Super-Stable Manifolds of Super-Saddle-Type Julia Sets in $\mathbb{C}^2$

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Abstract Assume that a complex analytic dynamical system $f : \mathbb{C}^2 \to \mathbb{C}^2$ maps the y-axis into itself and the mapping f restricted to the y-axis has a uniformly expanding Julia set. If f is critical and non-degenerate on the Julia set, then the Julia set has the "super-stable manifold" foliated by complex analytic curves.

A fixed point of a two-dimensional complex analytic dynamical system $f : \mathbb{C}^2 \to \mathbb{C}^2$ is called a *super-saddle* if one of the eigenvalues of the Jacobian matix of f at the fixed point is zero and the other eigenvalue has an absolute value greater than one.

Although the mapping is not diffeomorphic near the fixed point, the existence of the unstable manifold of the fixed point, which corresponds to the unstable eigen value, is known since the last century. The author [6] showed the existence of the "super-stable manifold" of the super-saddle-type fixed point. The "super-stable manifold" is the invariant manifold associated with the eigenvalue zero.

In this note, we consider a class of two-dimensional complex dynamical systems. We suppose that the dynamical system has a one-dimensional invariant sub-manifold, the system restricted to this sub-manifold has a compact Julia set, and that the system is super-attractive in the normal direction to the invariant sub-manifold. The conclusion we shall obtain is that the "super-stable manifold" forms a fiber bundle over the Julia set.

1. Super-Saddle-Type Julia Set

Suppose $f : \mathbb{C}^2 \to \mathbb{C}^2$ is complex analytic in a neighbourhood of the y-axis, $\mathbb{C}_y = \{0\} \times \mathbb{C} \subset \mathbb{C}^2$, and the y-axis is mapped into itself, *i.e.*,

$$\text{(A1)} \qquad f(\mathbb{C}_y) \subset \mathbb{C}_y.$$

Let us express f by its components as $f(x, y) = (f_1(x, y), f_2(x, y))$. The assumption (A1) above can be expressed as $f_1(0, y) = 0$. Next, we assume

$$\text{(A2)} \qquad \det Df = 0 \quad \text{on} \quad \mathbb{C}_y.$$

This assumption may be very particular. Let $J \subset \mathbb{C}_y$ denote the Julia set of $f|_{\mathbf{C}_y} : \mathbb{C}_y \to \mathbb{C}_y$. We assume

$$\text{(A1')} \qquad J \ \text{ is compact.}$$

For $p = (0, y_0) \in J$, we set

$$a(p) = f_2(0, y_0), \quad b(p) = \frac{\partial f_2}{\partial y}(0, y_0),$$

$$c(p) = \frac{\partial f_2}{\partial x}(0, y_0),$$

$$h_p(\xi, \eta) = f_2(\xi, y_0 + \eta) - a(p) - b(p)\eta - c(p)\xi.$$

Moreover, for a technical reason, we assume that the Julia set is uniformly hyperbolic, *i.e.*,

$$\text{(A3)} \qquad \exists \beta > 1, \quad \text{s.t.} \quad |b(p)| \geq \beta \ \text{ for } \ \forall p \in J.$$

From (A1), it follows that $\frac{\partial f_1}{\partial y}(0, y) = 0$. Hence if $p = (0, y) \in \mathbb{C}_y$ then

$$\det Df \ = \ \frac{\partial f_1}{\partial x}(0, y) \frac{\partial f_2}{\partial y}(0, y).$$

From (A3),

$$b(p) = \frac{\partial f_2}{\partial y}(0, y) \neq 0$$

holds in a neighborhood of $J \subset \mathbb{C}^2$, hence we have

$$\frac{\partial f_1}{\partial x}(0, y) \ = \ 0.$$

Therefore we can write

$$f_1(x, y) \ = \ x^2 g(x, y)$$

where $g(x, y)$ is complex analytic in a neighborhood of $\mathbb{C}_y$. For $p = (0, y_0) \in J$, we set

$$g_p(\xi, \eta) \ = \ g(\xi, y_0 + \eta).$$

As a non-degeneracy condition in the normal direction, we assume

$$\text{(A4)} \qquad g_p(0, 0) \neq 0 \ \text{ for } \ p \in J.$$

The "Julia set" J is said to be of *super-saddle-type* if all the conditions above are satisfied.

