Complex Ruelle Operator
and
Hyperbolic Complex Dynamical Systems

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1. Decomposition of Complex Ruelle operator

Let $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a hyperbolic rational mapping. We assume that all the attractive periodic points of R are fixed points, all the critical points of R are non-degenerate, and that the Julia set of R, J_R, is inculed in $\mathbb{C}$. Let N denote the number of attractive fixed points and let $a_1, \cdots, a_N$ denote the attractive fixed points. Let A_k denote the attractive basin of a_k. Let C_R denote the set of critical points of R.

For $k = 1, \cdots, N$, let γ_k denote an oriented multicurve in A_k, such that $\gamma_k = \partial\Omega_k$, where Ω_k is an open set satisfying $R^{-1}(\Omega_k) \subset \Omega_k$, $\Omega_k \cup A_k = \overline{\mathbb{C}}$, and $C_R \cap \Omega_k \cap A_k = \phi$. Let $\gamma = \cup_{k=1}^{N} \gamma_k$ and $\Omega = \cap_{k=1}^{N} \Omega_k$.

For open set $O \subset \overline{\mathbb{C}}$, let $\mathcal{O}_0(O)$ denote the space of functions $g : O \to \mathbb{C}$ holomorphic in O and has an analytic extention to a neighbourhood of the closure of O, and satisfies $g(\infty) = 0$ if ∞ belongs to the closure of O. We have the following decomposition of holomorphic functions. The direct sum in the theorem means the uniqueness of the decomposition.

Theorem 1.1

$$\mathcal{O}_0(\Omega) = \bigoplus_{k=1}^{N} \mathcal{O}_0(\Omega_k).$$

Proof Let $g \in \mathcal{O}_0(\Omega)$. Then g can be expressed as

$$g(x) = \frac{1}{2\pi i} \int_\gamma \frac{g(\tau)}{\tau - x} d\tau, \quad x \in \Omega.$$

For $k = 1, \cdots, N$, let $\Gamma_k : \mathcal{O}_0(\Omega) \to \mathcal{O}_0(\Omega_k)$ be defined by

$$(\Gamma_k g)(x) = \frac{1}{2\pi i} \int_{\gamma_k} \frac{g(\tau)}{\tau - x} d\tau, \quad x \in \Omega_k.$$

As $g(\tau)$ is bouded on γ_k, $\Gamma_k g$ is holomorphic in Ω_k and vanishes at the infinity. Hence $\Gamma_k g \in \mathcal{O}_0(\Omega_k)$. As $\gamma = \cup_{k=1}^N \gamma_k$, we have the decomposition

$$g = \sum_{k=1}^N \Gamma_k g.$$

To prove the uniqueness of the decomposition, assume $g_k \in \mathcal{O}_0(\Omega_k)$ for $k = 1, \cdots, N$, and

$$\sum_{k=1}^N g_k = 0.$$

Then g_k is holomorphic in Ω_k and at the same time it can be analytically extended to A_k, since $-g_k = \sum_{j \neq k} g_j$ is holomorphic in A_k. This shows that g_k is constant for $k = 1, \cdots, N$. However, g_k takes value zero at the infinity if the infinity belongs to the domain of its definition. Therefore, $g_k = 0$ for all $k = 1, \cdots, N$, except one. But the exceptional one must be zero since $\sum_{k=1}^N g_k = 0$.

THEOREM 1.2

$$\Gamma_k : \mathcal{O}_0(\Omega) \to \mathcal{O}_0(\Omega_k), \quad k = 1, \cdots, N$$

are projections.

PROOF For all $g \in \mathcal{O}_0(\Omega)$, $\Gamma_k g$ is holomorphic in Ω_k, hence we have $\Gamma_k^2 g = \Gamma_k g$. If $j \neq k$, then $\gamma_j \subset A_j \subset \Omega_k$. Therefore, $\Gamma_j \Gamma_k g = 0$ for all $g \in \mathcal{O}_0(\Omega)$. As we saw in the previous theorem, $\sum_{k=1}^N \Gamma_k = id$.

