Complex Ruelle Operator in a Parabolic Basin

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1. Parabolic basin and holomorphic quadratic differentials

In this note, we investigate the behavior of partial Ruelle operator associated to a parabolic basin of a complex dynamical system. Let $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational mapping of the Riemann sphere to itself. We assume that the infinity is a parabolic fixed point of R of the form :

$$R(z) = z + 1 + \frac{P(z)}{Q(z)}, \quad \deg P \leq \deg Q - 2,$$

where $P(z)$ and $Q(z)$ are polynomials without common factor. Let A_∞ denote the immediate parabolic basin of the infinity, and let $K = \mathbb{C} \setminus A_\infty$ and $\overline{K} = K \cup \{\infty\}$. We call K the filled Julia set of R. Further, we assume that all the critical points in A_∞ are non-degenerate, and the forward orbit of each critical point does not contain other critical points. For the sake of simplicity, we assume $\overline{K}$ is connected.

Let $\mathcal{O}_0(\overline{K})$ denote the space of functions $g : \overline{K} \to \mathbb{C}$ holomorphic in a neighborhood of $\overline{K}$ and $g(\infty) = 0$. The topology is defined as follows : sequence of functions $\{g_n\}$ in $\mathcal{O}_0(\overline{K})$ converges to some function g_∞ in $\mathcal{O}_0(\overline{K})$ if there exists a neighborhood of $\overline{K}$ such that $\{g_n\}$ are extendable to this neighborhood and the sequence converges to g_∞ uniformly in this neighborhood.

Let $\mathcal{O}(A_\infty)$ denote the space of holomorphic functions $f : A_\infty \to \mathbb{C}$ with the topology of local uniform convergence. We denote by $\mathcal{O}_0(A_\infty)$ the

set of holomorphic functions $f \in \mathcal{O}(A_\infty)$ satisfying $\lim_{z\to\infty} f(z) = 0$. We define the pairing of functions in these spaces.

DEFINITION 1.2 (pairing) For $g \in \mathcal{O}_0(\overline{K})$ and $f \in \mathcal{O}(A_\infty)$, Let

$$\langle f, g\rangle = \frac{1}{2\pi i}\int_\gamma f(\tau)g(\tau)d\tau,$$

where γ is a closed curve surrounding and passing near $\overline{K}$ with an orientation looking $\overline{K}$ on the left hand side. The contour courve γ should be chosen so that there is no critical point of R between $\partial\overline{K}$ and γ. The choice of γ depends on g, but the value of $\langle f, g\rangle$ does not depend on the choice, provided that the curve γ passes sufficiently near the filled Julia set $\overline{K}$.

PROPOSITION 1.3 Each $f \in \mathcal{O}(A_\infty)$ defines a continuous, holomorphic, and complex linear functional $\hat{f} : \mathcal{O}_0(\overline{K}) \to \mathbb{C}$ by $\hat{f}[g] = \langle f, g\rangle$ for $g \in \mathcal{O}_0(\overline{K})$.

Here, functional $\hat{f}$ is said to be holomorphic if $\hat{f}[g_\nu]$ is holomorphic with respect to ν for all holomorphic family $\{g_\nu\}$ in $\mathcal{O}_0(\overline{K})$.

PROOF Let $\{g_n\}$ be a sequence of functions in $\mathcal{O}_0(\overline{K})$ and assume g_n converges to 0 in $\mathcal{O}_0(\overline{K})$. Then by the definition of the topology of $\mathcal{O}_0(\overline{K})$, there exists a neighborhood U of $\overline{K}$ such that g_n are extendable to U and $\sup_{z\in U}|g_n(z)| \to 0$. Take a curve $\gamma \subset U$ and set $M = \sup_{\tau\in\gamma}|f(\tau)|$, and let $|\gamma|$ denote the length of γ. Then

$$|\langle f, g_n\rangle| = |\frac{1}{2\pi i}\int_\gamma f(\tau)g_n(\tau)d\tau| \le \frac{1}{2\pi}|\gamma|M\sup_{z\in U}|g_n(z)| \to 0.$$

Clearly by definition, the functional is complex linear and holomorphic in the sense above.

DEFINITION 1.4 The dual space $\mathcal{O}_0^*(\overline{K})$ is the space of continuous, holomorphic and complex linear functionals $F : \mathcal{O}_0(\overline{K}) \to \mathbb{C}$.