2. Constants and Definitions

For a positive real number ϵ, let

$$B_\epsilon \ = \ \{(\xi, \eta) \in \mathbb{C}^2 \mid |\xi|^2 + |\eta|^2 \leq \epsilon^2\}$$

and set

$$\alpha \ = \inf_{p \in J, (\xi, \eta) \in B_\epsilon} |g_p(\xi, \eta)|, \quad m_g \ = \sup_{p \in J, (\xi, \eta) \in B_\epsilon} |g_p(\xi, \eta)|,$$

$$m_h = \sup_{p\in J,(\xi,\eta)\in B_\epsilon}(|\frac{\partial^2 h_p}{\partial\xi^2}(\xi,\eta)|+|\frac{\partial^2 h_p}{\partial\xi\partial\eta}(\xi,\eta)|+|\frac{\partial^2 h_p}{\partial\eta^2}(\xi,\eta)|),$$
$$m_c = \sup_{p\in J}|c(p)|, \quad M_g = \sup_{p\in J,(\xi,\eta)\in B_\epsilon}|\frac{\partial g_p}{\partial\eta}(\xi,\eta)|.$$

We may assume these values are finite by taking sufficiently small $\epsilon > 0$. As $h_p(0,0) = 0$, $\frac{\partial h_p}{\partial\xi}(0,0) = 0$, $\frac{\partial h_p}{\partial\eta}(0,0) = 0$, the following proposition holds.

Proposition 1 For $p \in J$ and $(\xi,\eta) \in B_\epsilon$, we have

$$|\frac{\partial h_p}{\partial\xi}(\xi,\eta)| \le (|\xi|+|\eta|)m_h,$$

$$|\frac{\partial h_p}{\partial\eta}(\xi,\eta)| \le (|\xi|+|\eta|)m_h,$$
$$|h_p(\xi,\eta)| \le (|\xi|+|\eta|)^2 m_h.$$

Let $p = (0, y_0) \in J$. By regarding $(\xi,\eta) \in B_\epsilon$ as a local coordinate around p, we have
$$f(\xi, y_0+\eta) = (\xi^2 g_p(\xi,\eta), a(p)+c(p)\xi+b(p)\eta+h_p(\xi,\eta)).$$
Define $f_p(\xi,\eta) = (f_{p,1}(\xi,\eta), f_{p,2}(\xi,\eta))$ by

$$f_p(\xi,\eta) = f(\xi, y_0+\eta) - f(0,y_0).$$

It follows that
$$f_{p,1}(\xi,\eta) = \xi^2 g_p(\xi,\eta)$$
$$f_{p,2}(\xi,\eta) = c(p)\xi + b(p)\eta + h_p(\xi,\eta)$$
and that $f_p : B_\epsilon \to \mathbb{C}^2$, $f_p(0,0) = (0,0)$.

We choose sufficiently small $r_0 > 0$ and $u_0 > 0$ so that

$$\mathbb{D}_{r_0}\times\mathbb{D}_{u_0} \subset B_\epsilon, \quad f_p(\mathbb{D}_{r_0}\times\mathbb{D}_{u_0}) \subset B_\epsilon$$

hold for all $p \in J$.

Define positive real constants r and u by

$$u = \min(u_0, \frac{\beta-1}{8m_h}, 1)$$

$$r = \min(\frac{u}{2}, \frac{(\beta-1)u}{4m_c}, \frac{1}{2m_g}, \frac{\beta-1}{8uM_g})$$

. The following propositions hold.

Proposition 2 If $\xi \in \mathbb{D}_r$ and $\eta \in \partial\mathbb{D}_u$, then $|f_{p,2}(\xi,\eta)| > u$.

