Critical points and Julia sets of complex Hénon maps

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Abstract

Critical points of the Green function, defined in the unstable manifold of saddle points of complex dynamical systems, behave as a key factor for the structure of the Julia set. By numerical explorations using the so called "SaddleDrop" method, we see some features of the bifurcation behavior of Julia sets for the complex Hénon maps. Saddle drop method defines a kind of bifurcation set in the parameter space. In this note, we present some pictures showing the homoclinic and heteroclinic tangency obtained by following the concerned critical point, together with the bifurcation locus pictures.

0. Introduction

The so called "SaddleDrop" was a computer program created by K. Papadantonakis in 2000. He worked with J.Hubbard, J. Smilie, and E.Bedford to draw parameter space pictures for the complex Hénon map. This program produced very early pictures of parameter spaces and was quite helpful for researchers in dynamical systems. However, the program SaddleDrop was programed for MacOS of PowerPC, and, unfortunately, now it is difficult to find a computer to run it.

The Hénon map we consider is given by the formula

$$H_{b,c} : \mathbb{C}^2 \to \mathbb{C}^2, \quad \text{where} \quad H_{b,c}(x, y) = (x^2 + c + by, x).$$

This formula is different from the original Hénon map

$$(x, y) \mapsto (1 - ax^2 + y, bx),$$

which is conjugate to our map by scaling the coordinates $Y = by$ and $X = -ax$, with $c = -a$. Parameter $-b$ is the (complex) determinant of the jacobian matrix. SaddleDrop looks for the critical point of the Green function along the unstable manifold of a saddle fixed point of Hénon map. By Bedford and Smilie[3], presence of critical points of this Green function indicates the disconnectedness of Julia set. If the Green function has no critical points, then the Julia set is unstably connected. In this note, we try to reconstruct the SaddleDrop program. See [6] for an outline of SaddleDrop program.

1. Invariant sets and Green functions

Let us define invariant sets of Hénon map $H_{b,c}$ as

$$K^{\pm}_{b,c} := \{p \in \mathbb{C}^2 | \{H^{\pm n}_{b.c}(p)\}_{n\geq 0} \text{ is bounded}\},$$
$$J^{\pm}_{b,c} := \partial K^{\pm}_{b,c}, \quad U^{\pm}_{b.c} := \mathbb{C}^2 \setminus K^{\pm}_{b,c},$$
$$K_{b,c} := K^{+}_{b,c} \cap K^{-}_{b,c}, \quad J_{b,c} := J^{+}_{b,c} \cap J^{-}_{b,c}.$$

$J_{b,c}$ is called the Julia set of $H_{b,c}$.

Let $P_{b,c}$ be a saddle fixed point of $H_{b,c}$ and let $\lambda_{b,c}$ and $v_{b,c}$ be the eigenvalue and eigenvector of $DH_{b,c}(P_{b,c})$ with $|\lambda_{b,c}| > 1$. The unstable manifold $W^u(P_{b,c})$ of the saddle point $P_{b,c}$ is defined as

$$W^u(P_{b,c}) \;=\; \{p \in \mathbb{C}^2 : \lim_{n\to\infty} H^{\circ -n}_{b,c}(p) = P_{b,c}\}.$$

There exists an analytic injective immersion map $\gamma_{b,c} : \mathbb{C} \to W^u(P_{b,c})$ parametrizing the unstable manifold of the saddle point, satisfying

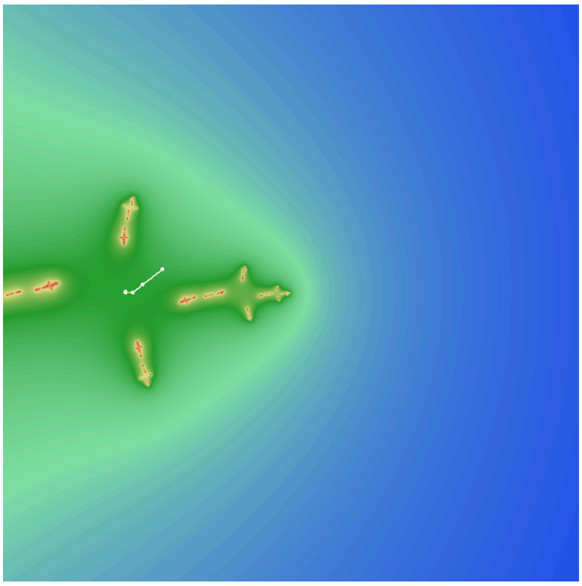
$$\gamma_{b,c}(0) = P_{b,c}, \quad D\gamma_{b,c}(0) = v_{b,c}, \quad \gamma_{b,c}(\lambda_{b,c} z) = H_{b,c}(\gamma_{b,c}(z)).$$

Note that these objects can be taken in an analytic manner with respect to parameter (b, c). According to H.Poincaré[8], these conditions determine the Taylor expansion of $\gamma_{b,c}(z)$ with infinite radius of convergence.

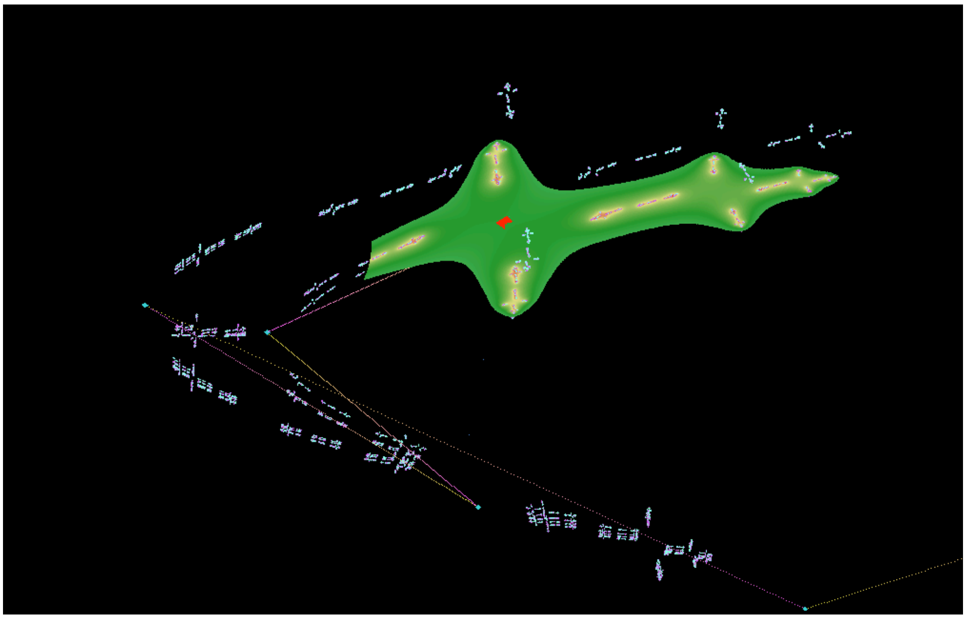
In order to visualize the set $K^+_{b,c}$, the Green function $G^+_{b,c} : \mathbb{C}^2 \to [0, \infty)$ of $K^+_{b,c}$ defined as

$$G^+_{b,c}(p) \;=\; \lim_{n\to\infty} \frac{1}{2^n} \log^+ ||H^{\circ n}_{b,c}(p)||$$

is used. This function is plurisubharmonic on $\mathbb{C}^2$ and pluriharmonic on $U^+_{b,c}$. Hence, the composed function $G^+_{b,c} \circ \gamma_{b,c}$ defines a subharmonic function on $\mathbb{C}$, which vanishes on $\gamma^{-1}(W^u(P_{b,c}) \cap K^+_{b,c})$ and is harmonic on $\gamma^{-1}_{b,c}(W^u(P_{b,c}) \cap U^+_{b,c})$.