DEFINITION 1.3 We define *complex Ruelle operator* $L : \mathcal{O}_0(\Omega) \to \mathcal{O}_0(\Omega)$ by

$$(Lg)(x) = \sum_{y \in R^{-1}(x)} \frac{g(y)}{(R'(y))^2}, \quad g \in \mathcal{O}_0(\Omega), \quad x \in \Omega.$$

Note that $R^{-1}(x) \subset \Omega$ and $R'(y) \neq 0$ as we assumed R is hyperbolic and Ω contains no critical points. As indicated by [1], the complex Ruelle operator can be expressed as an integral operator of the form :

$$(Lg)(x) = \frac{1}{2\pi i}\int_\gamma \frac{g(\tau)}{R'(\tau)(R(\tau)-x)}d\tau.$$

This formula is easily verified by applying the Cauchy's theorem about residues and it shows that $Lg \in \mathcal{O}_0(\Omega)$. Comparing L with the Perron-Frobenius operator, we see that the spectral radius of L is smaller than 1.

DEFINITION 1.4

$$L_{ij} : \mathcal{O}_0(\Omega_j) \to \mathcal{O}_0(\Omega_i) \quad \text{is defined by} \quad L_{ij} = \Gamma_i \circ L\,|_{\mathcal{O}_0(\Omega_j)}\,.$$

The Ruelle operator can be expressed as an $N \times N$ matrice of operators :

$$L = (L_{ij})$$

The components L_{ij} are computed as follows.

PROPOSITION 1.5 If $i \neq j$, then for $g_j \in \mathcal{O}_0(\Omega_j)$ and $x \in \Omega_i$,

$$(L_{ij}g_j)(x) = -\sum_{c \in C_R \cap A_i} \frac{g_j(c)}{R''(c)(R(c)-x)}.$$

PROOF As $g_j \in \mathcal{O}_0(\Omega_j)$ and $L_{ij}g_j$ is defined by

$$(L_{ij}g_j)(x) = \frac{1}{2\pi i}\int_{\gamma_i} \frac{g_j(\tau)}{R'(\tau)(R(\tau)-x)}d\tau,$$

we can apply the residue theorem to the complement of Ω_i. The residues at the critical points in A_i give the formula.

PROPOSITION 1.6 For $g_j \in \mathcal{O}_0(\Omega_j)$ and for $x \in \Omega_j$,

$$(L_{jj})g_j(x) = \sum_{y \in R^{-1}(x)} \frac{g_j(y)}{(R'(y))^2} + \sum_{c \in C_R \cap \Omega_j} \frac{g_j(c)}{R''(c)(R(c)-x)}.$$

PROOF In this case, we can apply the residue theorem to Ω_j.

2. Möbius transformation and complex Ruelle operator

In this section, we observe the behavior of the complex Ruelle operator under a coordinate change of the Riemenn sphere by a Möbius transformation.

Let $M : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a Möbius transformation of the Riemann sphere. Let $\alpha = M^{-1}(\infty)$, $\beta = M(\infty)$, and $\tilde{R} = M \circ R \circ M^{-1}$. We set $\tilde{\Omega}_k = M(\Omega_k)$,$\tilde{\Omega} = M(\Omega)$, and assume $\alpha \notin \Omega$. In order to avoid confusion, we denote the complex Ruelle operator defined in the previous section by L_R associated to the rational mapping R. Now, we define a "complex Ruelle operator" associated to the Möbius transformation M.

Definition 2.1

$$L_M : \mathcal{O}_0(\Omega) \to \mathcal{O}_0(\tilde{\Omega}) \qquad \text{is defined by } (L_M g)(\tilde{x}) = \frac{g \circ M^{-1}(\tilde{x})}{(M'(M^{-1}(\tilde{x})))^2},$$

$$\text{for} \;\; g \in \mathcal{O}_0(\Omega) \;\; \text{and} \;\; \tilde{x} \in \tilde{\Omega}.$$