PROPOSITION 1.5 For a functional $F \in \mathcal{O}_0^*(\overline{K})$,

$$f(\zeta) = F[\frac{1}{\zeta - z}], \quad \zeta \in A_\infty$$

defines a holomorphic function $f \in \mathcal{O}(A_\infty)$ and for $g \in \mathcal{O}_0(\overline{K})$,

$$F[g] = \langle f, g\rangle$$

holds.

PROOF For each $\zeta \in A_\infty$, $\frac{1}{\zeta - z} \in \mathcal{O}_0(\overline{K})$. It is a holomorphic family of holomorphic functions. Hence we have $f \in \mathcal{O}(A_\infty)$. Next, for $g \in \mathcal{O}_0(\overline{K})$, by applying the residue theorem, we have

$$g(z) = \frac{1}{2\pi i} \int_\gamma \frac{g(\tau)}{\tau - z} d\tau, \quad z \in \overline{K}$$

since $g(\infty) = 0$, the resudue at the infinity vanishes. Therefore,

$$F[g] = F[\frac{1}{2\pi i} \int_\gamma \frac{g(\tau)}{\tau - z} d\tau] = \frac{1}{2\pi i} \int_\gamma F[\frac{1}{\tau - z}] g(\tau) d\tau$$

$$= \frac{1}{2\pi i} \int_\gamma f(\tau) g(\tau) d\tau = \langle f, g \rangle.$$

Propositions 1.3 and 1.5 yield the following.

PROPOSITION 1.6 $\mathcal{O}_0^*(\overline{K})$ is isomorphic to $\mathcal{O}(A_\infty)$.

The isomorphism defined in proposition 1.5 is called the Cauchy transformation.

2. Complex Ruelle operator and its adjoint operator

We define a linear operator $L : \mathcal{O}_0(\overline{K}) \to \mathcal{O}_0(\overline{K})$ by

$$(Lg)(x) = \frac{1}{2\pi i} \int_\gamma \frac{g(\tau) d\tau}{R'(\tau)(R(\tau) - x)}, \quad g \in \mathcal{O}_0(\overline{K}), x \in \overline{K}.$$

We call this operator a *complex Ruelle operator*. More precisely, it is a component of a Ruelle operator for a perticular weight $(R'(z))^{-2}$ in the decompostion of the operator described in [4]. The coutour curve γ depends upon g. Observe that Lg is holomorphic in a neighborhood of $\overline{K}$ and $g(\infty) = 0$. Note that Lg can be expressed as

$$(Lg)(x) = \sum_{y \in R^{-1}(x)} \frac{g(y)}{(R'(y))^2} + \sum_{c \in C(R) \cap \overline{K}} \frac{g(c)}{R''(c)(R(c) - x)}$$

in a neighborhood of $\overline{K}$.

The dual operator $L^* : \mathcal{O}_0^*(\overline{K}) \to \mathcal{O}_0^*(\overline{K})$ defines the *adjoint Ruelle operator* $\mathcal{L}^* : \mathcal{O}(A_\infty) \to \mathcal{O}(A_\infty)$ through the Cauchy transformation described in the previous section.

PROPOSITION 2.1 The adjoint operator $\mathcal{L}^* : \mathcal{O}_0(A_\infty) \to \mathcal{O}_0(A_\infty)$ is given by

$$(\mathcal{L}^* f)(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(R(\tau))d\tau}{R'(\tau)(z-\tau)}, \quad f \in \mathcal{O}_0(A_\infty), z \in A_\infty.$$

Moreover,

$$(\mathcal{L}^* f)(z) = \frac{f(R(z))}{R'(z)} - \sum_{c \in C(R) \cap A_\infty} \frac{f(R(c))}{R''(c)(z-c)}.$$