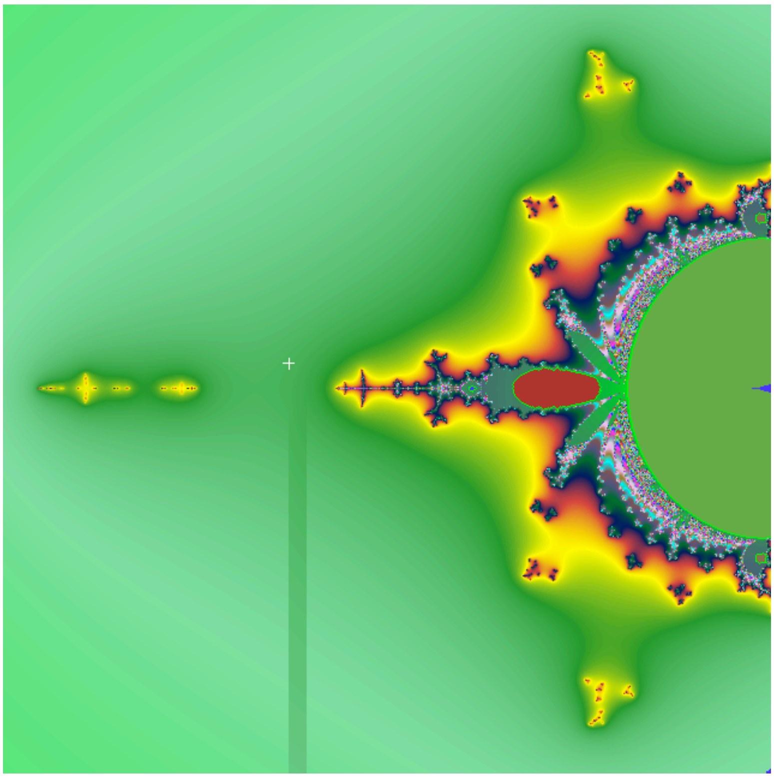


The above is a typical picture of the Green function on the unstable manifold of a saddle fixed point, $P_{b,c}$, which is the "β-fixed point". A critical point of the Green function is detected by Newton's method.

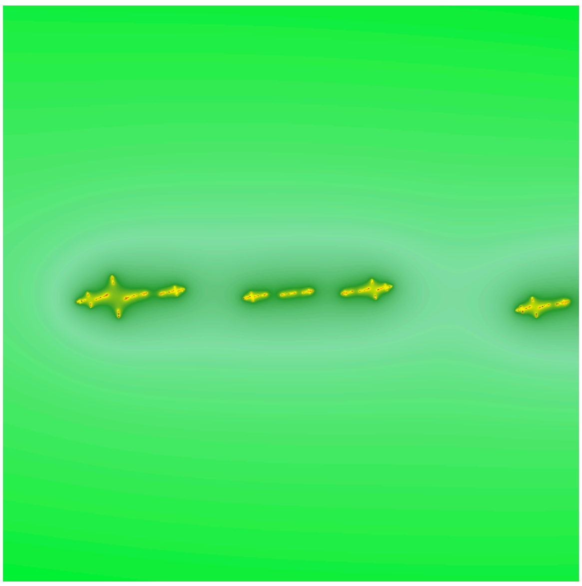


It is mapped by $\gamma_{b,c}$ to the unstable manifold $W^u(P_{b,c})$. The origin of the previous picture is mapped to the saddle point $P_{b,c}$. In the above picture, the region $\{z \in \mathbb{C} \mid G^+_{b,c} \circ \gamma_{b,c}(z) < \delta\}$ for some $\delta > 0$ of the previous picture is embedded in $\mathbb{C}^2$ by $\gamma_{b,c}$. Saddle periodic points in $J_{b,c}$ are plotted, too.

The critical point of Green function is a saddle critical point. Use this saddle critical point and try to follow the critical point, varying the parameter. Here, the parameter b is fixed and c is varied. Compute the value of the Green function for each parameter value, and color the pixels according to the value, to get the following picture.

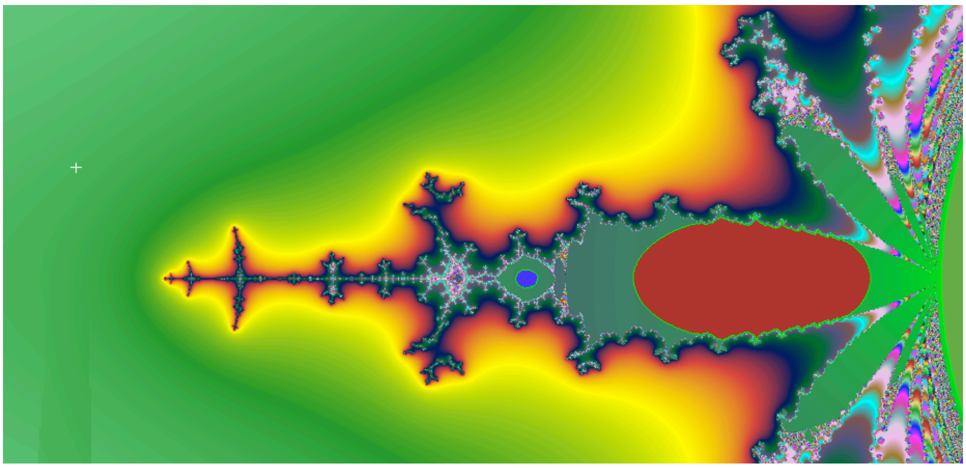


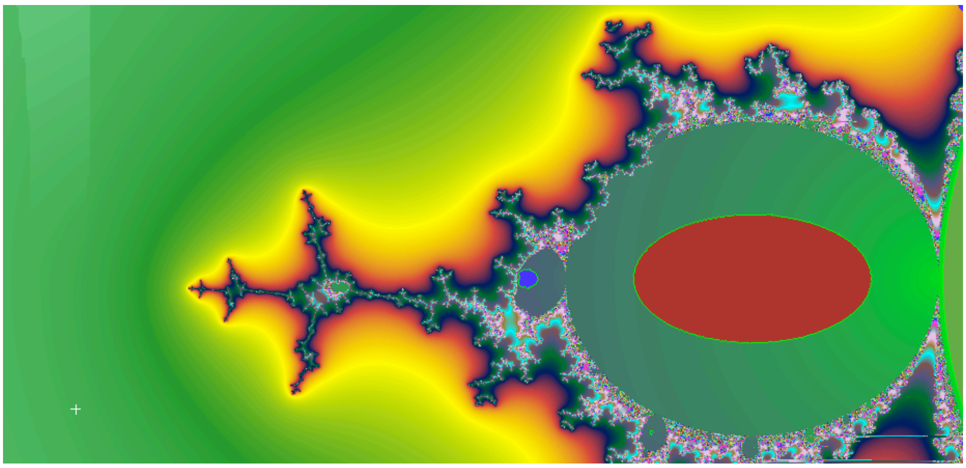
The picture of Green function on the unstable manifold of the other saddle point ("α-fixed point"), shown below, may be different from the previous picture of unstable manifold.



There are infinitely many critical points of the Green function, when they exist. The critical points are mapped to critical points by multiplication by the eigenvalue $\lambda_{b,c}$ of the saddle point. In the previous case, there are infinitely many critical points in different orbits. The obtained saddle drop picture depends on the choice of the critical point. The critical point of Green function depends analytically on the parameters b, c, and the set of critical points form a branched covering space over the space of parameters. The saddle drop picture should be considered as a slice of "Riemann surface".

Observe two slit lines in the saddle drop picture above. Observe also there are many "islands" apart from the main "Mandelbrot set". Observe the fingers and complicated structures in the enlarged pictures.