Proposition 2.2

$$L_{M^{-1}} = L_M^{-1}, \quad L_{\tilde{R}} = L_M \circ L_R \circ L_{M^{-1}}.$$

Proof First equality is easily verified by computing $L_{M^{-1}} \circ L_M$ and $L_M \circ L_{M^{-1}}$ directly. Second equality is easily verified similarly by the definition of the complex Ruelle operator. However, we would like to give a proof for the operator defined as an integral operator. Let $\tilde{g} \in \mathcal{O}_0(\tilde{\Omega})$. Then we have, for $x \in \Omega$ and $\tilde{x} \in \tilde{\Omega}$,

$$(L_{M^{-1}}\tilde{g})(x) = \frac{\tilde{g} \circ M(x)}{((M^{-1})'(M(x)))^2},$$

$$(L_R L_{M^{-1}}\tilde{g})(x) = \frac{1}{2\pi i}\int_\gamma \frac{(L_{M^{-1}}\tilde{g})(\tau)}{R'(\tau)(R(\tau)-x)}d\tau$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{\tilde{g} \circ M(\tau)}{R'(\tau)(R(\tau)-x)((M^{-1})' \circ M(\tau))^2}d\tau,$$

and

$$(L_M L_R L_{M^{-1}}\tilde{g})(\tilde{x})$$

$$= \frac{1}{(M' \circ M^{-1}(\tilde{x}))^2}\frac{1}{2\pi i}\int_\gamma \frac{\tilde{g} \circ M(\tau)d\tau}{R'(\tau)(R(\tau)-M^{-1}(\tilde{x}))((M^{-1})' \circ M(\tau))^2}$$

$$= \frac{1}{2\pi i}\int_{\gamma}\frac{\tilde{g}\circ M(\tau)(M'(\tau))^2 d\tau}{(M'\circ M^{-1}(\tilde{x}))^2 R'(\tau)(R(\tau)-M^{-1}(\tilde{x}))}.$$

On the other hand, by a change of variables $\sigma = M(\tau)$, we have

$$(L_{\tilde{R}}\tilde{g})(\tilde{x}) = \frac{1}{2\pi i}\int_{\tilde{\gamma}}\frac{\tilde{g}(\sigma)d\sigma}{\tilde{R}'(\sigma)(\tilde{R}(\sigma)-\tilde{x})}$$

$$= \frac{1}{2\pi i}\int_{\gamma}\frac{\tilde{g}\circ M(\tau)M'(\tau)d\tau}{M'(R(\tau))R'(\tau)(M^{-1})'\circ M(\tau)(M\circ R(\tau)-\tilde{x})}.$$

Hence we obtain

$$(L_M L_R L_{M^{-1}}\tilde{g})(\tilde{x}) - (L_{\tilde{R}}\tilde{g})(\tilde{x})$$
$$= \frac{1}{2\pi i}\int_{\gamma}\frac{\tilde{g}\circ M(\tau)(M'(\tau))^2}{R'(\tau)}\times$$
$$\times\left(\frac{1}{(M'\circ M^{-1}(\tilde{x}))^2(R(\tau)-M^{-1}(\tilde{x}))} - \frac{1}{M'\circ R(\tau)(M\circ R(\tau)-\tilde{x})}\right)d\tau.$$

As $R'(\tau)\neq 0$ and $M'\circ R(\tau)\neq 0$ for $\tau\in\Omega$, the integrand can have poles only at $\tau\in R^{-1}\circ M^{-1}(\tilde{x})\cap\Omega$. The residues at such points are, by setting $x = M^{-1}(\tilde{x})$ and $y\in R^{-1}(x)$, computed as

$$\frac{\tilde{g}\circ M(y)(M'(y))^2}{R'(y)}\left(\frac{1}{(M'(x))^2 R'(y)} - \frac{1}{M'\circ R(y)(M'\circ R(y)R'(y))}\right) = 0.$$

Hence the proposition follows.

Definition 2.3 Components $L_{M,ij}:\mathcal{O}_0(\Omega_j)\to\mathcal{O}_0(\tilde{\Omega}_i)$ is defined by $L_{M,ij}g_j = \tilde{\Gamma}_i L_M g_j$ for $g_j\in\mathcal{O}_0(\Omega_j)$, where $\tilde{\Gamma}_i:\mathcal{O}_0(\tilde{\Omega})\to\mathcal{O}_0(\tilde{\Omega}_i)$ denote the projection.