PROOF This is verified by a direct calculation. Let $\hat{f} \in \mathcal{O}_0^*(\overline{K})$ be a functional and $f \in \mathcal{O}(A_\infty)$ be the corresponding holomorphic function. Then we have

$$\begin{aligned}
(\mathcal{L}^* f)(z) &= (L^* \hat{f}) \left[\frac{1}{z-\zeta}\right] = \hat{f}\left[L\left[\frac{1}{z-\zeta}\right]\right] \\
&= \hat{f}\left[\frac{1}{2\pi i}\int_\gamma \frac{d\tau}{R'(\tau)(R(\tau)-\zeta)(z-\tau)}\right] \\
&= \frac{1}{2\pi i}\int_\gamma f(\zeta)d\zeta \left(\frac{1}{2\pi i}\int_\gamma \frac{d\tau}{R'(\tau)(R(\tau)-\zeta)(z-\tau)}\right) \\
&= \frac{1}{2\pi i}\int_\gamma \frac{d\tau}{R'(\tau)(z-\tau)} \left(\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{R(\tau)-\zeta}\right) \\
&= \frac{1}{2\pi i}\int_\gamma \frac{f(R(\tau))d\tau}{R'(\tau)(z-\tau)} \\
&= \frac{f(R(z))}{R'(z)} - \sum_{c \in C(R) \cap A_\infty} \mathrm{Res}_{\tau=c} \frac{f(R(\tau))}{R'(\tau)(z-\tau)} \\
&= \frac{f(R(z))}{R'(z)} - \sum_{c \in C(R) \cap A_\infty} \frac{f(R(c))}{R''(c)(z-c)}.
\end{aligned}$$

This proposition shows that the adjoint Ruelle operator decomposes into two parts. This decomposition is similar to that introduced in [1], and the analysis of spectrum below is almost same as described there. Let $A_R = \{z \in A_\infty \mid (R^{\circ n})'(z) \neq 0 \text{ for } n \geq 0\}$, and let $\mathcal{O}(A_R)$ denote the space of holomorphic functions on A_R with the topology of local uniform

convergence. Note that $\mathcal{O}(A_\infty) \subset \mathcal{O}(A_R)$. Define a linear operator $\mathcal{K} : \mathcal{O}(A_R) \to \mathcal{O}(A_R)$ by

$$(\mathcal{K}f)(z) = \frac{f(R(z))}{R'(z)}.$$

Let $\varphi : A_\infty \to \mathbb{C}$ denote the Fatou map defined by

$$\varphi(z) = \lim_{n\to\infty}(R^{\circ n}(z) - n), \quad z \in A_\infty.$$

Under our assumption on R, φ is holomorphic in A_∞ and stisfies function equation

$$\varphi \circ R(z) = \varphi(z) + 1, \quad z \in A_\infty$$

and

$$\varphi'(z) \neq 0 \quad \text{for} \quad z \in A_R.$$

Define a linear isomorphism $\mathcal{T} : \mathcal{O}(A_R) \to \mathcal{O}(A_R)$ by

$$(\mathcal{T}f)(z) = f(z)\varphi'(z).$$

The linear operater $\mathcal{K}$ is conjugate to $\mathcal{M} = \mathcal{T} \circ \mathcal{K} \circ \mathcal{T}^{-1}$ and $\mathcal{M} : \mathcal{O}(A_R) \to \mathcal{O}(A_R)$ is a very simple operator.

PROPOSITION 2.2

$$(\mathcal{M}h)(z) = h \circ R(z), \quad h \in \mathcal{O}(A_R).$$

PROOF By a direct computation.

$$(\mathcal{T}^{-1}h)(z) = h(z)(\varphi'(z))^{-1},$$

$$(\mathcal{K}\mathcal{T}^{-1}h)(z) = \frac{(\mathcal{T}^{-1}h)(R(z))}{R'(z)} = \frac{h(R(z))(\varphi'(R(z)))^{-1}}{R'(z)},$$

and, as we have $\varphi'(R(z))R'(z) = \varphi'(z)$ by differentiating the function equation $\varphi \circ R = \varphi + 1$,

$$(\mathcal{M}h)(z) = (\mathcal{T}\mathcal{K}\mathcal{T}^{-1}h)(z) = \frac{h(R(z))(\varphi'(R(z)))^{-1}}{R'(z)}\varphi'(z) = h(R(z)).$$