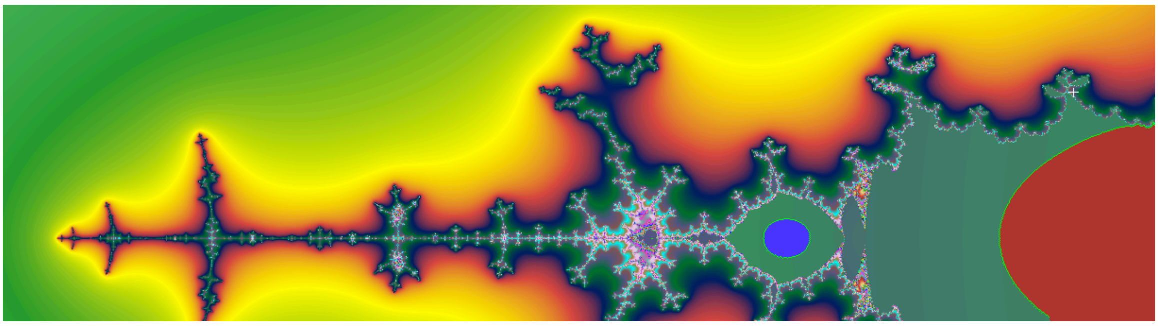




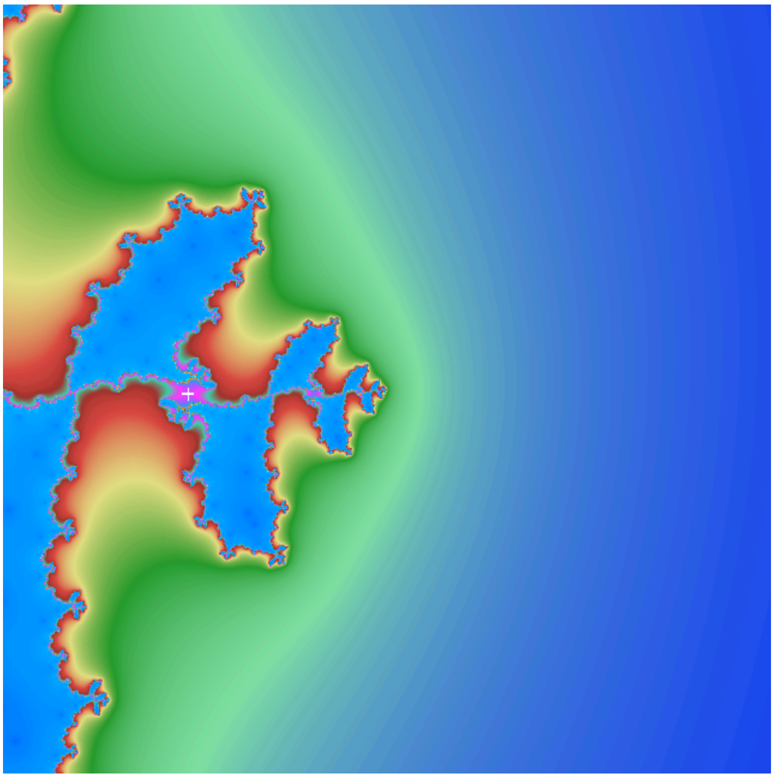
These pictures are enlargements of the previous saddle drop picture, representing same region, with different path of continuation of critical points. Observe the slits in these pictures. The Mandelbrot-like set is the locus where the Green function (on the branched covering space) vanishes. The critical point touches $J^+_{b,c}$ and disappears. This Mandelbrot-like set and so-called "finger" structure depends on the choice of the critical point.

2. Fingers and heteroclinic tangencies

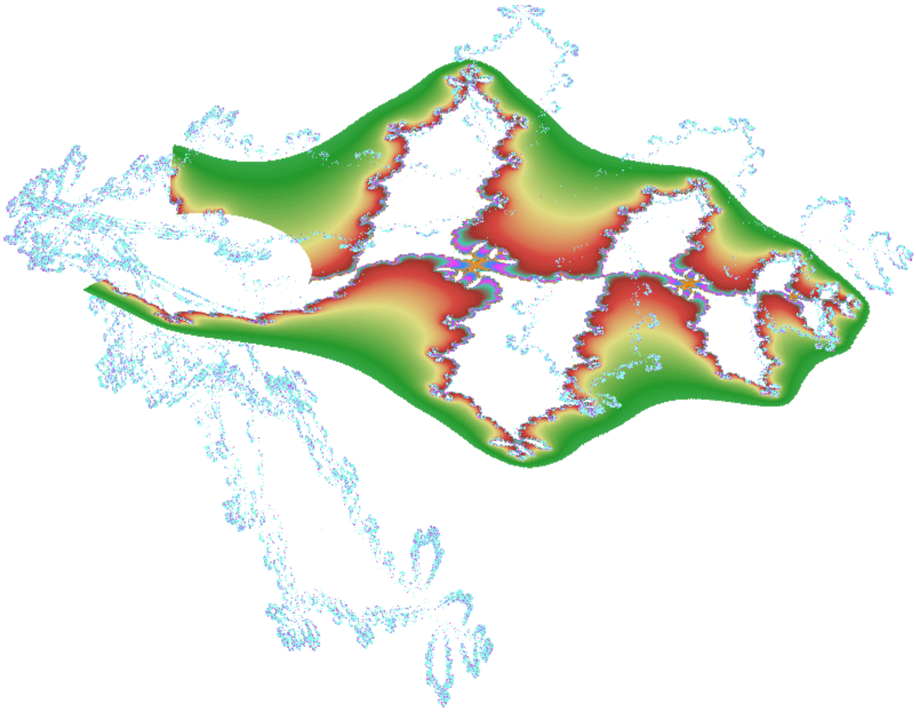
The boundary of the fingers are the locus where the unstable manifold becomes tangent to the lamination $J^+_{b,c}$ and the critical point collides with it at the tangent point. The following picture is an enlargement of the previous picture. Take a point near the boundary of a "finger".

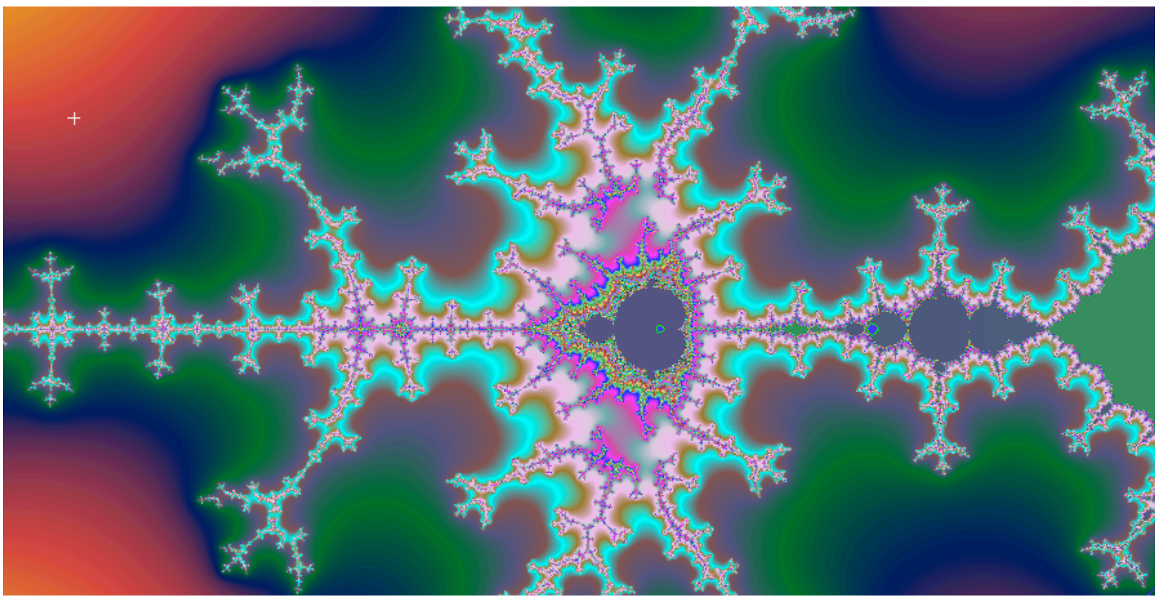


The unstable manifold picture is shown below.