Proposition 2.4 If $\infty\in\Omega_j$, then

$$(L_{M,jj}g_j)(\tilde{x}) = (L_M g_j)(\tilde{x}) + \mathrm{Res}_{\tau=\infty}\frac{g_j(\tau)}{M'(\tau)(M(\tau)-\tilde{x})}.$$

If $\infty\notin\Omega_j$, then

$$(L_{M,jj}g_j)(\tilde{x}) = (L_M g_j)(\tilde{x}).$$

If $i\neq j$ and $\infty\in\Omega_i$, then

$$L_{M,ij} = 0.$$

If $i\neq j$ and $\infty\notin\Omega_i$, then

$$(L_{M,ij}g_j)(\tilde{x}) = -\mathrm{Res}_{\tau=\infty}\frac{g_j(\tau)}{M'(\tau)(M(\tau)-\tilde{x})}.$$

PROOF These formulas are easily verified by a direct computation by applying the residue theorem.

3. Partial complex Ruelle operator

In this section, we examine a diagonal component of the complex Ruelle operator. The Fredholm determinant and the resolvent of the adjoint diagonal component $L_{ii} : \mathcal{O}_0(\Omega_i) \to \mathcal{O}_0(\Omega_i)$ of the complex Ruelle operator can be computed in a similar manner as is given by [1] and [2].

For the sake of simplicity, we assume $a_1 = \infty$, and a_1 is an attactive fixed point with eigenvalue σ satisfying $0 < |\sigma| < 1$. We define the partial Ruelle operator as follows.

DEFINITION 3.1 *Partial Ruelle operator* $L_{11} : \mathcal{O}_0(\Omega_1) \to \mathcal{O}_0(\Omega_1)$ is defined by

$$(L_{11}g)(x) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{g(\tau)d\tau}{R'(\tau)(R(\tau) - x)}.$$

An expicit formula for the partial Ruelle operator is given by proposition 1.6. We shall consider the dual operator.

DEFINITION 3.2 The *dual space* $\mathcal{O}_0^*(\Omega_1)$ of $\mathcal{O}_0(\Omega_1)$ is the space of continuous, complex linear, and holomorphic functional $F : \mathcal{O}_0(\Omega_1) \to \mathbb{C}$. The topology of $\mathcal{O}_0(\Omega_1)$ is understood as the uniform convergence in a neighborhood of the closure of Ω_1. A functional is said to be holomorphic if the value $F[g_\mu]$ is holomophic with respect to the parameter μ for a holomorphic family of functions g_μ.

PROPOSITION 3.3 For any $F \in \mathcal{O}_0^*(\Omega_1)$, there exists an $f \in \mathcal{O}_0(\overline{\mathbb{C}} \backslash \Omega_1)$, such that

$$F[g] = \frac{1}{2\pi i} \int_{\gamma_1} f(\tau)g(\tau)d\tau, \quad \text{for} \quad g \in \mathcal{O}_0(\Omega_1).$$

PROOF In fact, the so called Cauchy transform

$$f(z) = F[\frac{1}{z - \zeta}]$$

gives such a function. As $\frac{1}{z-\zeta}$ is a holomorphic family of functions in $\mathcal{O}_0(\Omega_1)$ parametrized by $z \in \overline{\mathbb{C}} \setminus \Omega_1$. $f(z)$ is holomorphic in $\overline{\mathbb{C}} \setminus \Omega_1$ and $f(\infty) = 0$, hence $f \in \mathcal{O}_0(\overline{\mathbb{C}} \setminus \Omega_1)$. For $g \in \mathcal{O}_0(\Omega_1)$,

$$F[g] = F[\frac{1}{2\pi i}\int_{\gamma_1} \frac{g(z)}{z-\zeta} dz]$$

$$= \frac{1}{2\pi i}\int_{\gamma_1} g(z) F[\frac{1}{z-\zeta}] dz = \frac{1}{2\pi i}\int_{\gamma_1} g(z) f(z) dz.$$

