Böttcher's Theorem and Super-Stable Manifolds for Multidimensional Complex Dynamical Systems

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Abstract Super-stable manifolds associated to zero eigenvalues of fixed points for multi-dimensional complex dynamical systems are discussed. Some super-attractive fixed points and "super-saddle" type fixed points have complex analytic super-stable manifolds.

In this note, we consider stable manifolds of super-attractive fixed points and stable manifolds of fixed points with a zero eigenvalue of multi-dimensional complex dynamical systems.

Let $f : \mathbb{C} \to \mathbb{C}$ be a complex analytic mapping. A point $p \in \mathbb{C}$ is called a *super-attractive* fixed point if $f(p) = p$ and $f'(p) = 0$. If f is not a constant function, the classical Böttcher's theorem asserts that f is analytically conjugate to a map $z \mapsto z^k$ for some integer $k > 1$ in a neighbourhood of p. In section 1, we recall this theorem.

Let us consider a complex 2-dimensional dynamical system $f : \mathbb{C}^2 \to \mathbb{C}^2$. Assume f is complex analytic in a neighborhood of the origin, $O = (0, 0)$, and that the origin is a fixed point of f , *i.e.*, $f(O) = O$. Furthermore, we assume both of the eigenvalues of the Jacobian matrix at the origin, DF_O, are zero. Such a fixed point is said to be *super-attractive*. Hubbard and Papadopol[3] studied the case of super-attractive fixed points for homogeneous polynomial maps and their perturbations.

In general, it is not possible to find an analytic change of coordinates around the super-attractive fixed point which transforms the dynamical system into a "simple normal form", for example, $(x, y) \mapsto (x^2, y^2)$. In section 2, we describe a class of dynamical systems which can be "normalized" by an analytic change of coordinates into the simplest normal form above.

In section 3, we shall discuss about "super-saddle" fixed points. Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a complex analytic mapping. A fixed point of f is said to be a *super-saddle* point if one of the eigenvalues of the Jacobian matrix at the fixed point is zero and the absolute value of the other eigenvalue is greater than 1. It is known that there exists the unstable manifold corresponding to the eigenvalue which is greater than 1 in modulus. We shall prove the existence of the "super-stable" manifold corresponding to the eigenvalue zero.

In the Appendix, we give some examples of 2-dimensional complex dynamical systems where multi-dimensional version of Böttcher's theorem can be applied. Computer generated pictures suggest that the basin of attraction of the super-attractive fixed point has "fractal" boundaries.

1. Böttcher's Theorem

In this section, we recall the classical Böttcher's theorem. Let $f : \mathbb{C} \to \mathbb{C}$ be complex analytic in a neighborhood of the origin, O , and assume O is a super-attractive fixed point, *i.e.*,

$$f(O) \;=\; O, \quad f'(O) \;=\; 0.$$

We assume f is not constant near O. By a linear change of coordinates around the origin if necessary, we may assume

$$f(z) \;=\; z^k + a_{k+1}z^{k+1} + \cdots$$

with $k \geq 2$.

THEOREM (Böttcher) If

$$f(z) \;=\; z^k + a_{k+1}z^{k+1} + \cdots$$

is complex analytic near the origin with $k \geq 2$, then there exist neighborhoods U, V of the origin and a complex analytic diffeomorphism

$$\varphi : U \to V, \quad \varphi(0) = 0, \quad \varphi'(0) = 1$$

such that

$$\varphi \circ f(z) \;=\; (\varphi(z))^k$$

holds near the origin.

PROOF For $r > 0$, let $\mathbb{D}_r = \{z \in \mathbb{C} \mid |z| < r\}$. By taking sufficiently small $r > 0$, we can assume the followings :

f is defined and complex analytic in $\mathbb{D}_r$.

closure$(f(\mathbb{D}_r)) \subset \mathbb{D}_r$.

For all $z \in \mathbb{D}_r$, $f^{\circ n}(z) \to 0$ $(n \to \infty)$, where $f^{\circ n}$ denotes the composite map defined by $f^{\circ n} = f^{\circ(n-1)} \circ f$.

O is the only critical point of f in $\mathbb{D}_r$.

$f(z) \neq 0$ if $z \in \mathbb{D}_r \setminus \{0\}$.

As $f : \mathbb{D}_r \to \mathbb{D}_r$ is a branched covering map of degree k, we can define a complex analytic function $f(z)^{\frac{1}{k}}$. This function is unique up to a k-th root of 1.

