Complex Ruelle Operator  
and  
Hyperbolic Complex Dynamical Systems  

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1. Decomposition of Complex Ruelle operator  

Let $R : \mathbb{C} \to \mathbb{C}$ be a hyperbolic rational mapping. We assume that all the attractive periodic points of $R$ are fixed points, all the critical points of $R$ are non-degenerate, and that the Julia set of $R$, $J_R$, is inculed in $\mathbb{C}$. Let $N$ denote the number of attractive fixed points and let $a_1, \cdots, a_N$ denote the attractive fixed points. Let $A_k$ denote the attractive basin of $a_k$. Let $C_R$ denote the set of critical points of $R$.

For $k = 1, \cdots, N$, let $\gamma_k$ denote an oriented multicurve in $A_k$, such that $\gamma_k = \partial \Omega_k$, where $\Omega_k$ is an open set satisfying $R^{-1}(\Omega_k) \subset \Omega_k$, $\Omega_k \cup A_k = \mathbb{C}$, and $C_R \cap \Omega_k \cap A_k = \phi$. Let $\gamma = \cup_{k=1}^{N} \gamma_k$ and $\Omega = \cap_{k=1}^{N} \Omega_k$.

For open set $O \subset \mathbb{C}$, let $\mathcal{O}_0(O)$ denote the space of functions $g : O \to \mathbb{C}$ holomorphic in $O$ and has an analytic extension to a neighbourhood of the closure of $O$, and satisfies $g(\infty) = 0$ if $\infty$ belongs to the closure of $O$. We have the following decomposition of holomorphic functions. The direct sum in the theorem means the uniqueness of the decomposition.

**Theorem 1.1**

$$\mathcal{O}_0(\Omega) = \bigoplus_{k=1}^{N} \mathcal{O}_0(\Omega_k).$$

**Proof** Let $g \in \mathcal{O}_0(\Omega)$. Then $g$ can be expressed as

$$g(x) = \frac{1}{2\pi i} \int_{\gamma} \frac{g(\tau)}{\tau - x} d\tau, \quad x \in \Omega.$$
For $k = 1, \cdots, N$, let $\Gamma_k : \mathcal{O}_0(\Omega) \to \mathcal{O}_0(\Omega_k)$ be defined by

$$(\Gamma_k g)(x) = \frac{1}{2\pi i} \int_{\gamma_k} \frac{g(\tau)}{\tau - x} d\tau, \quad x \in \Omega_k.$$ 

As $g(\tau)$ is bounded on $\gamma_k$, $\Gamma_k g$ is holomorphic in $\Omega_k$ and vanishes at the infinity. Hence $\Gamma_k g \in \mathcal{O}_0(\Omega_k)$. As $\gamma = \bigcup_{k=1}^{N} \gamma_k$, we have the decomposition

$$g = \sum_{k=1}^{N} \Gamma_k g.$$ 

To prove the uniqueness of the decomposition, assume $g_k \in \mathcal{O}_0(\Omega_k)$ for $k = 1, \cdots, N$, and

$$\sum_{k=1}^{N} g_k = 0.$$ 

Then $g_k$ is holomorphic in $\Omega_k$ and at the same time it can be analytically extended to $A_k$, since $-g_k = \sum_{j \neq k} g_j$ is holomorphic in $A_k$. This shows that $g_k$ is constant for $k = 1, \cdots, N$. However, $g_k$ takes value zero at the infinity if the infinity belongs to the domain of its definition. Therefore, $g_k = 0$ for all $k = 1, \cdots, N$, except one. But the exceptional one must be zero since $\sum_{k=1}^{N} g_k = 0$.

**Theorem 1.2**

$$\Gamma_k : \mathcal{O}_0(\Omega) \to \mathcal{O}_0(\Omega_k), \quad k = 1, \cdots, N$$

are projections.

**Proof** For all $g \in \mathcal{O}_0(\Omega)$, $\Gamma_k g$ is holomorphic in $\Omega_k$, hence we have $\Gamma_k^2 g = \Gamma_k g$. If $j \neq k$, then $\gamma_j \subset A_j \subset \Omega_k$. Therefore, $\Gamma_j \Gamma_k g = 0$ for all $g \in \mathcal{O}_0(\Omega)$. As we saw in the previous theorem, $\sum_{k=1}^{N} \Gamma_k = \text{id}$. 