If a complex number $\nu \neq 0$ is an eigenvalue of the operator $\mathcal{M}$ and $h_\nu \in \mathcal{O}(A_R)$ is an eigenfunction associated to ν, then h_ν must satisfy the function equation

$$(\mathcal{M}h_\nu)(z) = h_\nu(R(z)) = \nu h_\nu(z).$$

The Fatou function $\varphi : A_\infty \to \mathbb{C}$ has an inverse function $\psi = \varphi^{-1}$ defined for $\{x \in \mathbb{C} \mid \Re x > r\}$ for sufficiently large r. In this region, we have

$$h_\nu(\psi(x+1)) = \nu h_\nu(\psi(x)).$$

Hence, by taking an appropriate value for $\log \nu$,

$$p(x) = e^{-x \log \nu} h_\nu(\psi(x))$$

is a periodic function of x of period 1. This function $p(x)$ must be an entire function of period 1. We obtain an expression of the eigenfunction

$$h_\nu(z) = e^{\varphi(z) \log \nu} p(\varphi(z)).$$

The eigenfunction $f_\nu \in \mathcal{O}(A_R)$ of the operator $\mathcal{K}$ corresponding to h_ν is given by

$$f_\nu(z) = \frac{e^{\varphi(z) \log \nu} p(\varphi(z))}{\varphi'(z)}.$$

PROPOSITION 2.3 Any $\nu \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of $\mathcal{L}^*$, and its eigenfunction $f_\nu \in \mathcal{O}(A_\infty)$ is given by

$$f_\nu(z) = \frac{e^{\varphi(z) \log \nu} p(\varphi(z))}{\varphi'(z)},$$

where $\varphi : A_\infty \to \mathbb{C}$ is the Fatou function and $p : \mathbb{C} \to \mathbb{C}$ is an entire periodic function of period 1 satisfying $p(\varphi(c)) = 0$ for all critical point $c \in A_\infty$.

PROOF The Fatou function φ has critical points at the critical points of R and at the backward images of these critical points. As we assumed that the critical points of R are simple and the critical points do not collide, the function f_ν is holomorphic in A_∞. In case if critical points are not simple or collision of critical points occur, we pose appropriate degenrate zero conditions upon p at the corresponding points $\varphi(c)$. There exists entire periodic functions with prescribed zeroes at the images $\varphi(c)$ of critical points. For such periodic entire functions p, functions f_ν belong to $\mathcal{O}(A_\infty)$. And as $f_\nu(R(c)) = 0$ for all critical points $c \in C(R) \cap A_\infty$, they are also eigenfunctions of $\mathcal{L}^*$.

We define a subspace of $\mathcal{O}(A_\infty)$ which is invariant under the adjoint Ruelle operator $\mathcal{L}^*$.

DEFINITION 2.4

$$\mathcal{O}_1(A_\infty) = \{f \in \mathcal{O}(A_\infty) \mid \forall t > 0, \exists M > 0, \exists r > 0,$$
$$s.t. |f(z)| < M \quad \text{for} \quad \Re z > r \text{ and } |\Im z| < t\}.$$

PROPOSITION 2.5 The space $\mathcal{O}_1(A_\infty)$ is invariant under $\mathcal{L}^*$.

PROOF As $R(z) = z + 1 + O(z^{-2})$ near the infinity, we have $R'(z) = 1 + O(z^{-1})$. Therefore, by taking sufficiently large positive number $s > (\max_{c \in C(R) \cap A_\infty} \Re c) + 1$, we can assume

$$|R(z) - z - 1| \leq \frac{1}{2} \quad \text{and} \quad |R'(z) - 1| \leq \frac{1}{2}$$

holds for $\Re z > s$. If $f \in \mathcal{O}_1(A_\infty)$, then for any $t > 0$, we can find positive constants M_0 and r_0 such that $|f(z)| < M_0$ holds for $\Re z > r_0$ and $|\Im z| < t + 1$. Let

$$M_1 = 2M_0 + \sum_{c \in C(R) \cap A_\infty} \left| \frac{f(R(c))}{R''(c)} \right| (1 + |c|)$$

and $r_1 = \max(s, r_0, 2)$. Then we have

$$|(\mathcal{L}^* f)(z)| \leq \left| \frac{f(R(z))}{R'(z)} \right| + \sum_{c \in C(R) \cap A_\infty} \left| \frac{f(R(c))}{R''(c)} \right| \frac{1}{|z - c|}$$

$$\leq 2M_0 + \sum_{c \in C(R) \cap A_\infty} \left| \frac{f(R(c))}{R''(c)} \right| (1 + |c|) \leq M_1$$

for $\Re z > r_1$ and $|\Im z| < t$.