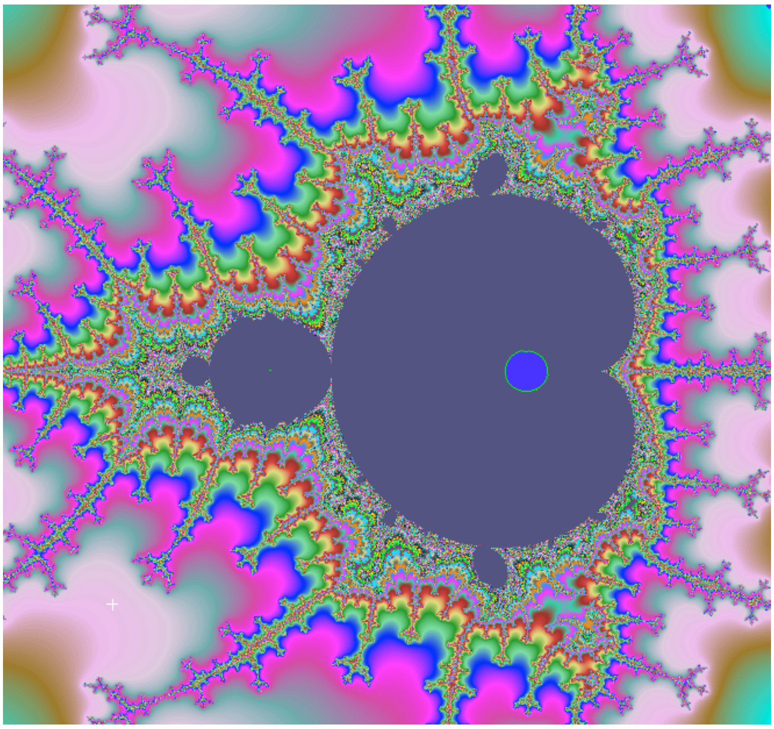


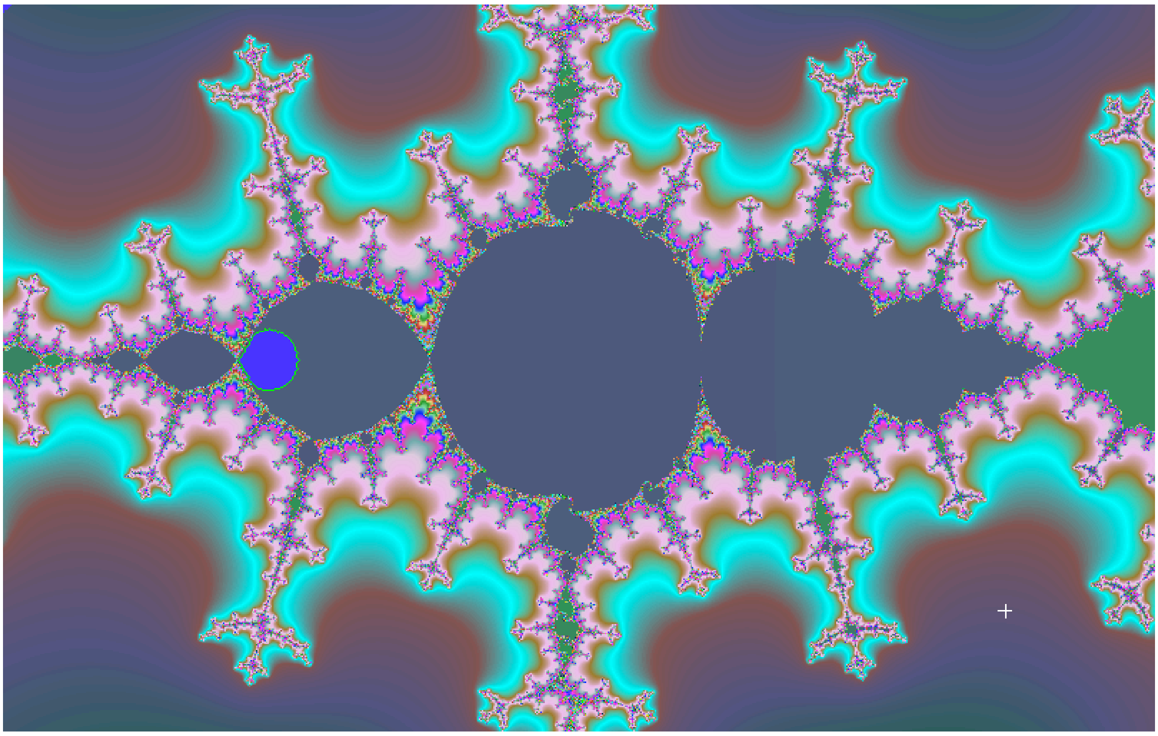
In this picture, we see that the critical point we follow comes near the boundary of the attractive basin of period 2. Note that the boundary of the basin is a slice of the set $W^u(P_{b,c}) \cap J_{b,c}$. But it does not resemble the Julia set of one dimensional quadratic map with an attracting cycle of period 2. In this picture, the critical point still exists, and hence the Julia set $J_{b,c}$ is not connected. Some part of this picture is embedded in $\mathbb{C}^2$ and shown in the next picture with periodic points. The unstable manifold $W^u(P_{b,c})$ is almost tangent to the stable lamination $J^+_{b,c}$.



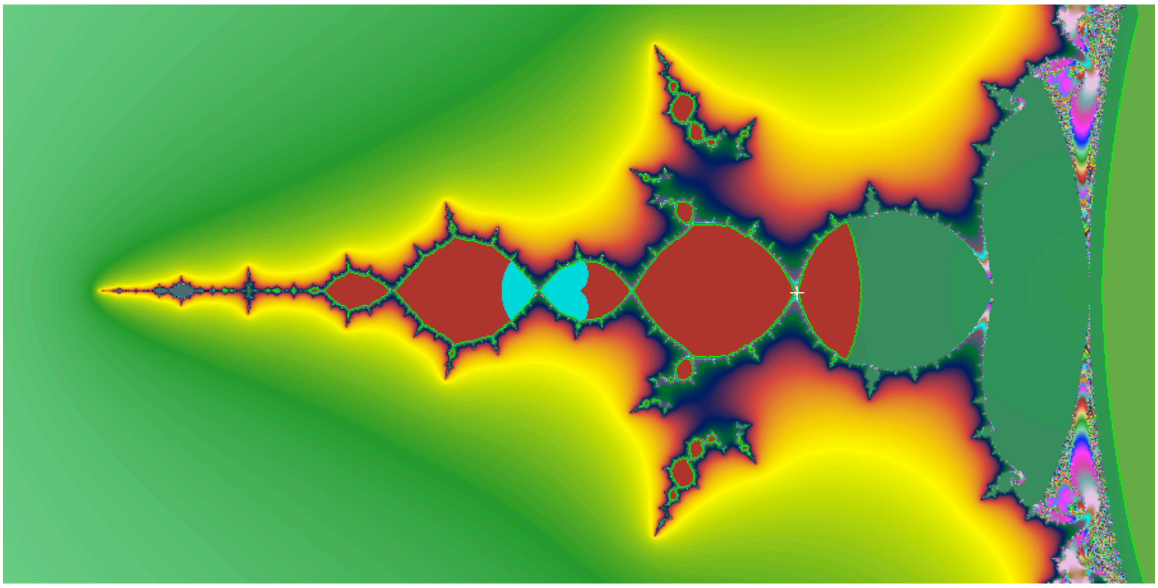


In saddle drop pictures, we find Mandelbrot-like set and Julia-like set. As in the case of cubic polynomials in one variable, the behavior of the concerned critical point is related to the structure of the bifurcation locus. However, by enlarging, we find they are different from those of one dimensional polynomials.