**Definition 1.3** We define complex Ruelle operator $L : \mathcal{O}_0(\Omega) \to \mathcal{O}_0(\Omega)$ by

$$(Lg)(x) = \sum_{y \in R^{-1}(x)} \frac{g(y)}{(R'(y))^2}, \quad g \in \mathcal{O}_0(\Omega), \quad x \in \Omega.$$ 

Note that $R^{-1}(x) \subset \Omega$ and $R'(y) \neq 0$ as we assumed $R$ is hyperbolic and $\Omega$ contains no critical points. As indicated by [1], the complex Ruelle
operator can be expressed as an integral operator of the form:

$$(Lg)(x) = \frac{1}{2\pi i} \int_{\gamma} \frac{g(\tau)}{R'(\tau)(R(\tau) - x)} d\tau.$$ 

This formula is easily verified by applying the Cauchy’s theorem about residues and it shows that $Lg \in \mathcal{O}_0(\Omega)$. Comparing $L$ with the Perron-Frobenius operator, we see that the spectral radius of $L$ is smaller than 1.

**Definition 1.4**

$L_{ij} : \mathcal{O}_0(\Omega_j) \rightarrow \mathcal{O}_0(\Omega_i)$ is defined by $L_{ij} = \Gamma_i \circ L \mid_{\mathcal{O}_0(\Omega_i)}$.

The Ruelle operator can be expressed as an $N \times N$ matrix of operators:

$L = (L_{ij})$

The components $L_{ij}$ are computed as follows.

**Proposition 1.5** If $i \neq j$, then for $g_j \in \mathcal{O}_0(\Omega_j)$ and $x \in \Omega_i$,

$$(L_{ij}g_j)(x) = -\sum_{c \in \mathcal{C}_R \cap A_i} \frac{g_j(c)}{R''(c)(R(c) - x)}.$$ 

**Proof** As $g_j \in \mathcal{O}_0(\Omega_j)$ and $L_{ij}g_j$ is defined by

$$(L_{ij}g_j)(x) = \frac{1}{2\pi i} \int_{\gamma_i} \frac{g_j(\tau)}{R'(\tau)(R(\tau) - x)} d\tau,$$

we can apply the residue theorem to the complement of $\Omega_i$. The residues at the critical points in $A_i$ give the formula.

**Proposition 1.6** For $g_j \in \mathcal{O}_0(\Omega_j)$ and for $x \in \Omega_j$,

$$(L_{jj}g_j)(x) = \sum_{y \in R^{-1}(x)} \frac{g_j(y)}{(R'(y))^2} + \sum_{c \in \mathcal{C}_R \cap \Omega_j} \frac{g_j(c)}{R''(c)(R(c) - x)}.$$ 

**Proof** In this case, we can apply the residue theorem to $\Omega_j$.

2. Möbius transformation and complex Ruelle operator
In this section, we observe the behavior of the complex Ruelle operator under a coordinate change of the Riemann sphere by a Möbius transformation.

Let \( M : \mathbb{C} \to \mathbb{C} \) be a Möbius transformation of the Riemann sphere. Let \( \alpha = M^{-1}(\infty) \), \( \beta = M(\infty) \), and \( \hat{R} = M \circ R \circ M^{-1} \). We set \( \tilde{\Omega}_k = M(\Omega_k) \), \( \tilde{\Omega} = M(\Omega) \), and assume \( \alpha \notin \Omega \). In order to avoid confusion, we denote the complex Ruelle operator defined in the previous section by \( L_R \) associated to the rational mapping \( R \). Now, we define a “complex Ruelle operator” associated to the Möbius transformation \( M \).