Proof
$$\begin{aligned}|f_{p,2}(\xi,\eta)| &= |c(p)\xi + b(p)\eta + h_p(\xi,\eta)| \\ &\ge \beta u - m_c r - (u+r)^2 m_h\end{aligned}$$

$$
\begin{aligned}
&> \beta u - \frac{(\beta-1)u}{2} - 4u^2 m_h \\
&\geq \beta u - \frac{(\beta-1)u}{4} - 4u\frac{\beta-1}{8} \\
&= \frac{\beta+3}{4}u \\
&> u
\end{aligned}
$$

PROPOSITION 3 If $(\xi,\eta) \in \mathbb{D}_r \times \mathbb{D}_u$, then

$$
|f_{p,1}(\xi,\eta)| \leq \frac{r}{2}, \quad |\frac{\partial f_{p,2}}{\partial \eta}(\xi,\eta)| \geq \frac{3}{4}(\beta-1)+1 > 1
$$

hold.

PROOF

$$
|f_{p,1}(\xi,\eta)| = |\xi|^2 |g_p(\xi,\eta)| \leq r^2 m_g \leq \frac{r}{2}.
$$

$$
\begin{aligned}
|\frac{\partial f_{p,2}}{\partial \eta}(\xi,\eta)| &= |b(p) + \frac{\partial h_p}{\partial \eta}(\xi,\eta)| \geq \beta - (|\xi|+|\eta|)m_h \\
&\geq \beta - (u+r)m_h \geq \beta - 2um_h \\
&\geq \beta - \frac{\beta-1}{4} = \frac{3}{4}(\beta-1)+1 > 1.
\end{aligned}
$$

3. Pull back of a graph

For $p \in J$, let $q = f(p)$. Analytic curves passing by q can be pulled back by the analytic map $f : \mathbb{C}^2 \to \mathbb{C}^2$ to obtain an analytic curve passing by p . Let us consider this operation in this section. Let

$$
\begin{aligned}
X_p &= \{\varphi : \mathbb{D}_r \to \mathbb{D}_u \mid \varphi : \text{analytic}\}, \\
X_q &= \{\varphi : \mathbb{D}_r \to \mathbb{D}_u \mid \varphi : \text{analytic}\},
\end{aligned}
$$

and difine the topology by the supremum norm :

$$
\|\varphi\| = \sup_{(\xi,\eta)\in \mathbb{D}_r \times \mathbb{D}_u} |\varphi(\xi,\eta)|.
$$

For $\varphi_q \in X_q$, the graph $\{(\xi,\eta) \in \mathbb{D}_r \times \mathbb{D}_u \mid \eta = \varphi_q(\xi)\}$ of φ_q defines an analytic curve passing near q by identifying $\mathbb{D}_r \times \mathbb{D}_u \subset B_\epsilon$ as a local coordinate around q. The pull-back $\psi_p \in X_p$ given by f is defined by the equation

$$
f_{p,2}(\xi, \psi_p(\xi)) = \varphi_q(f_{p,1}(\xi, \psi_p(\xi))).
$$

As we shall prove in the following paragraphs, ψ_p is well defined and $\psi_p \in X_p$.

LEMMA 4 For $\varphi_q \in X_q$ and $\xi \in \mathbb{D}_r$, there exists a unique $\eta \in \mathbb{D}_u$ such that

$$
f_{p,2}(\xi,\eta) = \varphi_q(f_{p,1}(\xi,\eta)).
$$

Proof Let

$$N(\eta) \;=\; \eta - \frac{1}{b(p)}(f_{p,2}(\xi,\eta) - \varphi_q(f_{p,1}(\xi,\eta))).$$

Then

$$\begin{aligned}
|N(\eta)| \;&=\; |\eta - \frac{1}{b(p)}(c(p)\xi + b(p)\eta + h_p(\xi,\eta) - \varphi_q(f_{p,1}(\xi,\eta)))| \\
&= \frac{1}{|b(p)|}|c(p)\xi + h_p(\xi,\eta) - \varphi_q(f_{p,1}(\xi,\eta))| \\
&< \frac{1}{\beta}(m_c r + 4u^2 m_h + u) \\
&\le \frac{1}{\beta}(\frac{\beta-1}{4}u + \frac{\beta-1}{2}u + u) \;=\; \frac{3\beta+1}{4\beta}u \;<\; u.
\end{aligned}$$

Hence

$$N : \mathbb{D}_u \to \mathbb{D}_{\frac{3\beta+1}{4\beta}u} \subset \mathbb{D}_u.$$