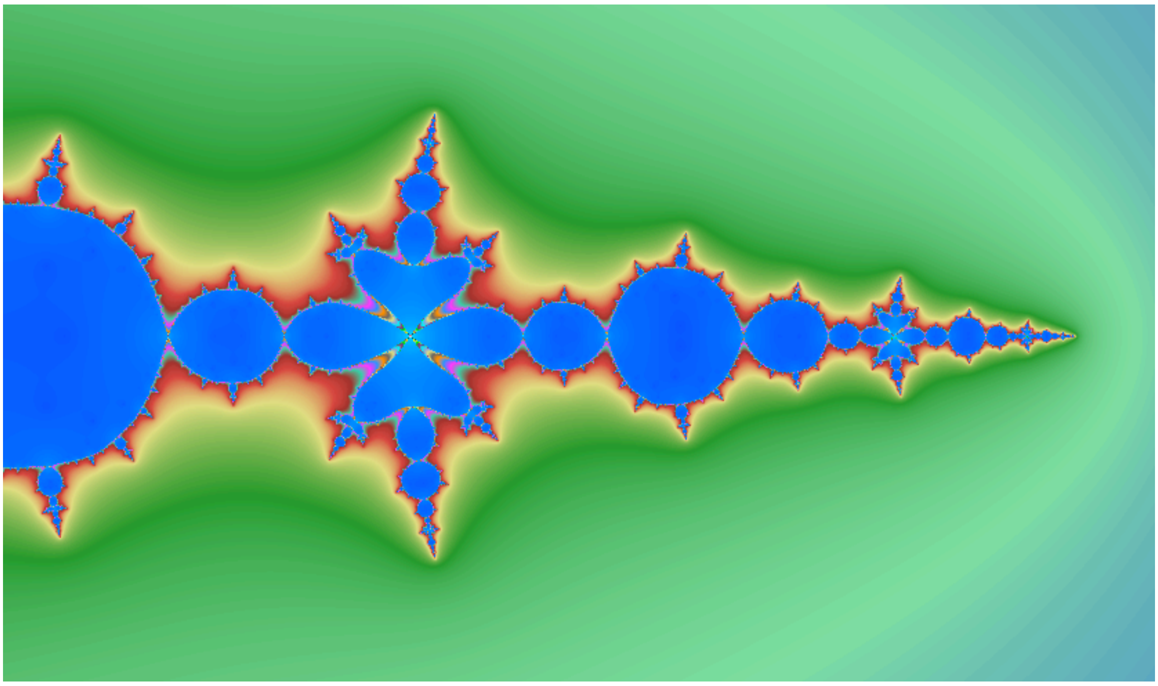




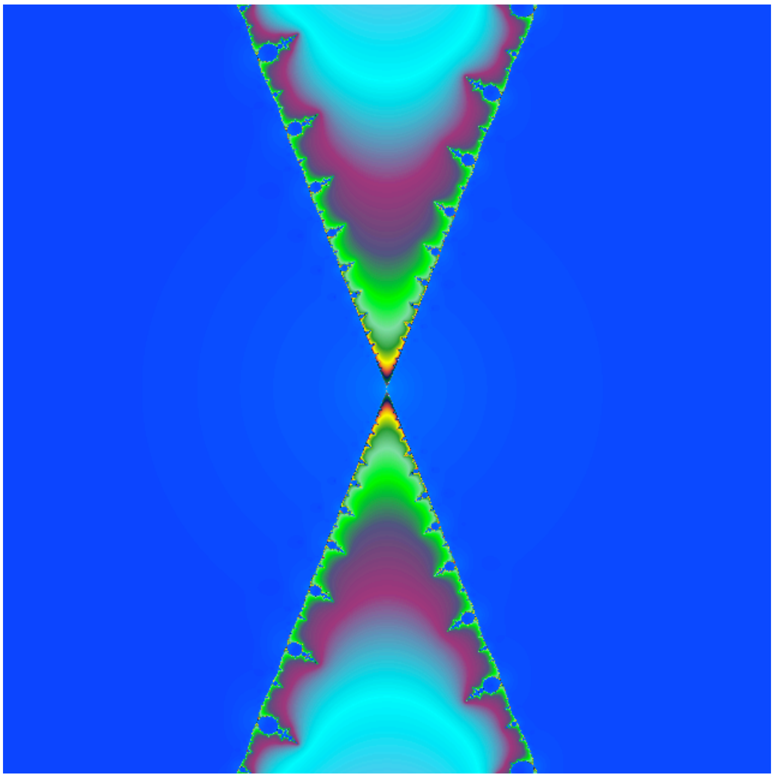
Heteroclinic tangencies are found where the Julia-set-like bifurcation locus is pinched. Take a parameter, for example, from the cursor location of the following picture, representing a c-slice with $b = -0.2$. The selected parameter are $b = -0.2, c = -1.2181$.



The unstable manifold picture for the β-saddle $P_{b,c}$ is as follows. The concerned critical point touches the $J^+_{b,c}$ at the "four-petals point" in this picture.



In this case, the picture of the unstable manifold of the α-saddle point is different, which is shown below.

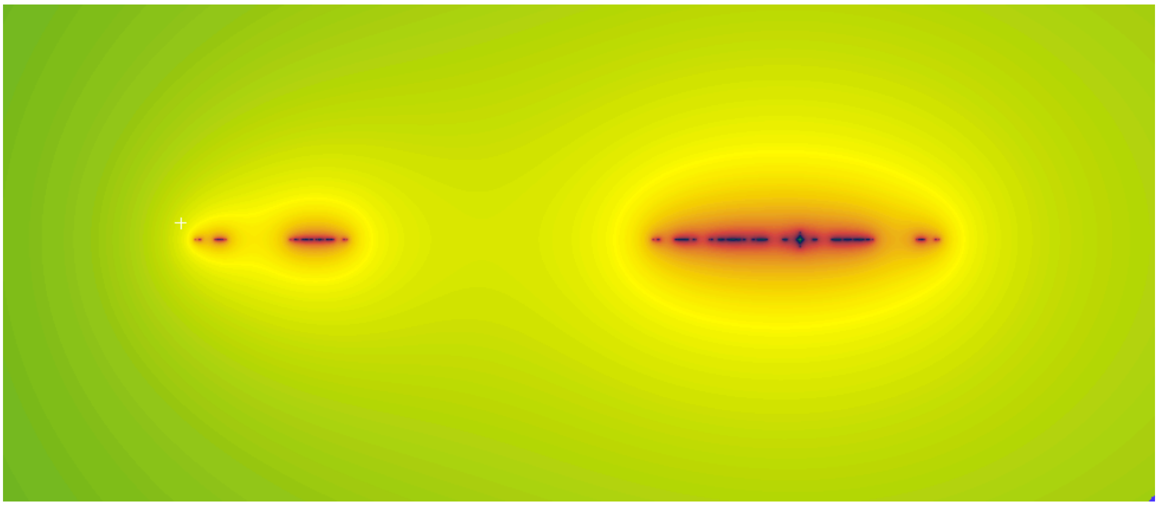


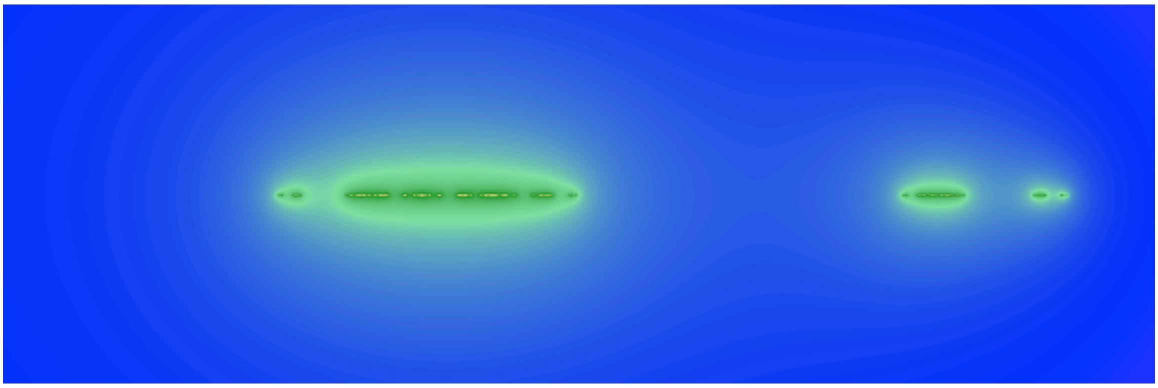
3. Antennas and homoclinic tangencies

By Bedford and Smilie[4], in the real parameter space, the first bifurcation from the real horseshoe takes place when homoclinic or heteroclinic tangency occurs. More precisely, for $b > 0$, *i.e.* $\det(DH_{b,c}) < 0$, heteroclinic tangency of the unstable manifold of α-saddle point, $W^u(Q_{b,c})$, and the stable manifold of the β-saddle point, $W^s(P_{b,c})$ takes place. And for $b < 0$, *i.e.* $\det(DH_{b,c}) > 0$, homoclinic tangency of $W^u(P_{b,c})$ and $W^s(P_{b,c})$.

So, we fix $b = -0.3$, for example, and we look for a critical point of the Green function on the unstable manifold $W^u(P_{b,c})$ touching the Julia set at the stable manifold $W^s(P_{b,c})$ for parameters near the "antenna" in the parameter space. The β-saddle point is located at the end of the Julia set (when the unstable eigenvalue is real), the homoclinic tangency locus looks like the end of an antenna. The next is a saddle drop picture of a critical point in $W^u(P_{b,c})$ for $b = -0.3$ near $c = -2.0$. The antenna of the Mandelbrot set near $c = -2.0$ is broken into fragments.