Note that such function $f(z) \in \mathcal{O}_0(\overline{\mathbb{C}} \setminus \Omega_1)$ is unique since

$$\frac{1}{2\pi i}\int_{\gamma_1} f(\tau)\frac{1}{z-\tau} d\tau = f(z).$$

PROPOSITION 3.4 The dual operator $L_{11}^* : \mathcal{O}_0^*(\Omega_1) \to \mathcal{O}_0^*(\Omega_1)$ is represented by integral operator $\mathcal{L}_{11}^* : \mathcal{O}_0(\overline{\mathbb{C}} \setminus \Omega_1) \to \mathcal{O}_0(\overline{\mathbb{C}} \setminus \Omega_1)$ defined, for $f \in \mathcal{O}_0(\overline{\mathbb{C}} \setminus \Omega_1)$ and $z \in \overline{\mathbb{C}} \setminus \Omega_1$, by

$$(\mathcal{L}_{11}^* f)(z) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(R(\tau))d\tau}{R'(\tau)(z-\tau)}$$

$$= \frac{f(R(z))}{R'(z)} - \sum_{c \in C_R \cap A_1} \frac{f(R(c))}{R''(c)(z-c)}.$$

PROOF The proof is almost same as in [1]. By a direct computation, we have

$$\begin{aligned}
(\mathcal{L}_{11}^* f)(z) &= (L_{11}^* F)[\tfrac{1}{z-\zeta}] = F[L_{11}[\tfrac{1}{z-\zeta}]] \\
&= F[\tfrac{1}{2\pi i}\textstyle\int_{\gamma_1} \frac{d\tau}{R'(\tau)(R(\tau)-\zeta)(z-\tau)}] \\
&= \tfrac{1}{2\pi i}\textstyle\int_{\gamma_1} f(\zeta)d\zeta \tfrac{1}{2\pi i}\int_{\gamma_1} \frac{d\tau}{R'(\tau)(R(\tau)-\zeta)(z-\tau)} \\
&= \tfrac{1}{2\pi i}\textstyle\int_{\gamma_1} \frac{d\tau}{R'(\tau)(z-\tau)} \tfrac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)d\zeta}{R(\tau)-\zeta} \\
&= \tfrac{1}{2\pi i}\textstyle\int_{\gamma_1} \frac{f(R(\tau))d\tau}{R'(\tau)(z-\tau)} \\
&= \tfrac{f(R(z))}{R'(z)} - \textstyle\sum_{c \in C_R \cap A_1} \mathrm{Res}_{\tau=c} \frac{f(R(\tau))}{R'(\tau)(z-\tau)} \\
&= \tfrac{f(R(z))}{R'(z)} - \textstyle\sum_{c \in C_R \cap A(\infty)} \frac{f(R(c))}{R''(c)(z-c)}.
\end{aligned}$$

Note that $\mathcal{L}_{11}^* f \in \mathcal{O}_0(\overline{\mathbb{C}} \setminus \Omega_1)$ and the poles at the critical ponts in the last line of the above calculation cancel out.

4. Fredholm determinant of the adjoint Ruelle operator

In this section, we compute the Fredholm determinant and the resolvent of the adjoint operator $\mathcal{L}_{11}^*$. The calculation is almost same as in [1].

Let n_1 denote the number of critical points in A_1. And let $\{c_1, \cdots, c_{n_1}\}$ be the critical points in A_1. Let

$$H(x, z; \lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{(R^{\circ n})'(z)(R^{\circ n}(z) - x)},$$

and let

$$M(\lambda) = \left(\delta_{ij} + \frac{\lambda}{R''(c_i)} H(c_j, R(c_i); \lambda)\right)_{i,j=1}^{n_1}$$

be an $n_1 \times n_1$ matrice.

THEOREM 4.1 The Fredholm determinant $D_{11}(\lambda)$ of $\mathcal{L}_{11}^*$ is given by

$$D_{11}(\lambda) = \prod_{n=1}^{\infty} (1 - \sigma^{n+1}\lambda) \det M(\lambda).$$

It is meromorphic in $\mathbb{C}$ and holomorphic for $|\lambda| < |\sigma|^{-2}$. $\mathcal{L}_{11}^*$ has no essential spectrum.