Next, let us show that $N : \mathbb{D}_u \to \mathbb{D}_u$ is a contraction mapping. Note that $|\varphi_q'| < \frac{4u}{r}$ holds in $\mathbb{D}_{\frac{r}{2}}$, since $\varphi_q : \mathbb{D}_r \to \mathbb{D}_u$ is complex analytic in $\mathbb{D}_r$. For $\eta_1, \eta_2 \in \mathbb{D}_u$, we have

$$\begin{aligned}
|N(\eta_1) - N(\eta_2)| &\le \frac{1}{\beta}(|h_p(\xi,\eta_1) - h_p(\xi,\eta_2)| + |\varphi_q(f_{p,1}(\xi,\eta_1)) - \varphi_q(f_{p,1}(\xi,\eta_2))|) \\
&\le \frac{1}{\beta}(2um_h|\eta_1-\eta_2| + \frac{4u}{r}r^2 M_g|\eta_1-\eta_2|) \\
&\le \frac{1}{\beta}(\frac{\beta-1}{4} + \frac{\beta-1}{2})|\eta_1-\eta_2| \\
&= \frac{3}{4}\frac{\beta-1}{\beta}|\eta_1-\eta_2|.
\end{aligned}$$

As $\frac{3(\beta-1)}{4\beta} < 1$, we see N is a contraction mapping.

This lemma assures the existence and uniqueness of the solution of equation $N(\eta) = \eta$ for each $\xi \in \mathbb{D}_r$. Let $\psi_p(\xi)$ denote the solution. $\psi_p(\xi)$ is nothing but the implicit function defined by

$$f_{p,2}(\xi, \psi_p(\xi)) \;=\; \varphi_q(f_{p,1}(\xi, \psi_p(\xi))).$$

As $\psi_p : \mathbb{D}_r \to \mathbb{D}_u$ is an analytic function, $\psi_p \in X_p$.

We denote the mapping defined by the pull-back operation by

$$\Gamma_p : X_q \to X_p,$$

where $q = f(p)$.

4. Graph bundle and invariant section

Consider the set of graphs X_p for each point p of the Julia set J of super-saddle type. And let

$$X_J \;=\; \bigcup_{p \in J} X_p.$$

We regard X_J as a fiber bundle over J with X_p the fiber at p and introduce the (product) topology in a natural way. Let Ξ denote the space of continuous cross-sections of the fiber bundle X_J. For $\varphi \in \Xi$, $\varphi_p \in X_p$ denotes the graph at $p \in J$ given by the section.

Define a norm $\|\varphi\|$ on Ξ by

$$\|\varphi\| \;=\; \sup_{p\in J} \|\varphi_p\|,$$

where $\|\varphi_p\|$ is the supremum norm on X_p. We define the graph transformation map $\Gamma : \Xi \to \Xi$ by

$$(\Gamma(\varphi))_p \;=\; \Gamma_p(\varphi_{f(p)})$$

for $\varphi \in \Xi$ and $p \in J$. We obtain the following theorem.

Theorem 5 $\quad \Gamma : \Xi \to \Xi$ is a contraction map

Proof Let $\varphi^{(1)}, \varphi^{(2)} \in \Xi$ and $\psi^{(1)} = \Gamma(\varphi^{(1)})$, $\psi^{(2)} = \Gamma(\varphi^{(2)})$. Let $q = f(p)$. Then, $\psi_p^{(1)} = \Gamma_p(\varphi_q^{(1)})$, $\psi_p^{(2)} = \Gamma_p(\varphi_q^{(2)})$. And they satisfy

$$f_{p,2}(\xi, \psi_p^{(1)}(\xi)) \;=\; \varphi_q^{(1)}(f_{p,1}(\xi, \psi_p^{(1)}(\xi)))$$

and

$$f_{p,2}(\xi, \psi_p^{(2)}(\xi)) \;=\; \varphi_q^{(2)}(f_{p,1}(\xi, \psi_p^{(2)}(\xi)))$$