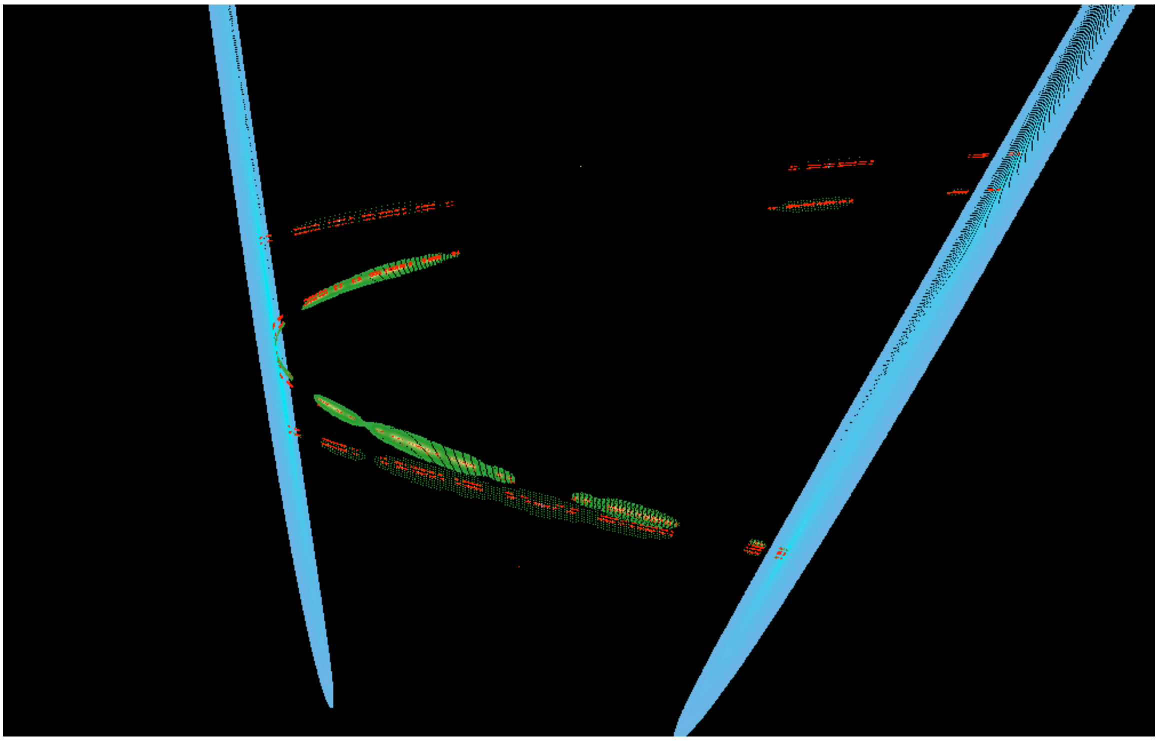
An enlargement near the top of the antenna.





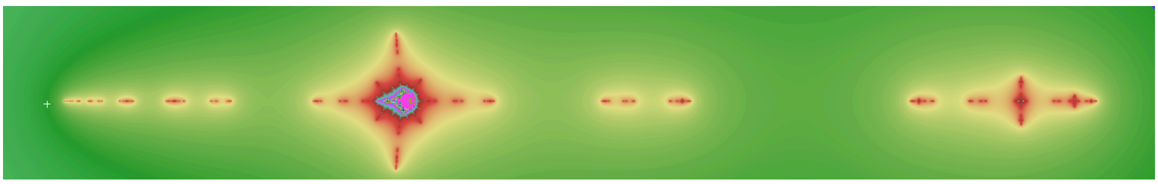
Picture of $W^u(P_{b,c})$ above, and picture of $W^s(P_{b,c})$ below.

Next picture shows how they are embedded in $\mathbb{C}^2$ with Julia set.

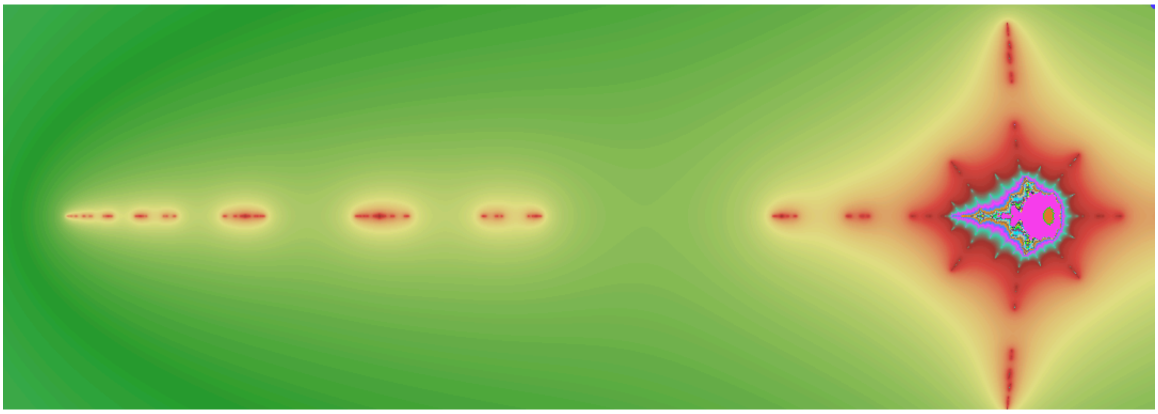


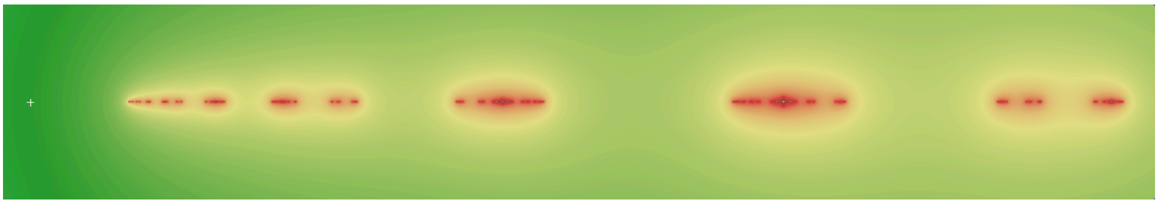
4. First bifurcation from real horseshoe by heteroclinic tangency

Bedfor and Smilie[4] showed that similar bifurcation from a real horseshoe occurs with a heteroclinic tangency of $W^u(Q_{b,c})$ and $W^s(P_{b,c})$ in the case $b > 0$, *i.e.* $\det(DH_{b,c}) < 0$. The following picture is a saddle drop picture, for $b = 0.3$, of a critical point corresponding to the the first heteroclinic tangency in $W^u(Q_{b,c})$. The leftmost end point of the bifurcation locus should be the first bifurcation locus.

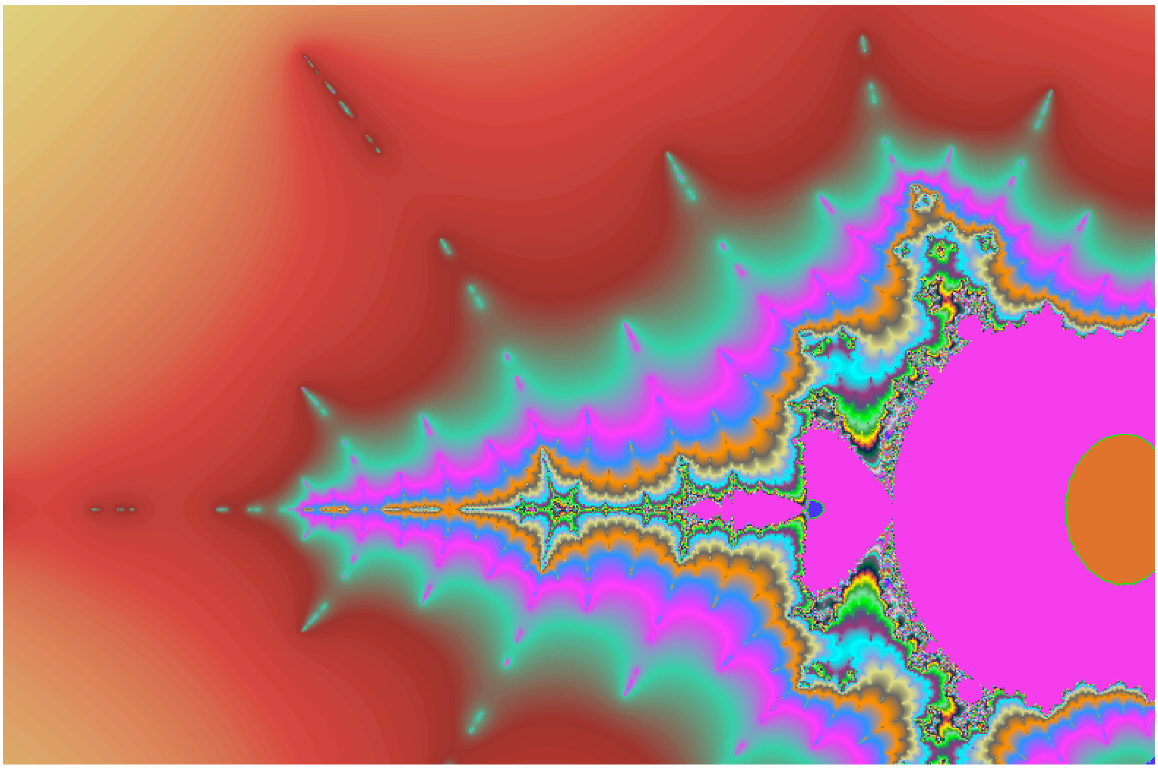


Followings are successive enlargements of the above picture. Observe the fractal structure and many gaps, with a tiny Mandlbrot-like set (the critical point is attracted to a period three attractor here).