**Definition 2.1**

\[
L_M : \mathcal{O}_0(\Omega) \to \mathcal{O}_0(\tilde{\Omega}) \quad \text{is defined by} \quad (L_M g)(\tilde{x}) = \frac{g \circ M^{-1}(\tilde{x})}{(M'(M^{-1}(\tilde{x})))^2},
\]

for \( g \in \mathcal{O}_0(\Omega) \) and \( \tilde{x} \in \tilde{\Omega} \).

**Proposition 2.2**

\[
L_{M^{-1}} = L_M^{-1}, \quad L_{\hat{R}} = L_M \circ L_R \circ L_{M^{-1}}.
\]

**Proof** First equality is easily verified by computing \( L_{M^{-1}} \circ L_M \) and \( L_M \circ L_{M^{-1}} \) directly. Second equality is easily verified similarly by the definition of the complex Ruelle operator. However, we would like to give a proof for the operator defined as an integral operator. Let \( \tilde{g} \in \mathcal{O}_0(\tilde{\Omega}) \). Then we have, for \( x \in \Omega \) and \( \tilde{x} \in \tilde{\Omega} \),

\[
(L_{M^{-1}} \tilde{g})(x) = \frac{\tilde{g} \circ M(x)}{((M^{-1})'(M(x)))^2},
\]

\[
(L_R L_{M^{-1}} \tilde{g})(x) = \frac{1}{2\pi i} \int_{\gamma} \frac{(L_{M^{-1}} \tilde{g})(\tau)}{R'(\tau)(R(\tau) - x)} \, d\tau
\]

\[
= \frac{1}{2\pi i} \int_{\gamma} \frac{\tilde{g} \circ M(\tau)}{R'(\tau)(R(\tau) - x)((M^{-1})'(M(\tau)))^2} \, d\tau,
\]

and

\[
(L_M L_R L_{M^{-1}} \tilde{g})(\tilde{x}) = \frac{1}{(M' \circ M^{-1}(\tilde{x}))^2} \frac{1}{2\pi i} \int_{\gamma} \frac{\tilde{g} \circ M(\tau) d\tau}{R'(\tau)(R(\tau) - M^{-1}(\tilde{x}))((M^{-1})'(M(\tau)))^2}.
\]
\[
= \frac{1}{2\pi i} \oint_{\gamma} \frac{\tilde{g} \circ M(\tau)(M'(\tau))^2 d\tau}{(M' \circ M^{-1}(\tilde{x}))^2 R'(\tau)(R(\tau) - M^{-1}(\tilde{x}))}.
\]

On the other hand, by a change of variables \(\sigma = M(\tau)\), we have
\[
(L_{\tilde{R}} \tilde{g})(\tilde{x}) = \frac{1}{2\pi i} \oint_{\gamma} \frac{\tilde{g}(\sigma)d\sigma}{\tilde{R}(\sigma)(\tilde{R}(\sigma) - \tilde{x})}
\]
\[
= \frac{1}{2\pi i} \oint_{\gamma} \frac{\tilde{g} \circ M(M'(\tau))d\tau}{M'(R(\tau))R'(\tau)(M^{-1} \circ M(\tau)(M \circ R(\tau) - \tilde{x})).}
\]

Hence we obtain
\[
(L_M L_R L_{M^{-1}} \tilde{g})(\tilde{x}) - (L_{\tilde{R}} \tilde{g})(\tilde{x})
\]
\[
= \frac{1}{2\pi i} \int_{\gamma} \frac{\tilde{g} \circ M(M'(\tau))^2}{R'(\tau)} \times
\]
\[
\left(\frac{1}{(M' \circ M^{-1}(\tilde{x}))^2(R(\tau) - M^{-1}(\tilde{x}))} - \frac{1}{M' \circ R(\tau)(M \circ R(\tau) - \tilde{x})}\right) d\tau.
\]

As \(R'(\tau) \neq 0\) and \(M' \circ R(\tau) \neq 0\) for \(\tau \in \Omega\), the integrand can have poles only at \(\tau \in R^{-1} \circ M^{-1}(\tilde{x}) \cap \Omega\). The residues at such points are, by setting \(x = M^{-1}(\tilde{x})\) and \(y \in R^{-1}(x)\), computed as
\[
\frac{\tilde{g} \circ M(y)(M'(y))^2}{R'(y)} \left(\frac{1}{(M'(x))^2 R'(y)} - \frac{1}{M' \circ R(y)(M' \circ R(y)R'(y))}\right) = 0.
\]

Hence the proposition follows.