for all $\xi \in \mathbb{D}_r$. We have

$$\begin{aligned}
|f_{p,2}(\xi, \psi_p^{(1)}(\xi)) - f_{p,2}(\xi, \psi_p^{(2)}(\xi))| \;&=\; |\int_{\psi_p^{(2)}(\xi)}^{\psi_p^{(1)}(\xi)} \frac{\partial f_{p,2}}{\partial \eta}(\xi, \eta) d\eta| \\
&= |\int_{\psi_p^{(2)}(\xi)}^{\psi_p^{(1)}(\xi)} (b(p) + \frac{\partial h_p}{\partial \eta}(\xi, \eta)) d\eta| \\
&= |b(p)(\psi_p^{(1)}(\xi) - \psi_p^{(2)}(\xi)) + \int_{\psi_p^{(2)}(\xi)}^{\psi_p^{(1)}(\xi)} \frac{\partial h_p}{\partial \eta}(\xi, \eta) d\eta| \\
&\geq \beta|\psi_p^{(1)}(\xi) - \psi_p^{(2)}(\xi)| - (u + r) m_h |\psi_p^{(1)}(\xi) - \psi_p^{(2)}(\xi)| \\
&\geq (\beta - \frac{\beta - 1}{4})|\psi_p^{(1)}(\xi) - \psi_p^{(2)}(\xi)| \\
&= (\frac{3}{4}(\beta - 1) + 1)|\psi_p^{(1)}(\xi) - \psi_p^{(2)}(\xi)|,
\end{aligned}$$

and on the other hand, we have

$$\begin{aligned}
&|\varphi_q^{(1)}(f_{p,1}(\xi, \psi_p^{(1)}(\xi))) - \varphi_q^{(2)}(f_{p,1}(\xi, \psi_p^{(2)}(\xi)))| \\
&\leq |\varphi_q^{(1)}(f_{p,1}(\xi, \psi_p^{(1)}(\xi))) - \varphi_q^{(2)}(f_{p,1}(\xi, \psi_p^{(1)}(\xi)))| \\
&\quad + |\varphi_q^{(2)}(f_{p,1}(\xi, \psi_p^{(1)}(\xi))) - \varphi_q^{(2)}(f_{p,1}(\xi, \psi_p^{(2)}(\xi)))| \\
&\leq \|\varphi_q^{(1)} - \varphi_q^{(2)}\| + \frac{2u}{r} r^2 M_g |\psi_p^{(1)}(\xi) - \psi_p^{(2)}(\xi)|
\end{aligned}$$

$$\leq \|\varphi^{(1)} - \varphi^{(2)}\| + \frac{\beta-1}{4}|\psi_p^{(1)}(\xi) - \psi_p^{(2)}(\xi)|.$$

Hence,

$$(\frac{1}{2}(\beta-1)+1)|\psi_p^{(1)}(\xi) - \psi_p^{(2)}(\xi)| \leq \|\varphi^{(1)} - \varphi^{(2)}\|$$

holds for all $p \in J$ and $\xi \in \mathbb{D}_r$. Therefore, we have

$$\|\psi^{(1)} - \psi^{(2)}\| \leq \frac{1}{\frac{1}{2}(\beta-1)+1}\|\varphi^{(1)} - \varphi^{(2)}\|,$$

which implies that $\Gamma : \Xi \to \Xi$ is a contraction mapping.

This contraction mapping has a unique fixed point. We denote the unique invariant section by $\sigma \in \Xi$. Thus we obtained the following theorem.

THEOREM 6 There exists a unique $\sigma \in \Xi$ such that $\Gamma(\sigma) = \sigma$.

5. Super-stable manifold

As is easily verified, the family of analytic curves given by the invariant section $\sigma \in \Xi$, which is obtained in the preceeding section, satisfies $\sigma_p(0) = 0$ for all $p \in J$. For each $p = (0, y_0) \in J$, we denote by W_p the complex analytic curve passing by p corresponding to the invariant section, *i.e.*,

$$W_p = \{(\xi, y_0 + \eta) \in \mathbb{C}^2 \mid \xi \in \mathbb{D}_r, \eta \in \mathbb{D}_u, \eta = \sigma_p(\xi)\}$$

Let

$$W_J = \bigcup_{p \in J} W_p$$

and we call it the *local super-stable manifold* of J . W_J is a fiber bundle over J and each fiber is an open disk analytically embedded in $\mathbb{C}^2$. The union of preimages of W_J by f is called the *super-stable manifold* of J.