For $z \in \mathbb{D}_r$, let

$$\varphi_0(z) \;=\; z$$

and

$$\varphi_1(z) \;=\; (f(z))^{\frac{1}{k}} \quad (\varphi_1'(0) \;=\; 1).$$

Similarly, define complex analytic maps $\varphi_n : \mathbb{D}_r \to \mathbb{C}$ for $n = 1, 2, \cdots$ by

$$\varphi_n(z) \;=\; (f^{\circ n}(z))^{\frac{1}{k^n}} \quad (\varphi_n'(0) \;=\; 1).$$

As we shall prove in the next paragraph, φ_n converges uniformly in $\mathbb{D}_r$ as $n \to \infty$. If we set

$$\varphi = \lim_{n\to\infty} \varphi_n$$

then

$$\varphi \circ f(z) = \lim_{n\to\infty} (f^{\circ n}(f(z)))^{\frac{1}{k^n}} = \lim_{n\to\infty} ((f^{\circ(n+1)}(z))^{\frac{1}{k^{n+1}}})^k = (\varphi(z))^k.$$

Hence, $\varphi : \mathbb{D}_r \to \varphi(\mathbb{D}_r)$ is the analytic coordinate transformation we need.

Let us verify the uniform convergence of φ_n in $\mathbb{D}_r$. Define $H : \mathbb{D}_r \to \mathbb{C}$ by

$$H(z) = \frac{\varphi_1(z)}{z}, \quad H(0) = 1.$$

H is complex analytic and $H(z) \neq 0$ in $\mathbb{D}_r$. For $m > 1$, let $H(z)^{\frac{1}{m}}$ denote the branch satisfying $H(0)^{\frac{1}{m}} = 1$. As

$$\varphi_1(z) = (f(z))^{\frac{1}{k}} = zH(z)$$

and

$$\frac{\varphi_{n+1}(z)}{\varphi_n(z)} = \frac{(f^{\circ(n+1)}(z))^{\frac{1}{k^{n+1}}}}{(f^{\circ n}(z))^{\frac{1}{k^n}}} = (\frac{(f(f^{\circ n}(z)))^{\frac{1}{k}}}{f^{\circ n}(z)})^{\frac{1}{k^n}}$$

$$= (\frac{\varphi_1(f^{\circ n}(z))}{f^{\circ n}(z)})^{\frac{1}{k^n}} = (H(f^{\circ n}(z)))^{\frac{1}{k^n}},$$

we have

$$\varphi_{n+1}(z) = z\prod_{i=0}^{n} \frac{\varphi_{i+1}(z)}{\varphi_i(z)} = z\prod_{i=0}^{n}(H(f^{\circ i}(z)))^{\frac{1}{k^i}}.$$

Therefore

$$\log(\frac{\varphi_{n+1}(z)}{z}) = \sum_{i=0}^{n} \log(H(f^{\circ i}(z))^{\frac{1}{k^i}}) = \sum_{i=0}^{n} \frac{1}{k^i} \log(H(f^{\circ i}(z))).$$

As $f^{\circ i}(z) \in \mathbb{D}_r$ for $z \in \mathbb{D}_r,\ i = 0, 1, 2, \cdots$, we see $\log(H(f^{\circ i}(z)))$ is uniformly bounded in $\mathbb{D}_r$. Hence the convergence of φ_n is uniform in $\mathbb{D}_r$.

2. Multi-dimensional Böttcher's Theorem

In this section, we consider the case of dimension two. Similar theory holds for higher dimensional cases.

Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be complex analytic in a neighborhood of the origin, $O = (0, 0)$. Suppose the origin is a fixed point of F, *i.e.*, $F(O) = O$. Let

$$F(x, y) = (f_1(x, y),\ f_2(x, y)).$$

We assume that the x-axis, $\{(x, 0)\}$, and the y-axis $\{(0, y)\}$ are invariant under F, *i.e.*,

$$f_2(x, 0) = 0 \quad \text{and} \quad f_1(0, y) = 0$$

holds for all x and y near the origin. We assume

$$f_1(x,0) \;=\; x^2 + h.o.t., \quad f_2(0,y) \;=\; y^2 + h.o.t.$$

Moreover, we assume $\det(DF) = 0$ along the x-axis and the y-axis.

These assumptions are quite special and may appear to be quite artificial. However, if we want F to be analytically conjugated to mappping $(x,y) \mapsto (x^2, y^2)$, we must impose at least infinitely many algebraic relations between the Taylor coefficients of F in order to have a conjugacy map as formal power series.