**Definition 2.3** Components \(L_{M,i,j}: O_0(\Omega_j) \rightarrow O_0(\hat{\Omega}_i)\) is defined by \(L_{M,i,j}g_j = \tilde{\Gamma}_i L_M g_j\) for \(g_j \in O_0(\Omega_j)\), where \(\tilde{\Gamma}_i: O_0(\hat{\Omega}) \rightarrow O_0(\hat{\Omega}_i)\) denote the projection.

**Proposition 2.4** If \(\infty \in \Omega_j\), then
\[
(L_{M,ij}g_j)(\tilde{x}) = (L_M g_j)(\tilde{x}) + \operatorname{Res}_{\tau = \infty} \frac{g_j(\tau)}{M'(\tau)(M(\tau) - \tilde{x})}.
\]

If \(\infty \not\in \Omega_j\), then
\[
(L_{M,ij}g_j)(\tilde{x}) = (L_M g_j)(\tilde{x}).
\]

If \(i \neq j\) and \(\infty \in \Omega_i\), then
\[
L_{M,ij} = 0.
\]

If \(i \neq j\) and \(\infty \not\in \Omega_i\), then
\[
(L_{M,ij}g_j)(\tilde{x}) = -\operatorname{Res}_{\tau = \infty} \frac{g_j(\tau)}{M'(\tau)(M(\tau) - \tilde{x})}.
\]
PROOF These formulas are easily verified by a direct computation by applying the residue theorem.

3. Partial complex Ruelle operator

In this section, we examine a diagonal component of the complex Ruelle operator. The Fredholm determinant and the resolvent of the adjoint diagonal component $L_{ii} : \mathcal{O}_0(\Omega_i) \rightarrow \mathcal{O}_0(\Omega_i)$ of the complex Ruelle operator can be computed in a similar manner as is given by [1] and [2].

For the sake of simplicity, we assume $\omega_1 = \infty$, and $\omega_1$ is an attractive fixed point with eigenvalue $\sigma$ satisfying $0 < |\sigma| < 1$. We define the partial Ruelle operator as follows.

**DEFINITION 3.1 Partial Ruelle operator** $L_{11} : \mathcal{O}_0(\Omega_1) \rightarrow \mathcal{O}_0(\Omega_1)$ is defined by

$$(L_{11}g)(x) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{g(\tau)d\tau}{R'(\tau)(R(\tau) - x)}.$$ 

An explicit formula for the partial Ruelle operator is given by proposition 1.6. We shall consider the dual operator.

**DEFINITION 3.2 The dual space** $\mathcal{O}^*_0(\Omega_1)$ of $\mathcal{O}_0(\Omega_1)$ is the space of continuous, complex linear, and holomorphic functional $F : \mathcal{O}_0(\Omega_1) \rightarrow \mathbb{C}$. The topology of $\mathcal{O}_0(\Omega_1)$ is understood as the uniform convergence in a neighborhood of the closure of $\Omega_1$. A functional is said to be holomorphic if the value $F[g_\mu]$ is holomorphic with respect to the parameter $\mu$ for a holomorphic family of functions $g_\mu$.

**PROPOSITION 3.3** For any $F \in \mathcal{O}^*_0(\Omega_1)$, there exists an $f \in \mathcal{O}_0(\overline{\mathbb{C}} \setminus \Omega_1)$, such that

$$F[g] = \frac{1}{2\pi i} \int_{\gamma_1} f(\tau)g(\tau)d\tau, \quad \text{for} \quad g \in \mathcal{O}_0(\Omega_1).$$