This theorem follows immediately from proposition 4.3 below. We assume that the backward orbits of critical points do not intersect with the curve γ_1. Let

$$\Omega_R = \overline{\mathbb{C}} \setminus (\Omega_1 \cup \bigcup_{n=0}^{\infty} \{z \in \Omega_1 \mid R'(R^{\circ n}(z)) = 0\}).$$

Then $\mathcal{O}_0(\Omega_1) \subset \mathcal{O}_0(\Omega_R)$. Define $\mathcal{L}_R^* : \mathcal{O}_0(\Omega_R) \to \mathcal{O}_0(\Omega_R)$ by

$$(\mathcal{L}_R^* f)(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(R(\tau))d\tau}{R'(\tau)(z - \tau)}$$

for $f \in \mathcal{O}_0(\Omega_R)$. We see immediately that the image of $\mathcal{L}_R^*$ is included in $\mathcal{O}_0(\Omega_1)$, and $\mathcal{L}_R^*$ and $\mathcal{L}_{11}^*$ coincide on $\mathcal{O}_0(\Omega_1)$. Therefore, $\mathcal{L}_R^*$ and $\mathcal{L}_{11}^*$ has the same spectrum. Define a operator $\mathcal{K} : \mathcal{O}_0(\Omega_R) \to \mathcal{O}_0(\Omega_R)$ by

$$(\mathcal{K} f)(z) = \frac{f(R(z))}{R'(z)}, \quad f \in \mathcal{O}_0(\Omega_R), z \in \Omega_R.$$

PROPOSITION 4.2 The spectrum of $\mathcal{K}$ is $\{\sigma^{n+1}\}_{n=1}^{\infty}$, and the Fredholm determinant is given by

$$\det(I - \lambda\mathcal{K}) = \prod_{n=1}^{\infty} (1 - \sigma^{n+1}\lambda).$$

The eigenfunction $f_n(z)$ for eigenvalue $\lambda^{-1} = \sigma^{n+1}$ is given by

$$f_n(z) = \frac{1}{\varphi'(z)(\varphi(z))^n},$$

where holomorphic function $\varphi : \Omega_R \to \overline{\mathbb{C}}$ is the Schröder's function of the form $\varphi(z) = z + a_0 + \frac{a_2}{z} + \cdots + \frac{a_n}{z^n} + \cdots$ near the infinity and satisfying the Schröder's equation $\varphi(R(z)) = \sigma^{-1}\varphi(z)$.

Proof The fact that $f_n(z)$ is an eigenfunction for eigenvalue σ^{n+1} is immediately verified by using the Shröder's equation. Eigenfunction $f_n(z)$ can be extended to Ω_R by using the function equations

$$\mathcal{K}f_n = \sigma^{n+1} f_n \quad \text{and} \quad (\mathcal{K}f_n)(z) = \frac{f_n(R(z))}{R'(z)}.$$

As the eigenfunctions $\{f_n\}_{n=1}^{\infty}$ form a complete basis of $\mathcal{O}_0(\Omega_R^{\infty})$, where Ω_R^{∞} denotes the connected component of Ω_R containing the infinity, and as the eigenfunctions are determined from a germ at the infinity of the eigenfunction by the function equation above, the Fredholm determinant is given by the formula in the proposition.

Define linear maps $\mathcal{G} : \mathcal{O}_0(\Omega_R) \to \mathbb{C}^{n_1}$ and $\mathcal{F} : \mathbb{C}^{n_1} \to \mathcal{O}_0(\Omega_R)$ by

$$\mathcal{G}f = \left(\frac{f(R(c_j))}{R''(c_j)} \right)_{j=1}^{n_1}, \quad f \in \mathcal{O}_0(\Omega_R),$$

$$\mathcal{F}\alpha = \sum_{j=1}^{n_1} \frac{\alpha_j}{z - c_j}, \quad \alpha = (\alpha_j) \in \mathbb{C}^{n_1}.$$

We have

$$\mathcal{L}_R^* = \mathcal{K} - \mathcal{F}\mathcal{G}.$$

The Fredholm determinant of the adjoint Ruelle operator $\mathcal{L}_R^*$ is computed as follows.

Proposition 4.3

$$D_{11}(\lambda) = \det(I - \lambda\mathcal{L}_R^*) = \det(I - \lambda\mathcal{K}) \det M(\lambda),$$

where $M(\lambda) = I_{n_1} + \lambda\mathcal{G}(I - \lambda\mathcal{K})^{-1}\mathcal{F}$.