PROPOSITION 2.6 The adjoint operator $\mathcal{L}^*$ restricted to the subspace $\mathcal{O}_1(A_\infty)$ has a continuum of eigenvalues $\{\nu \in \mathbb{C} \mid 0 < |\nu| \leq 1\}$. The eigenfunctions are as given in proposition 2.3.

3. Discrete eigenvalues of the operator

In this section, we apply the perturbation method described in [1] to our case. Let ℓ denote the number of critical points of R in A_∞, and

let $C(R) \cap A_\infty = \{c_1, \cdots, c_\ell\}$. Define linear maps $\mathcal{G} : \mathcal{O}(A_R) \to \mathbb{C}^\ell$ and $\mathcal{F} : \mathbb{C}^\ell \to \mathcal{O}(A_R)$ by

$$\mathcal{G}f = \left(\frac{f(R(c_j))}{R''(c_j)} \right)_{j=1,\cdots,\ell}, \qquad f \in \mathcal{O}(A_R),$$

and

$$\mathcal{F}(\alpha_j) = \sum_{j=1}^{\ell} \frac{\alpha_j}{z - c_j}, \qquad (\alpha_j) \in \mathbb{C}^\ell.$$

The adjoint operator $\mathcal{L}^*$ can be expressed as

$$\mathcal{L}^* = \mathcal{K} - \mathcal{F}\mathcal{G}.$$

As $\ker \mathcal{G} = \{f \in \mathcal{O}(A_R) \mid f(R(c_j)) = 0, j = 1, \cdots, \ell\}$, We see that

$$\mathcal{L}^* \mid_{\ker \mathcal{G}} = \mathcal{K} \mid_{\ker \mathcal{G}}$$

and

$$\mathcal{O}(A_R)/\ker \mathcal{G} \simeq \mathbb{C}^\ell.$$

We define an $\ell \times \ell$ matrice $M(\lambda)$ by

$$M(\lambda) = I_\ell + \lambda \mathcal{G} \left(\sum_{k=0}^{\infty} \lambda^k \mathcal{K}^k \right) \mathcal{F}.$$

As

$$(\mathcal{K}^k f)(z) = \frac{f(R^{\circ k}(z))}{(R^{\circ k})'(z)}$$

the (i, j)-component of $M(\lambda)$ is given by

$$\delta_{ij} + \sum_{k=1}^{\infty} \frac{\lambda^k}{(R^{\circ k})''(c_i)(R^{\circ k}(c_i) - c_j)}.$$

Note that $M(\lambda)$ is holomorphic for $|\lambda| < 1$, since critical points c_i are in the parabolic basin A_∞.

PROPOSITION 3.1 If $\det M(\lambda) = 0$ holds for some λ with $0 < |\lambda| < 1$ and there exists an eigenvector $u \in \ker M(\lambda) \setminus \{0\}$ satisfying $M(\lambda)u = 0$, then

$$V = \sum_{k=0}^{\infty} \lambda^k \mathcal{K}^k \mathcal{F} u$$

satisfies

$$\mathcal{L}^* V = \frac{1}{\lambda} V.$$

Moreover, $V \in \mathcal{O}_1(A_\infty)$.

PROOF Let $u = (\alpha_j)$ and $v = \mathcal{F}u = \sum_{j=1}^{\ell} \frac{\alpha_j}{z-c_j}$. Clearly, v belongs to $\mathcal{O}(A_R)$, since

$$\mathcal{K}^k v = \mathcal{T}^{-1}\mathcal{M}^k\mathcal{T}v = \mathcal{T}^{-1}((\mathcal{T}v)\circ R^{\circ k})$$

and V converges uniformly on compact subsets of A_R. Next we show that $V \in \mathcal{O}(A_\infty)$. V may have poles at critical point c_i or at its backward images by R. The residue of $\mathcal{K}^k v$ at critical point c_i is given by

$$\mathrm{Res}_{z=c_i}\mathcal{K}^k v = \mathrm{Res}_{z=c_i}\frac{v(R^{\circ k}(z))}{(R^{\circ k})'(z)} = \mathrm{Res}_{z=c_i}\sum_{j=1}^{\ell}\frac{\alpha_j}{(R^{\circ k})'(z)(R^{\circ k}(z)-c_j)}$$

$$= \sum_{j=1}^{\ell}\frac{\alpha_j}{(R^{\circ k})''(c_i)(R^{\circ k}(c_i)-c_j)}.$$