**PROOF** In fact, the so called Cauchy transform

$$f(z) = F[\frac{1}{z - \zeta}]$$

gives such a function. As $\frac{1}{z - \zeta}$ is a holomorphic family of functions in $\mathcal{O}_0(\Omega_1)$ parametrized by $z \in \overline{\mathbb{C}} \setminus \Omega_1$. $f(z)$ is holomorphic in $\overline{\mathbb{C}} \setminus \Omega_1$ and $f(\infty) = 0$, ...
hence $f \in O_0(\overline{C} \setminus \Omega_1)$. For $g \in O_0(\Omega_1)$,

$$F[g] = F[\frac{1}{2\pi i} \int_{\gamma_1} \frac{g(z)}{z - \zeta} dz]$$

$$= \frac{1}{2\pi i} \int_{\gamma_1} g(z) F[\frac{1}{z - \zeta}] dz = \frac{1}{2\pi i} \int_{\gamma_1} g(z) f(z) dz.$$

Note that such function $f(z) \in O_0(\overline{C} \setminus \Omega_1)$ is unique since

$$\frac{1}{2\pi i} \int_{\gamma_1} f(\tau) \frac{1}{z - \tau} d\tau = f(z).$$

**Proposition 3.4** The dual operator $L^*_1 : O_0^*(\Omega_1) \to O_0^*(\Omega_1)$ is represented by integral operator $L^*_1 : O_0(\overline{C} \setminus \Omega_1) \to O_0(\overline{C} \setminus \Omega_1)$ defined, for $f \in O_0(\overline{C} \setminus \Omega_1)$ and $z \in \overline{C} \setminus \Omega_1$, by

$$(L^*_1 f)(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(R(\tau)) d\tau}{R'(\tau)(z - \tau)}$$

$$= \frac{f(R(z))}{R'(z)} - \sum_{c \in C \cap A_2} \frac{f(R(c))}{R''(c)(z - c)}.$$

**Proof** The proof is almost same as in [1]. By a direct computation, we have

$$(L^*_1 f)(z) = (L^*_1 F)[\frac{1}{z - \zeta}] = F[L_1[\frac{1}{z - \zeta}]]$$

$$= F[\frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\tau)}{R'(\tau)(\tau - \zeta)(\tau - z)} d\tau]$$

$$= \frac{1}{2\pi i} \int_{\gamma_1} f(\zeta) d\zeta \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\tau)}{R'(\tau)(\tau - \zeta)(\tau - z)} d\tau$$

$$= \frac{1}{2\pi i} \int_{\gamma_1} \frac{d\tau}{R'(\tau)(\tau - z)} 2\pi i \int_{\gamma_1} \frac{f(\zeta)}{R''(\tau)(\tau - \zeta)} d\tau$$

$$= \frac{f(R(z))}{R'(z)} - \sum_{c \in C \cap A_1} \frac{f(R(c))}{R''(c)(z - c)}.$$

Note that $L^*_1 f \in O_0(\overline{C} \setminus \Omega_1)$ and the poles at the critical points in the last line of the above calculation cancel out.

**4. Fredholm determinant of the adjoint Ruelle operator**

In this section, we compute the Fredholm determinant and the resolvent of the adjoint operator $L^*_1$. The calculation is almost same as in [1].
Let $n_1$ denote the number of critical points in $A_1$. And let $\{c_1, \ldots, c_{n_1}\}$ be the critical points in $A_1$. Let

$$H(x, z; \lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{(R^{*n})(z)(R^{*n}(z) - x)},$$

and let

$$M(\lambda) = \left( \delta_{ij} + \frac{\lambda}{R''(c_i)} H(c_j, R(c_i); \lambda) \right)_{i,j=1}^{n_1}$$

be an $n_1 \times n_1$ matrice.

**Theorem 4.1** The Fredholm determinant $D_{11}(\lambda)$ of $\mathcal{L}^*_1$ is given by

$$D_{11}(\lambda) = \prod_{n=1}^{\infty} (1 - \sigma^{n+1}\lambda) \det M(\lambda).$$

It is meromorphic in $\mathbb{C}$ and holomorphic for $|\lambda| < |\sigma|^{-2}$. $\mathcal{L}^*_1$ has no essential spectrum.