THEOREM 7 Let $p = (0, y_0) \in J$ and $(\xi, \eta) \in \mathbb{D}_r \times \mathbb{D}_u$.
1) If $\eta = \sigma_p(\xi)$ (*i.e.*, $(\xi, y_0 + \eta) \in W_p$), then

$$\lim_{n \to \infty} \mathrm{dist}(f^n(\xi, y_0 + \eta), f^n(p)) = 0.$$

2) If $\eta \neq \sigma_p(\xi)$, then

$$\mathrm{dist}(f^n(\xi, y_0 + \eta), f^n(p)) > u$$

for some positive integer n .

PROOF First, consider the case $\eta = \sigma_p(\xi)$. Let $(\xi_0, \eta_0) = (\xi, \eta)$ and let

$$(\xi_{k+1}, \eta_{k+1}) = f_{p_k}(\xi_k, \eta_k), \qquad k = 0, 1, \cdots,$$

where $p_k = f^k(p)$. This sequence of points corresponds to the orbit of $(\xi_0, y_0 + \eta_0)$. As σ is the invariant section, we see

$$\eta_k = \sigma_{p_k}(\xi_k).$$

On the other hand, $|\xi_{k+1}| \le \frac{1}{2}|\xi_k|$, $(k = 0, 1, 2, \cdots)$ implies $\lim_{k\to\infty} \xi_k = 0$. For $k > 0$, we have $\xi_k \in \mathbb{D}_{\frac{r}{2}}$. Hence,

$$|\eta_k| \le \frac{4u}{r}|\xi_k|.$$

Therefore, $\lim_{k\to\infty} \eta_k = 0$. That is to say

$$\lim_{n\to\infty} \mathrm{dist}(f^n(\xi, y_0 + \eta), p_n) \;=\; 0.$$

Next, consider the case $\eta \neq \sigma_p(\xi)$. Similarly as in the case above, let $(\xi_0, \eta_0) = (\xi, \eta)$ and define (ξ_k, η_k) for $k = 0, 1, 2, \cdots$. In this case, the sequence is defined only for k with $\xi_k \in \mathbb{D}_r$ and $\eta_k \in \mathbb{D}_u$. Let $p_k = (0, y_k)$. We examine the distance between the point $(\xi_k, y_k + \eta_k)$ and the curve W_{p_k}. For this purpose, consider $|\eta_k - \sigma_{p_k}(\xi_k)|$. We have

$$\begin{aligned}
|\eta_{k+1} - \sigma_{p_{k+1}}(\xi_{k+1})| \;&=\; |f_{p_k,2}(\xi_k, \eta_k) - \sigma_{p_{k+1}}(f_{p_k,1}(\xi_k, \eta_k))| \\
&\ge |f_{p_k,2}(\xi_k, \eta_k) - f_{p_k,2}(\xi_k, \sigma_{p_k}(\xi_k))| - |f_{p_k,2}(\xi_k, \sigma_{p_k}(\xi_k)) - \sigma_{p_{k+1}}(f_{p_k,1}(\xi_k, \eta_k))| \\
&\ge (\frac{3}{4}(\beta - 1) + 1)|\eta_k - \sigma_{p_k}(\xi_k)| - |\sigma_{p_{k+1}}(f_{p_k,1}(\xi_k, \sigma_{p_k}(\xi_k))) - \sigma_{p_{k+1}}(f_{p_k,1}(\xi_k, \eta_k))| \\
&\ge (\frac{3}{4}(\beta - 1) + 1)|\eta_k - \sigma_{p_k}(\xi_k)| - \frac{4u}{r} r^2 M_g |\eta_k - \sigma_{p_k}(\xi_k)| \\
&\ge (\frac{1}{4}(\beta - 1) + 1)|\eta_k - \sigma_{p_k}(\xi_k)|,
\end{aligned}$$

which proves the theorem.

6. Harmonic function on the super-stable manifold

There exists a real valued continuous function defined on the super-stable manifold W_J of the Julia set J of super-saddle type. The continuous function is well adapted to the dynamics of the system. We discuss about this "potential function" in this section.