Proof

$$\begin{aligned}\det(I-\lambda\mathcal{L}_R^*) &= \det(I-\lambda\mathcal{K}+\lambda\mathcal{F}\mathcal{G})\\ &= \det(I-\lambda\mathcal{K})\det(I+\lambda(I-\lambda\mathcal{K})^{-1}\mathcal{F}\mathcal{G})\\ &= \det(I-\lambda\mathcal{K})\det(I_{n_1}+\lambda\mathcal{G}(I-\lambda\mathcal{K})^{-1}\mathcal{F})\\ &= \det(I-\lambda\mathcal{K})\det M(\lambda).\end{aligned}$$

The $n_1 \times n_1$ matrice $M(\lambda)$ is computed as follows.

$$\begin{aligned}M(\lambda) &= I_{n_1}+\lambda\mathcal{G}(I-\lambda\mathcal{K})^{-1}\mathcal{F}\\ &= I_{n_1}+\lambda\mathcal{G}(\textstyle\sum_{n=0}^{\infty}\lambda^n\mathcal{K}^n)\mathcal{F}\\ &= I_{n_1}+\lambda\mathcal{G}\left(\sum_{n=0}^{\infty}\frac{\lambda^n}{(R^{\circ n})'(z)(R^{\circ n}(z)-c_j)}\right)_{j=1}^{n_1}\\ &= I_{n_1}+\lambda\left(\frac{1}{R''(c_i)}\sum_{n=0}^{\infty}\frac{\lambda^n}{(R^{\circ n})'(c_i)(R^{\circ n}(c_i)-c_j)}\right)_{i,j=1}^{n_1}\\ &= \left(\delta_{ij}+\frac{\lambda}{R''(c_i)}H(c_j,R(c_i);\lambda)\right)_{i,j=1}^{n_1}.\end{aligned}$$

As the spectrum of $\mathcal{K}$ is $\{\sigma^{n+1}\}_{n=1}^{\infty}$, $M(\lambda)$ is meromorphic in $\mathbb{C}$ and holomorphic in $\{\lambda \mid |\lambda|<\sigma^{-2}\}$. This completes the proof of the proposition 4.3 and the Theorem 4.1.

5. The resolvent of the partial adjoint Ruelle operator

The resolvent of the parttial adjoint Ruelle operator $\mathcal{L}_{11}^*$ can be computed in an analougous manner as in [1] and [2]. First, we compute the resolvent function of operator $\mathcal{K}$.

Proposition 5.1 The function $H(x,z;\lambda)$ defined in the previous section is the resolvent function of $\mathcal{K}$, *i.e.*,

$$H(x,z;\lambda)=(I-\lambda\mathcal{K})^{-1}\frac{1}{z-x}$$

$$=\sum_{n=0}^{\infty}\lambda^n\mathcal{K}^n\frac{1}{z-x}=\sum_{n=0}^{\infty}\frac{\lambda^n}{(R^{\circ n})'(z)(R^{\circ n}(z)-x)},$$

where $x\in\Omega_1$, $z\in\Omega_R$ and $\lambda\in\mathbb{C}\setminus\{\sigma^{-k}\}_{k=2}^{\infty}$. $H(x,z;\lambda)$ is holomorphic in x,z and λ.

Proof As is easily observed, we have

$$\mathcal{K}^n f(z)=\frac{f(R^{\circ n}(z))}{(R^{\circ n})'(z)}.$$

Let $f_n : \Omega_R \to \mathbb{C}, n = 1, 2, \cdots$, be the complete system of eigenfunctions of $\mathcal{K}$ given by propositon 4.2. For each $x \in \Omega_1$, we can expand the function $(z-x)^{-1} \in \mathcal{O}_0(\Omega_R)$ in the form

$$\frac{1}{z-x} = \sum_{n=1}^{\infty} b_n(x) f_n(z).$$

Observe that $b_n(x)$ is holomorpic in Ω_1. With this expression, we have

$$(I - \lambda\mathcal{K})^{-1}\frac{1}{z-x} = \sum_{n=0}^{\infty} \lambda^n \mathcal{K}^n \frac{1}{z-x}$$

$$= \sum_{n=0}^{\infty} \lambda^n \mathcal{K}^n \sum_{k=1}^{\infty} b_k(x) f_k(z) = \sum_{k=1}^{\infty} b_k(x) \sum_{n=0}^{\infty} \lambda^n \mathcal{K}^n f_k(z)$$

$$= \sum_{k=1}^{\infty} b_k(x) \sum_{n=0}^{\infty} (\lambda\sigma^{k+1})^n f_k(z) = \sum_{k=1}^{\infty} b_k(x) \frac{1}{1-\lambda\sigma^{k+1}} f_k(z).$$