Hence we have

$$\mathrm{Res}_{z=c_i}V(z) = \alpha_i + \sum_{k=1}^{\infty}\sum_{j=1}^{\ell}\frac{\lambda^k\alpha_j}{(R^{\circ k})''(c_i)(R^{\circ k}(c_i)-c_j)}$$

$$= \sum_{j=1}^{\ell}\left(\delta_{ij} + \sum_{k=1}^{\infty}\frac{\lambda^k}{(R^{\circ k})''(c_i)(R^{\circ k}(c_i)-c_j)}\right)\alpha_j = 0.$$

Therefore V is regular at critical points c_i and consequently it is regular at the backward images of the critical points. This implies that $V \in \mathcal{O}(A_\infty)$. Furthermore V belongs also to $\mathcal{O}_1(A_\infty)$. For, as we assumed $R(z) = z + 1 + O(z^{-2})$, for any $t > 0$, we can find some $t_1 > t$ and $r > 0$ such that if $\Re z > r$ and $|\Im z| < t$ then $\frac{1}{2} < |\varphi'(z)| < \frac{3}{2}$, $\Re(R^{\circ k}(z)) > r$ and $|\Im(R^{\circ k}(z))| < t_1$ holds for $k = 1, 2, \cdots$. Let $m = \sup_{\Re z > r, |\Im z| < t_1}|v(z)\varphi'(z)|$. As

$$\mathcal{T}V = \sum_{k=0}^{\infty}\lambda^k\mathcal{T}\mathcal{K}^k v = \sum_{k=0}^{\infty}\lambda^k\mathcal{M}^k\mathcal{T}v = \sum_{k=0}^{\infty}\lambda^k(\mathcal{T}v)\circ R^{\circ k},$$

we have

$$|V(z)| \le 2\sum_{k=0}^{\infty}|\lambda|^k m = \frac{2m}{1-|\lambda|}.$$

Hence $V \in \mathcal{O}_1(A_\infty)$.

We have also

$$\lambda\mathcal{L}^*V = \lambda(\mathcal{K} - \mathcal{F}\mathcal{G})V$$

$$= \sum_{k=1}^{\infty}\lambda^k\mathcal{K}^k v - \lambda\mathcal{F}\mathcal{G}\left(\sum_{k=0}^{\infty}\lambda^k\mathcal{K}^k\right)\mathcal{F}u$$

$$= \sum_{k=1}^{\infty} \lambda^k \mathcal{K}^k v + \mathcal{F}\left(I_\ell - M(\lambda)\right) u$$
$$= \sum_{k=1}^{\infty} \lambda^k \mathcal{K}^k v + \mathcal{F} u = V.$$

Hence V is an eigenfunction of $\mathcal{L}^*$.

4. Eigenfunctions of L corresponding to the discrete eigenvalues

In this section, we consider eigenfunctions for the Ruelle operator L itself. As we saw in the previous section, the adjoint operator has a continuum of eigenvalues. In order to distinguish eigenvalues and eigenfunctions, we have to examine the eigenspaces for each eigenvalues. The Cauchy's integral formula

$$g(z) = \frac{1}{2\pi i} \int_\gamma \frac{g(\zeta)}{\zeta - z} d\zeta$$

indicates that rational functions of the form

$$\chi_\eta(z) = \frac{1}{z - \eta}$$

form a "basis" of the function space $\mathcal{O}_0(\overline{K})$. For $\eta \in A_\infty$, χ_η belongs to $\mathcal{O}_0(\overline{K})$. The image $L\chi_\eta$ is computed as follows.

PROPOSITION 4.1 If $\eta \in A_\infty \setminus C(R)$, then

$$(L\chi_\eta)(x) = \sum_{y \in R^{-1}(x)} \frac{1}{(R'(y))^2} \chi_\eta(y) - \sum_{c \in C(R) \cap \overline{K}} \frac{1}{R''(c)(c - \eta)} \chi_{R(c)}(x)$$

and

$$L\chi_\eta = \frac{1}{R'(\eta)} \chi_{R(\eta)} + \sum_{j=1}^{\ell} \frac{1}{R''(c_j)(c_j - \eta)} \chi_{R(c_j)}.$$

PROOF These formulas are directly verified by applying the residue formula to domains inside and outside of the contour courve γ.