Theorem 8 There exists a positive, real-valued continuous function $\rho : W_J \setminus J \to \mathbb{R}_+$ defined on $W_J \setminus J$, such that ρ is harmonic on each fiber $W_p \setminus \{p\}$, and such that for all $z \in W_J \setminus J$,

$$\rho(f(z)) \;=\; 2\rho(z)$$

holds.

Proof For $(x, y) \in W_J \subset \mathbb{C}^2$, let

$$\chi_0(x, y) \;=\; -\log|x|.$$

As $|x| < r$, $\chi_0(x, y) > \log\frac{1}{r} > 0$. Obviously, $\chi_0(x, y)$ is continuous on $W_J \setminus J$, and is harmonic on each fiber. Next, let

$$\chi_1(x, y) \;=\; -\frac{1}{2}\log|f_1(x, y)|.$$

We have

$$\chi_1(x, y) \;=\; -\frac{1}{2}\log|x^2 g(x, y)|$$

$$= -\log|x| - \frac{1}{2}\log|g(x,y)|$$
$$= \chi_0(x,y) - \frac{1}{2}\log|g(x,y)|.$$

For $n \geq 1$, let

$$\chi_{n+1}(x,y) \;=\; -\frac{1}{2^{n+1}}\log|f_1(f^{\circ n}(x,y))|.$$

We denote as $f^{\circ n}(x,y) = (x_n, y_n)$. Then,

$$\chi_{n+1}(x,y) \;=\; -\frac{1}{2^{n+1}}\log|x_n^2 g(x_n,y_n)|$$
$$= -\frac{1}{2^n}\log|x_n| - \frac{1}{2^{n+1}}\log|g(x_n,y_n)|$$
$$= \chi_n(x,y) - \frac{1}{2^{n+1}}\log|g(x_n,y_n)|.$$

Thus, we obtain

$$\chi_{n+1}(x,y) \;=\; -\log|x| - \sum_{k=0}^{n}\frac{1}{2^{k+1}}\log|g(x_k,y_k)|.$$

For $(x_0,y_0) \in W_J \setminus J$, we see $(x_k,y_k) \in W_J \setminus J$ for $k = 0,1,2,\cdots$. Recall that $\alpha \leq |g(x,y)| \leq m_g$ holds on W_J. When $n \to \infty$, $\chi_n(x,y) + \log|x|$ converges uniformly in $W_J \setminus J$. Let

$$\rho(x,y) \;=\; \lim_{n\to\infty}\chi_n(x,y).$$

Then for $z \in W_J \setminus J$, we have

$$\rho(f(z)) \;=\; \lim_{n\to\infty}\chi_n(f(z))$$
$$= \lim_{n\to\infty} -\frac{1}{2^{n+1}}\log|f_1(f^{\circ n}(f(z)))|$$
$$= \lim_{n\to\infty} -\frac{1}{2^{n+1}}\log|f_1(f^{\circ(n+1)}(z))|$$
$$= 2\lim_{n\to\infty}\chi_n(z)$$
$$= 2\rho(z).$$

This completes the proof.

7. Numerical experiments

Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be an analytic map satisfying the conditions in section 1. Let $A(O)$ denote the attractive basin of the origin defined as

$$A(O) \;=\; \{(x,y) \in \mathbb{C}^2 \mid \lim_{n\to\infty} f^{\circ n}(x,y) \;=\; O\}.$$

The Julia set J of super-saddle type is included in the boundary of the attractive basin $A(O)$. In this case, the super-stable manifold of J is included in the bondary of $A(O)$, too. If, moreover, f maps the x-axis into the x-axis, and if the Julia set in the x-axis is of super-saddle type, then the super-stable manifold of the latter Julia set is included in the boundary of $A(O)$ as well. Hence, the attractive basin $A(O)$ can have different forms as its boundary. The boundary of the attractive basin is not "self-similar", if the "self-similarity" means that any small portion of the set includes a miniature of the set itself.