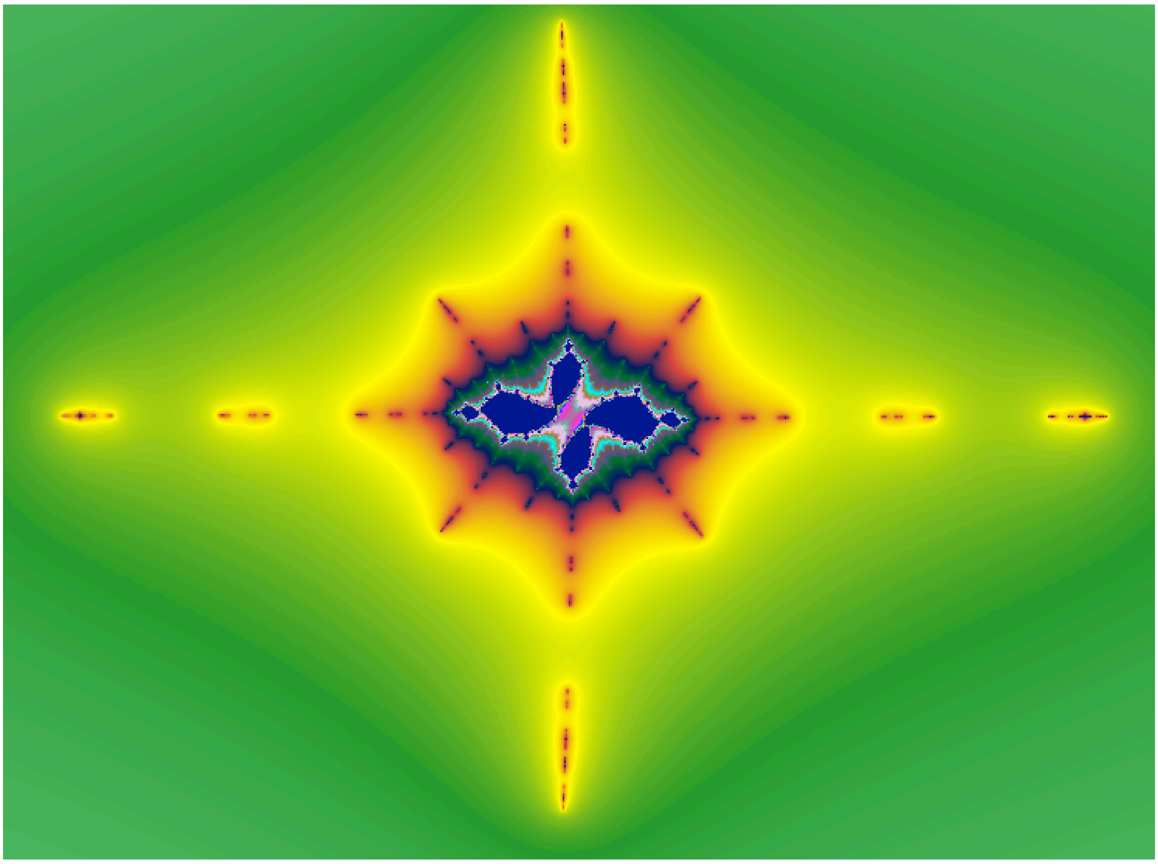




For real parameters b, c, at least some ordering of critical points is possible, and "following a critical point of first tangency" makes sense. But in our saddle drop picture,"looking for the critical point of slowest escape rate" becomes difficult, since there is no global ordering of critical points.

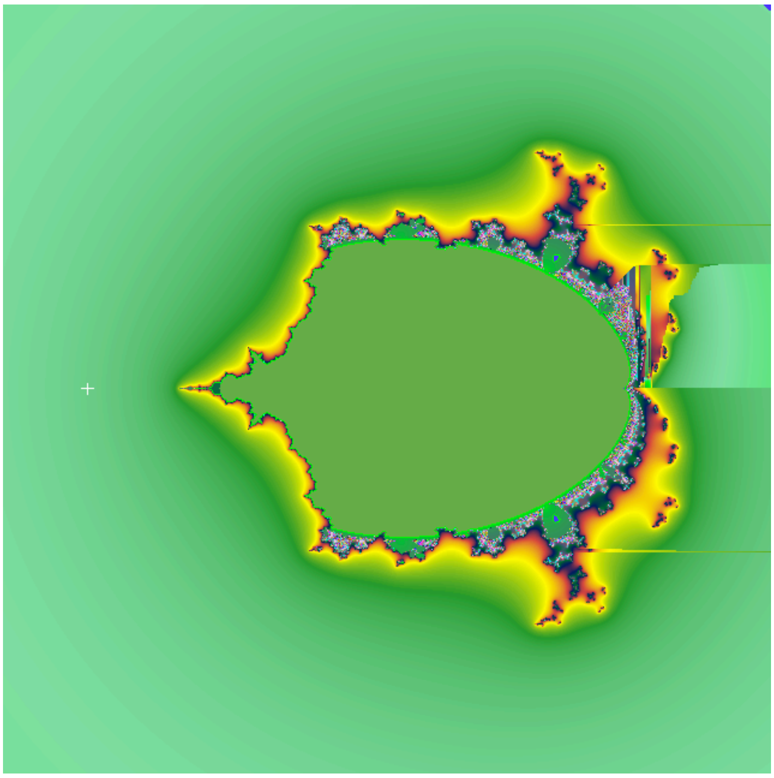


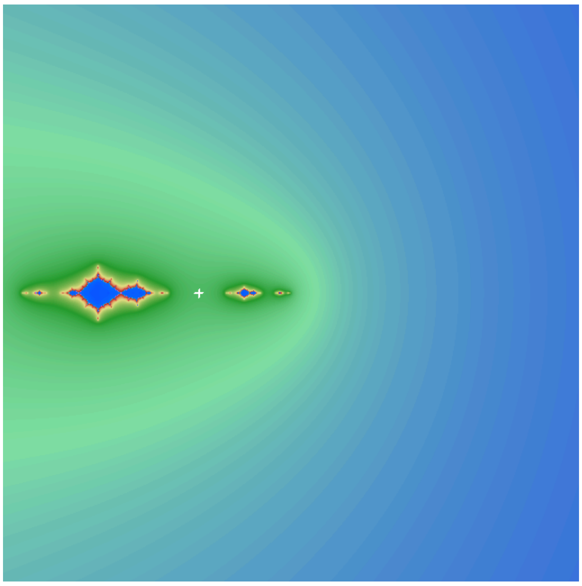
The above is an enlargement of the "Mandelbrot-like" region of the saddle drop picture. Some defect of the Mandelbrot set suggests the "finger" phenomena, *i.e.*, the critical point collides with a basin of attraction of a periodic attractor. So, if we follow the critical point and go into a fijord of the "finger", we get a picture as follows. In this picture, we see that the critical point we follow continues to exist but the Julia set is not a Cantor set any more.