This theorem follows immediately from proposition 4.3 below. We assume that the backward orbits of critical points do not intersect with the curve $\gamma_1$. Let

$$\Omega_R = \mathbb{C} \setminus (\Omega_1 \cup \bigcup_{n=0}^{\infty} \{z \in \Omega_1 \mid R'(R^{*n}(z)) = 0\}).$$

Then $\mathcal{O}_0(\Omega_1) \subset \mathcal{O}_0(\Omega_R)$. Define $\mathcal{L}^*_R : \mathcal{O}_0(\Omega_R) \to \mathcal{O}_0(\Omega_R)$ by

$$(\mathcal{L}^*_R f)(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(R(\tau))d\tau}{R'(\tau)(z - \tau)}$$

for $f \in \mathcal{O}_0(\Omega_R)$. We see immediately that the image of $\mathcal{L}^*_R$ is included in $\mathcal{O}_0(\Omega_1)$, and $\mathcal{L}^*_R$ and $\mathcal{L}^*_1$ coincide on $\mathcal{O}_0(\Omega_1)$. Therefore, $\mathcal{L}^*_R$ and $\mathcal{L}^*_1$ has the same spectrum. Define a operator $\mathcal{K} : \mathcal{O}_0(\Omega_R) \to \mathcal{O}_0(\Omega_R)$ by

$$(\mathcal{K} f)(z) = \frac{f(R(z))}{R'(z)}, \quad f \in \mathcal{O}_0(\Omega_R), z \in \Omega_R.$$ 

**Proposition 4.2** The spectrum of $\mathcal{K}$ is $\{\sigma^{n+1}\}_{n=1}^{\infty}$, and the Fredholm determinant is given by

$$\det(I - \lambda \mathcal{K}) = \prod_{n=1}^{\infty} (1 - \sigma^{n+1}\lambda).$$
The eigenfunction \(f_n(z)\) for eigenvalue \(\lambda^{-1} = \sigma^{n+1}\) is given by

\[
f_n(z) = \frac{1}{\varphi'(z)(\varphi(z))^{n}},
\]

where holomorphic function \(\varphi : \Omega_R \rightarrow \overline{\mathbb{C}}\) is the Schröder’s function of the form \(\varphi(z) = z + a_0 + \frac{a_0}{z} + \cdots + \frac{a_n}{z^n} + \cdots\) near the infinity and satisfying the Schröder’s equation \(\varphi(R(z)) = \sigma^{-1}\varphi(z)\).

**Proof** The fact that \(f_n(z)\) is an eigenfunction for eigenvalue \(\sigma^{n+1}\) is immediately verified by using the Shröder’s equation. Eigenfunction \(f_n(z)\) can be extended to \(\Omega_R\) by using the function equations

\[
Kf_n = \sigma^{n+1}f_n \quad \text{and} \quad (Kf_n)(z) = \frac{f_n(R(z))}{R'(z)}.
\]

As the eigenfunctions \(\{f_n\}_{n=1}^{\infty}\) form a complete basis of \(\mathcal{O}_0(\Omega_R)\), where \(\Omega_R^\infty\) denotes the connected component of \(\Omega_R\) containing the infinity, and as the eigenfunctions are determined from a germ at the infinity of the eigenfunction by the function equation above, the Fredholm determinant is given by the formula in the proposition.

Define linear maps \(G : \mathcal{O}_0(\Omega_R) \rightarrow \mathbb{C}^{n_1}\) and \(F : \mathbb{C}^{n_1} \rightarrow \mathcal{O}_0(\Omega_R)\) by

\[
Gf = \left(\frac{f(R(c_j)))}{R'(c_j)}\right)_{j=1}^{n_1}, \quad f \in \mathcal{O}_0(\Omega_R),
\]

\[
F\alpha = \sum_{j=1}^{n_1} \frac{\alpha_j}{z - c_j}, \quad \alpha = (\alpha_j) \in \mathbb{C}^{n_1}.
\]

We have

\[
\mathcal{L}_R^* = K - FG.
\]

The Fredholm determinant of the adjoint Ruelle operator \(\mathcal{L}_R^*\) is computed as follows.