We executed numerical experiments for the case where the x-axis and the y-axis are respectively invariant under the mapping f, and the Julia sets in each of the axes are of super-saddle type. We denote these Julia sets by J_x and J_y respectively. Note that $J_x \subset \mathbb{C}_y$ and $J_y \subset \mathbb{C}_y$. In [5], we showed the case where J_x and J_y are unit circles. The mappings used there are as follows.

$$f(x, y) = (x^2 - 4x^2y, y^2 + 2.7ixy^2)$$

$$f(x, y) = (x^2 - 4x^2y + x^3, y^2 + 2.7ixy^2)$$
$$f(x, y) = (x^2 - 4x^2y + x^3, y^2 + 2.7ixy^2 - y^3)$$

Figire ! of this article is for the dynamical system defined by

$$f(x, y) = (3x^2 - 2x^3 - 4x^2y, y^2 + 2.7ixy^2).$$

In this case, the Julia set J_x is same as the usual Julia set in $\mathbb{C}$ of the dynamical system $x \mapsto 3x^2 - 2x^3$, which has two super-attractive fixed points at $x = 0$ and $x = 1$. J_y is the unit circle. In this figure, we can observe portions of the boundary of the attractive basin $A(O)$, which resemble the two different Julia sets. The region represented in the figure is the rectangular region given by the following.

Fig.1: $\quad y = 0.9, \quad -0.9 \le \Re x \le 0.9, \quad -0.72 \le \Im x \le 0.72$

Figures 2 and 3 are for the case with

$$f(x, y) = (x^2 - 4x^2y, -0.9y + y^2 + 2.7ixy^2).$$

In this case, the origin is an attractive fixed point with eigenvalues 0 and -0.9. The figures represent the regions indicted in the following..

Fig.2: $\quad y = 0.7, \quad -0.9 \le \Re x \le 0.9, \quad -0.72 \le \Im x \le 0.72$

Fig.3: $\quad y = 0.7, \quad 0.4697 \le \Re x \le 0.7, \quad -0.05309 \le \Im x \le -0.05285$

Figure 4 corresponds to the case with

$$f(x, y) = (3x^2 - 4x^2y - 2x^3, (0.7 + 0.2i)y + y^2 + 2.7ixy^2).$$

The region is as follows.

Fig.4: $\quad y = 0.5, \quad -0.4 \le \Re x \le 0.6, \quad -0.1 \le \Im x \le 0.7$

Figures 5 to 8 are for

$$f(x, y) = (3x^2 - 4x^2y - 2x^3, 3y^2 + 2.7ixy^2 - 2y^3).$$

This dynamical system has three super-attractive fixed points. The regions represented by the figures are as follows.

Fig.5: $y = 0.5, \quad -0.7 \leq \Re x \leq 0.7, \quad -0.56 \leq \Im x \leq 0.56$

Fig.6: $y = 0.5, \quad 0.4 \leq \Re x \leq 0.44, \quad 0.464 \leq \Im x \leq 0.496$

Fig.7: $y = 0.5, \quad 0.427133 \leq \Re x \leq 0.47135, \quad 0.4812912 \leq \Im x \leq 0.4812928$

Fig.8: $y = 0.5, \quad 0.4245 \leq \Re x \leq 0.4385, \quad 0.5364 \leq \Im x \leq 0.5476.$

References

[1] Böttcher, Bull. Kasan Math. Soc. **14** (1905) p176.

[2] J. Hadamard : Sur l'itération et les solutions asymptotiques des équations différentielles, Bulletin de la Société Mathématiques de France, Tome 29, 1901.

[3] J. H. Hubbard and P. Papadopol, Superattractive fixed points in $\mathbb{C}^n$, preprint.

[4] H. Poincaré : Sur une classe nouvelle des transcendentes uniformes, Journal de Mathématiques, 4^e série, tome VI, Fasc. IV, 1890.

[5] S. Ushiki : Unstable manifolds of analytic dynamical systems, Journal of Mathematics of Kyoto University, Vol. 21, No. 4, pp763-785, 1981.

[6] S. Ushiki : Böttcher's Theorem and Super-Stable Manifolds for Multidimensional Complex Dynamical Systems, in "Structure and Bifurcations of Dynamical Systems" ed. S. Ushiki, Advanced Series for Dynamical Systems, Vol. 11, World Scientific Publishers, 1992.

Fig.1

Fig.2

Fig.3

Fig.4

Fig.5

Fig.6

Fig.7

Fig.8