^+} \otimes_{\mathbb{Z}_2} O_P^{E_\bullet^-}$.

The orientation bundle of $\mathcal{M}_P^{KW}$ is the pull-back of $O_P^{E_\bullet} \to \mathcal{B}_P$ under the forgetful map $\mathcal{M}_P^{KW} \to \mathcal{B}_P$. Hence we obtain:

Theorem (Joyce–T–Upmeier)

Let X be a closed, oriented, smooth four-manifold, and let $P \to X$ be a principal G-bundle over X, where G is a connected Lie group.

1) Choose an orientation on $H^2(X)$ and on $\mathfrak{g}$, and a *Spin*^{c}-structure $\mathfrak{s}$ on X. Then we can construct a canonical orientation on $\mathcal{M}_P^{KW}$ for all principal G-bundles $P \to X$, as a derived manifold.
2) If G is simply-connected, or if $G = U(m)$, then the orientation on $\mathcal{M}_P^{KW}$ in the above 1) is independent of the choice of *Spin*^{c}-structure $\mathfrak{s}$.

II. Vafa–Witten equations on closed four-manifolds

Let X be a closed, oriented, smooth four-manifold, and let $P \to X$ be a principal G-bundle over X, where G is a connected Lie group.

For $(A, B, C) \in \mathcal{A}_P \times \Gamma(\mathfrak{g}_P \otimes \Lambda^+_X) \times \Gamma(\mathfrak{g}_P \otimes \Lambda^0_X)$, we consider the following equations:

$$d_A^* B + d_A C = 0, \text{and}$$
$$F_A^+ + [B, C] + [B.B] = 0,$$

where $[B.B] \in \Gamma(\mathfrak{g}_P \otimes \Lambda^+_X)$.

The gauge group $\mathcal{G}_P = \mathrm{Aut}\,(P)$ acts on pairs $(A, B, C) \in \mathcal{A}_P \times \Gamma(\mathfrak{g}_P \otimes \Lambda^+_X) \times \Gamma(\mathfrak{g}_P \otimes \Lambda^0_X)$. We say that (A, B, C) is *irreducible* if the stabiliser group in $\mathcal{G}_P$ is trivial.

Denote by $\mathcal{M}_P^{VW}$ the moduli space of gauge equivalence classes of irreducible solutions (A, B, C) to the Vafa–Witten equations.

The elliptic operator $E_\bullet = (E_0, E_1, D)$ in this case is

$$D : \Gamma(\Lambda^0_X \oplus \Lambda^+_X \oplus \Lambda^1_X) \to \Gamma(\Lambda^1_X \oplus \Lambda^0_X \oplus \Lambda^+_X),$$

where

$$D := \begin{pmatrix} d & d^* & 0 \\ 0 & 0 & d^* \\ 0 & 0 & d^+ \end{pmatrix}.$$

So, $E_\bullet = E^+_\bullet \oplus (E^+_\bullet)^*$. Thus, the orientation bundle $O^{E_\bullet}_P$ becomes $O^{E_\bullet}_P \cong O^{E^+_\bullet}_P \otimes_{\mathbb{Z}_2} O^{(E^+_\bullet)^*}_P \cong O^{E^+_\bullet}_P \otimes_{\mathbb{Z}_2} (O^{E^+_\bullet}_P)^* \cong \mathcal{B}_P \times \mathbb{Z}_2$, namely it is canonically trivial. As in the case of the Kapustin–Witten equations, the orientation bundle of $\mathcal{M}^{VW}_P$ is the pull-back of $O^{E_\bullet}_P \to \mathcal{B}_P$ under the forgetful map $\mathcal{M}^{VW}_P \to \mathcal{B}_P$. Hence we obtain:

Theorem (Joyce–T–Upmeier)

The Vafa–Witten moduli spaces $\mathcal{M}^{VW}_P$ have canonical orientations for all G and principal G-bundles $P \to X$, as a derived manifold.