This shows that $H(x, z; \lambda)$ has an analytic extension to the domain $\Omega_1 \times \Omega_R \times (\mathbb{C} \setminus \{\sigma^{-k}\}_{k=2}^{\infty})$.

The resolvent function $E(x, z; \lambda)$ is defined by

$$E(x, z; \lambda) = (I - \lambda\mathcal{L}_{11}^*)^{-1}\frac{1}{z-x}, \quad x \in \Omega_1, z \in \overline{\mathbb{C}} \setminus \Omega_1, \lambda \in \mathbb{C}, E(x, \infty; \lambda) = 0.$$

$E(x, z; \lambda)$ is holomorphic in x and z, and meromorphic in λ.

Proposition 5.2

$$(I - \lambda\mathcal{L}_{11}^*)^{-1} = (I - \lambda\mathcal{K})^{-1} - \lambda(I - \lambda\mathcal{K})^{-1}\mathcal{F}(M(\lambda))^{-1}\mathcal{G}(I - \lambda\mathcal{K})^{-1},$$

where $M(\lambda) = I_{n_1} + \lambda\mathcal{G}(I - \lambda\mathcal{K})^{-1}\mathcal{F}$ is an $n_1 \times n_1$ matrice.

Proof By a direct computation.

$$\begin{aligned}
(I - \lambda\mathcal{L}_{11}^*)^{-1} &= (I - \lambda\mathcal{K} + \lambda\mathcal{F}\mathcal{G})^{-1} \\
&= (I - \lambda\mathcal{K})^{-1}(I + \lambda\mathcal{F}\mathcal{G}(I - \lambda\mathcal{K})^{-1})^{-1} \\
&= (I - \lambda\mathcal{K})^{-1}(I + \textstyle\sum_{k=1}^{\infty}(-\lambda)^k(\mathcal{F}\mathcal{G}(I - \lambda\mathcal{K})^{-1})^k) \\
&= (I - \lambda\mathcal{K})^{-1}(I - \lambda\mathcal{F}\textstyle\sum_{k=0}^{\infty}(-\lambda\mathcal{G}(I - \lambda\mathcal{K})^{-1})^k\mathcal{G}(I - \lambda\mathcal{K})^{-1}) \\
&= (I - \lambda\mathcal{K})^{-1} - \lambda(I - \lambda\mathcal{K})^{-1}\mathcal{F}(M(\lambda))^{-1}\mathcal{G}(I - \lambda\mathcal{K})^{-1}.
\end{aligned}$$

As is easily verified, $M(\lambda)$ in this proposition is same as $M(\lambda)$ given in the beginning of the previous section.

Let

$$H_1(z; \lambda) = ((I - \lambda\mathcal{K})^{-1}\frac{1}{z-c_i})_{i=1}^{n_1} = (H(c_i, z; \lambda))_{i=1}^{n_1}$$

be a row vector and let

$$H_2(x;\lambda) = \mathcal{G}(I - \lambda\mathcal{K})^{-1}\frac{1}{z-x} = \mathcal{G}H(x,z;\lambda) = \left(\frac{H(x,R(c_i);\lambda)}{R''(c_i)}\right)_{i=1}^{n_1}$$

be a column vector. With all these things together, we find the explicit expression of the resolvent function $E(x,z;\lambda)$.

THEOREM 5.3

$$E(x,z;\lambda) = H(x,z;\lambda) + \lambda H_1(z;\lambda)(M(\lambda))^{-1}H_2(x;\lambda).$$

The resolvent function is holomorphic in $x \in \Omega_1$, and in $z \in \Omega_R$, and meromorphic in $\lambda \in \mathbb{C}$. The poles are the zeros of the Fredholm determinant $D_{11}(\lambda)$.

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