**Proposition 4.3**

\[
D_{11}(\lambda) = \det(I - \lambda \mathcal{L}_R^*) = \det(I - \lambda K) \det M(\lambda),
\]

where \(M(\lambda) = I + \lambda G(I - \lambda K)^{-1}F\).
PROOF

\[
\det(I - \lambda L_R^*) = \det(I - \lambda \mathcal{K} + \lambda F G) \\
= \det(I - \lambda \mathcal{K}) \det(I + \lambda(I - \lambda \mathcal{K})^{-1} F G) \\
= \det(I - \lambda \mathcal{K}) \det(I_{n_1} + \lambda G(I - \lambda \mathcal{K})^{-1} F) \\
= \det(I - \lambda \mathcal{K}) \det M(\lambda).
\]

The \(n_1 \times n_1\) matrix \(M(\lambda)\) is computed as follows.

\[
M(\lambda) = I_{n_1} + \lambda G(I - \lambda \mathcal{K})^{-1} F \\
= I_{n_1} + \lambda G(\sum_{n=0}^{\infty} \lambda^n \mathcal{K}^n) F \\
= I_{n_1} + \lambda G\left(\sum_{n=0}^{\infty} \frac{\lambda^n}{(R^n)'(z)(R^n(z) - c_j)}\right)_{j=1}^{n_1} \\
= I_{n_1} + \lambda \left(\sum_{n=0}^{\infty} \frac{\lambda^n}{(R^n)'(c_i)(R^n(c_i) - c_j)}\right)_{i,j=1}^{n_1} \\
= (\delta_{ij} + \frac{\lambda}{R^n(c_i)} H(c_j, R(c_i); \lambda))_{i,j=1}^{n_1}.
\]

As the spectrum of \(\mathcal{K}\) is \(\{\sigma^{n+1} \}_{n=1}^{\infty}\), \(M(\lambda)\) is meromorphic in \(\mathbb{C}\) and holomorphic in \(\{\lambda \mid |\lambda| < \sigma^{-2}\}\). This completes the proof of the proposition 4.3 and the Theorem 4.1.

5. The resolvent of the partial adjoint Ruelle operator

The resolvent of the partial adjoint Ruelle operator \(L_R^*\) can be computed in an analogous manner as in [1] and [2]. First, we compute the resolvent function of operator \(\mathcal{K}\).

**Proposition 5.1** The function \(H(x, z; \lambda)\) defined in the previous section is the resolvent function of \(\mathcal{K}\), i.e.,

\[
H(x, z; \lambda) = (I - \lambda \mathcal{K})^{-1} \frac{1}{z - x}
\]

\[
= \sum_{n=0}^{\infty} \lambda^n \mathcal{K}^n \frac{1}{z - x} = \sum_{n=0}^{\infty} \frac{\lambda^n}{(R^n)'(z)(R^n(z) - x)},
\]

where \(x \in \Omega_1\), \(z \in \Omega_R\) and \(\lambda \in \mathbb{C} \setminus \{\sigma^{-k} \}_{k=2}^{\infty}\). \(H(x, z; \lambda)\) is holomorphic in \(x, z\) and \(\lambda\).

**Proof** As is easily observed, we have

\[
\mathcal{K}^n f(z) = \frac{f(R^n(z))}{(R^n)'(z)}.
\]
Let $f_n : \Omega_R \to \mathbb{C}, n = 1, 2, \ldots,$ be the complete system of eigenfunctions of $\mathcal{K}$ given by proposition 4.2. For each $x \in \Omega_1$, we can expand the function
$(z - x)^{-1} \in \mathcal{O}_0(\Omega_R)$ in the form
$$
\frac{1}{z - x} = \sum_{n=1}^{\infty} b_n(x) f_n(z).
$$

Observe that $b_n(x)$ is holomorphic in $\Omega_1$. With this expression, we have
$$
(I - \lambda \mathcal{K})^{-1} \frac{1}{z - x} = \sum_{n=0}^{\infty} \lambda^n \mathcal{K}^n \frac{1}{z - x}
$$
$$
= \sum_{n=0}^{\infty} \lambda^n \mathcal{K}^n \sum_{k=1}^{\infty} b_k(x) f_k(z) = \sum_{k=1}^{\infty} b_k(x) \sum_{n=0}^{\infty} \lambda^n \mathcal{K}^n f_k(z)
$$
$$
= \sum_{k=1}^{\infty} b_k(x) \sum_{n=0}^{\infty} (\lambda \sigma^{k+1})^n f_k(z) = \sum_{k=1}^{\infty} b_k(x) \frac{1}{1 - \lambda \sigma^{k+1}} f_k(z).
$$