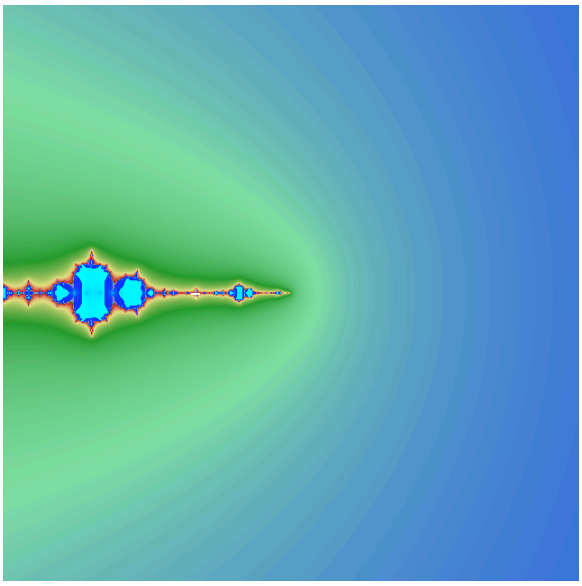


5. Boundary of connectedness locus

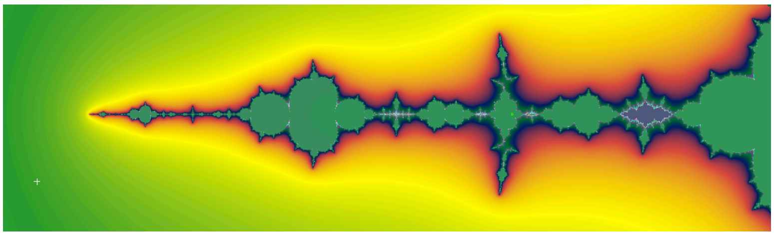
By Bedford and Smilie[3], the Julia set $J_{b,c}$ is connected if and only if there is no critical point of the Green function on the unstable manifold $W^u(P_{b,c})$ (or of some point of $J_{b,c}$). However, it is not easy to find the boundary of connectedness locus. The following is a saddle drop picture of $W^u(P_{b,c})$ for $b = -0.3$. The critical point in the unstable manifold in the left picture is followed. When the parameter in the saddle drop picture is taken from the end point of the antenna in the real axis, the Julia set appears to be connected as shown in the right picture of unstable manifold.



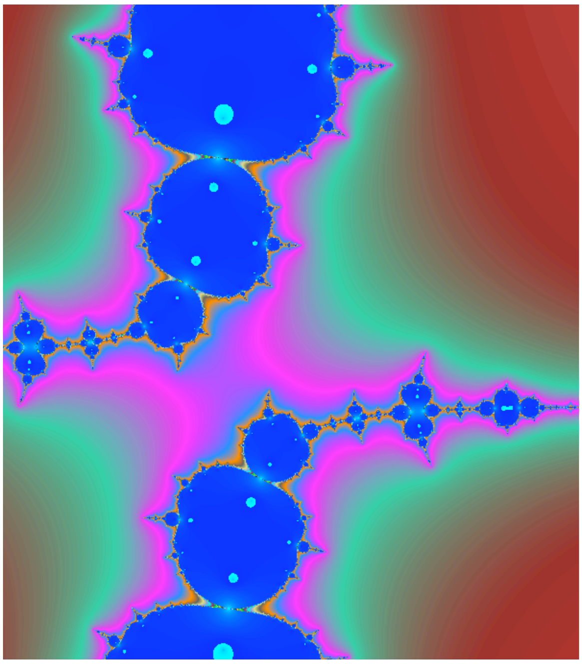




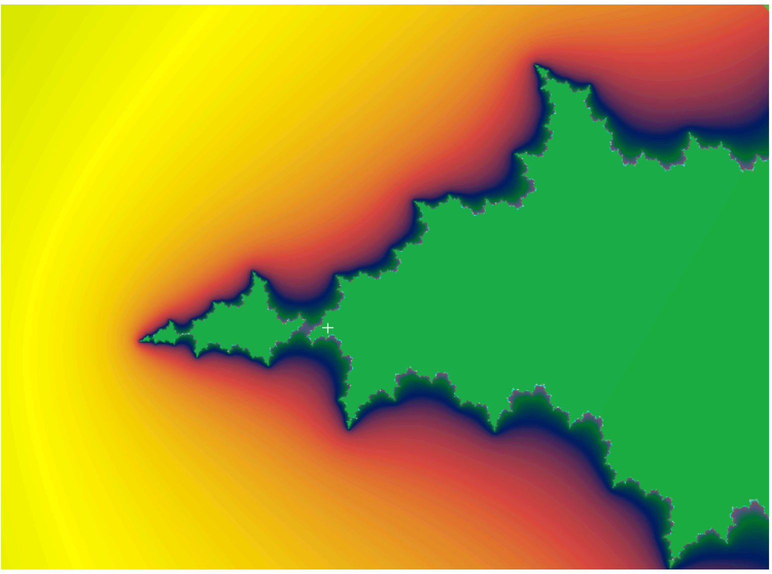
The parameter of the right picture above was taken from a point near the end of the "antenna", whose enlargement is shown below.



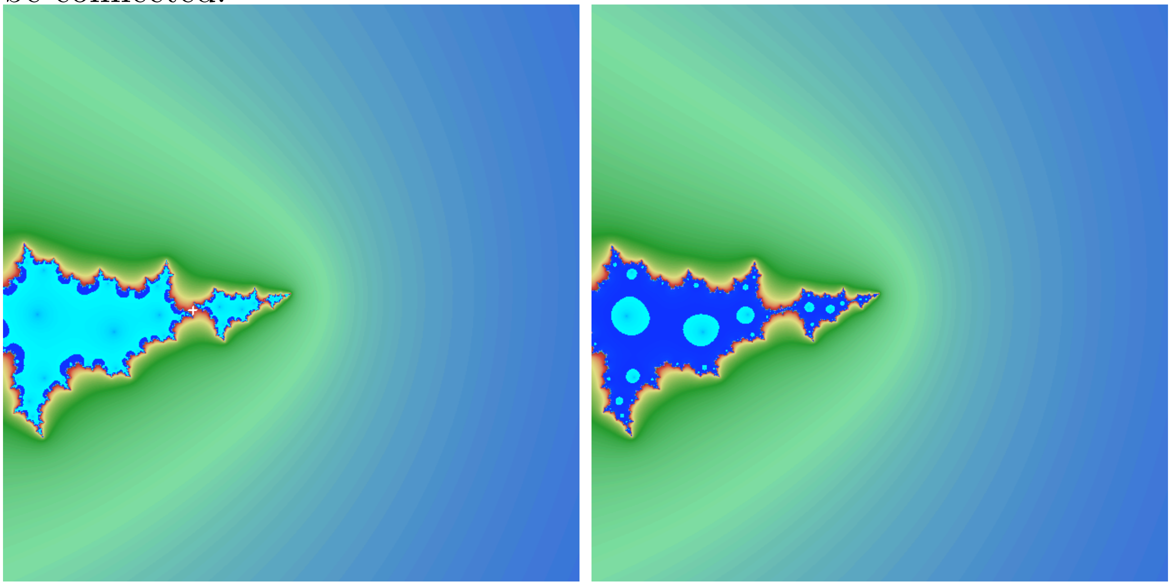
However, an enlargement of the unstable manifold picture shows that the concerning critical point is located in the real axis, but at the same time, other critical points coexist in the non real region.



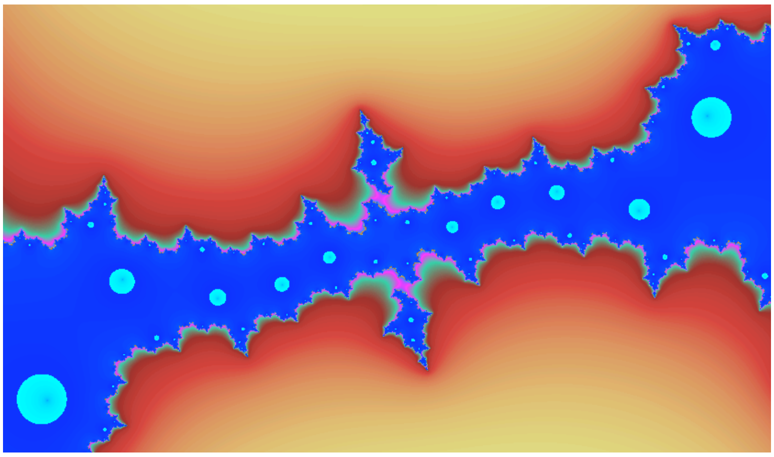
Let us try to inspect some parameters near the boundary of the parameter region where the concerned critical point breaks down. The following is a portion of the saddle drop picture. Take a parameter near a "channel".



The following pictures are for parameters taken from the "channel", and from the "peninsula". In the left picture, the Julia set is disconnected, since there is a critical point. In the right picture, the Julia set appears to be connected.



But further inspection by enlarging the region near the critical point reveals the disccnnectedness.



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[9] SaddleDrop and related pictures:
http://www.math.cornell.edu/~dynamics/.