Let us consider a formal sum of the following form.

$$U = \sum_{i=1}^{\ell} \sum_{k=1}^{\infty} \alpha_{i,k} \chi_{R^{\circ k}(c_i)}, \qquad \alpha_{i,k} \in \mathbb{C}.$$

The space of functions of this form is invariant under L. In this space, we can formulate a formal eigen equation

$$LU = \frac{1}{\lambda}U.$$

By a formal computation, we obtain an equation for λ as follows.

PROPOSITION 4.2 If the eigen equation has a solution, then λ satisfies $\det N(\lambda) = 0$, where $N(\lambda)$ is an $\ell \times \ell$-matrice

$$N(\lambda) = \left(\delta_{ij} + \frac{\lambda}{R''(c_i)} \sum_{k=0}^{\infty} \frac{\lambda^k}{(R^{\circ k}(R(c_j)) - c_i)(R^{\circ k})'(R(c_j))}\right).$$

PROOF This is verified by a straightforward computation.

PROPOSITION 4.3

$$\det N(\lambda) = \det M(\lambda).$$

PROOF The (i,j)-component of $M(\lambda)$ is given by

$$\delta_{ij} + \sum_{k=1}^{\infty} \frac{\lambda^k}{(R^{\circ k})''(c_i)(R^{\circ k}(c_i) - c_j)}$$

$$= \delta_{ij} + \sum_{k=1}^{\infty} \frac{\lambda^k}{(R^{\circ (k-1)})'(R(c_i))R''(c_i)(R^{\circ (k-1)}(R(c_i)) - c_j)}$$

$$= \delta_{ij} + \frac{\lambda}{R''(c_i)} \sum_{k=0}^{\infty} \frac{\lambda^k}{(R^{\circ k})'(R(c_i))(R^{\circ k}(R(c_i)) - c_j)}.$$

Let S denote the diagonal $\ell \times \ell$-matrice whose (i,i)-component is $\lambda/R''(c_i)$, and let W denote the $\ell \times \ell$-matrice whose (i,j)-component is

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{(R^{\circ k})'(R(c_i))(R^{\circ k}(R(c_i)) - c_j)}.$$

Then we see that

$$M(\lambda) = I_\ell + SW \quad \text{and} \quad {}^tN(\lambda) = I_\ell + WS.$$

Hence we have $\det M(\lambda) = \det N(\lambda)$.

Finally, we compute the eigenfunction for the eigenvalue λ^{-1}.

PROPOSITION 4.4 Formal eigenfunction of the Ruelle operator L is given by

$$U = \sum_{i=1}^{\ell} \sum_{k=1}^{\infty} \alpha_{i,k} \chi_{R^{\circ k}(c_i)},$$

where $(\alpha_{1,1}, \cdots, \alpha_{\ell,1})$ is a vector in the kernel of $N(\lambda)$ and

$$\alpha_{i,k} = \frac{\lambda^{k-1} \alpha_{i,1}}{(R^{\circ(k-1)})'(R(c_i))} \quad \text{for} \quad k = 1, 2, \cdots.$$

Note that the obtained eigen function converges as a meromorphic function if $|\lambda| < 1$. However, the limit function does not belong to the space $\mathcal{O}_0(\overline{K})$.

References

[1] G.M.Levin, M.L.Sodin, and P.M.Yuditski: A Ruelle Operator for a Real Julia Set, Communications in Mathematical Physics, 141, 119-132(1991).

[2] G.Levin, M.Sodin, and P.Yuditski: Ruelle operators with rational weights for Julia sets, Journal d'analyse mathématiques, Vol. 63(1994),303-331.

[3] M.Tsujii: A transversality condition for quadratic family at Collet-Eckmann parameter, Problems in Complex Dynamical Systems, RIMS Kokyuroku 1042,pp99-101,1998.

[4] S.Ushiki: Complex Ruelle Operator and Hyperbolic Complex Dynamical Systems, pp50-61,RIMS Kokyuroku 1072, "Invariants of Dynamical Systems and Applications", 1998.