This shows that $H(x, z; \lambda)$ has an analytic extension to the domain $\Omega_1 \times \Omega_R \times (\mathbb{C} \setminus \{\sigma^{-k}\}_{k=2}^{\infty})$.

The resolvent function $E(x, z; \lambda)$ is defined by
$$
E(x, z; \lambda) = (I - \lambda \mathcal{L}_{11}^*)^{-1} \frac{1}{z - x}, \quad x \in \Omega_1, z \in \mathbb{C} \setminus \Omega_1, \lambda \in \mathbb{C}, E(x, \infty; \lambda) = 0.
$$

$E(x, z; \lambda)$ is holomorphic in $x$ and $z$, and meromorphic in $\lambda$.

**Proposition 5.2**

$$(I - \lambda \mathcal{L}_{11}^*)^{-1} = (I - \lambda \mathcal{K})^{-1} - \lambda (I - \lambda \mathcal{K})^{-1} \mathcal{F}(M(\lambda))^{-1} \mathcal{G}(I - \lambda \mathcal{K})^{-1},$$

where $M(\lambda) = I_{n_1} + \lambda \mathcal{G}(I - \lambda \mathcal{K})^{-1} \mathcal{F}$ is an $n_1 \times n_1$ matrix.

**Proof** By a direct computation.

$$(I - \lambda \mathcal{L}_{11}^*)^{-1} = (I - \lambda \mathcal{K} + \lambda \mathcal{F} \mathcal{G})^{-1}
$$
$$
= (I - \lambda \mathcal{K})^{-1} (I + \lambda \mathcal{F} \mathcal{G}(I - \lambda \mathcal{K})^{-1})^{-1}
$$
$$
= (I - \lambda \mathcal{K})^{-1} (I + \sum_{k=1}^{\infty} (-\lambda)^k (\mathcal{F} \mathcal{G}(I - \lambda \mathcal{K})^{-1})^k)
$$
$$
= (I - \lambda \mathcal{K})^{-1} (I - \lambda \mathcal{F} \sum_{k=0}^{\infty} (-\lambda)^k (I - \lambda \mathcal{K})^{-1})^k \mathcal{G}(I - \lambda \mathcal{K})^{-1}
$$
$$
= (I - \lambda \mathcal{K})^{-1} - \lambda (I - \lambda \mathcal{K})^{-1} \mathcal{F}(M(\lambda))^{-1} \mathcal{G}(I - \lambda \mathcal{K})^{-1}.
$$

As is easily verified, $M(\lambda)$ in this proposition is same as $M(\lambda)$ given in the beginning of the previous section.

Let
$$
H_1(z; \lambda) = ((I - \lambda \mathcal{K})^{-1} \frac{1}{z - c_i})_{i=1}^{n_1} = (H(c_i, z; \lambda))_{i=1}^{n_1}
$$
be a row vector and let
\[ H_2(x; \lambda) = G(I - \lambda K)^{-1} \frac{1}{z - x} = GH(x, z; \lambda) = \left( \frac{H(x, R(c_i); \lambda)}{R''(c_i)} \right)_{i=1}^{n_1} \]
be a column vector. With all these things together, we find the explicit expression of the resolvent function \( E(x, z; \lambda) \).

**Theorem 5.3**

\[ E(x, z; \lambda) = H(x, z; \lambda) + \lambda H_1(z; \lambda)(M(\lambda))^{-1}H_2(x; \lambda). \]

The resolvent function is holomorphic in \( x \in \Omega_1 \), and in \( z \in \Omega_R \), and meromorphic in \( \lambda \in \mathbb{C} \). The poles are the zeros of the Fredholm determinant \( D_{11}(\lambda) \).

**References**