Under the assumptions above, we can apply the Böttcher's theorem to normalize the mapping on the x-axis and the y-axis respectively, we can rewrite the mapping F in the form

$$\begin{aligned} f_1(x,y) &= x^2(1 + y g_1(x,y)) \\ f_2(x,y) &= y^2(1 + x g_2(x,y)) \end{aligned}$$

in a neighborhood of the origin, where $g_1(x,y)$ and $g_2(x,y)$ are complex analytic in the neighborhood of the origin. Let $\psi : \mathbb{C}^2 \to \mathbb{C}^2$ denote the "normal form" mapping $\psi(x,y) = (x^2, y^2)$.

THEOREM If $F : \mathbb{C}^2 \to \mathbb{C}^2$ is in the form above, then there exists a complex analytic diffeomorphism $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ defined in a neighborhood of the origin satisfying

$$\Phi(0,0) \;=\; (0,0), \quad D\Phi_{(0,0)} \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

such that

$$\Phi \circ F \;=\; \psi \circ \Phi$$

holds near the origin.

PROOF Since

$$DF_O \;=\; \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

there exists a neighborhood $U \subset \mathbb{C}^2$ of the origin, satisfying $\mathrm{closure}(F(U)) \subset U$ and that for any $(x,y) \in U$, $\lim_{n\to\infty} F^{\circ n}(x,y) = O$ holds. Moreover, we can assume

$$|y g_1(x,y)| < \frac{1}{2}, \quad |x g_2(x,y)| < \frac{1}{2}$$

for all $(x,y) \in U$. We shall denote the components of $F^{\circ n}$ as

$$F^{\circ n}(x,y) \;=\; (F_1^{\circ n}(x,y),\ F_2^{\circ n}(x,y)) \;=\; (x_n,\ y_n).$$

First, let us construct the first component Φ_1 of Φ . Let $\varphi_0(x,y) = x$ and define $\varphi_n(x,y) : U \to \mathbb{C}$ by

$$\varphi_n(x,y) \;=\; (F_1^{\circ n}(x,y))^{\frac{1}{2^n}}$$

for $n = 1, 2, \cdots$. Here, we choose the branch of the right hand side satisfying $\frac{\partial \varphi_n}{\partial x}(O) = 1$. As F maps the y-axis into itself, φ_n is complex analytic in the neighborhood. Let us verify that φ_n converges uniformly in U . We see

$$\frac{\varphi_{n+1}(x,y)}{\varphi_n(x,y)} \;=\; \frac{(F_1^{\circ(n+1)}(x,y))^{\frac{1}{2^{n+1}}}}{(F_1^{\circ n}(x,y))^{\frac{1}{2^n}}} \;=\; \left(\frac{(f_1(F^{\circ n}(x,y)))^{\frac{1}{2}}}{F_1^{\circ n}(x,y)}\right)^{\frac{1}{2^n}}$$

$$= \left(\frac{(f_1(x_n, y_n))^{\frac{1}{2}}}{x_n}\right)^{\frac{1}{2^n}} \;=\; \left(\frac{(x_n^2(1 + y_n g_1(x_n, y_n)))^{\frac{1}{2}}}{x_n}\right)^{\frac{1}{2^n}} \;=\; (1 + y_n g_1(x_n, y_n))^{\frac{1}{2^{n+1}}}.$$

As $|yg_1(x, y)| < \frac{1}{2}$ holds in the neighborhood U ,

$$\varphi_{n+1}(x, y) \;=\; x \prod_{j=0}^{n} (1 + y_j g(x_j, y_j))^{\frac{1}{2^{j+1}}}$$

is uniformly convergent in U , where $(x_0, y_0) = (x, y)$. Hence, by setting

$$\lim_{n\to\infty} \varphi_n \;=\; \Phi_1 \;,$$

Φ_1 is complex analytic in U and satisfies the function equation

$$\Phi_1 \circ F \;=\; \Phi_1^2.$$

Similarly, the second component Φ_2 can be defined. Therefore, by setting

$$\Phi(x, y) \;=\; (\Phi_1(x, y), \Phi_2(x, y)),$$

the function equation

$$\Phi \circ F \;=\; \psi \circ \Phi$$

holds near the origin.

Note that a similar theorem holds for higher degree cases as the following.

Theorem Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a complex analytic mapping defined near the origin. Suppose F is of the form

$$F(x, y) \;=\; (x^k(1 + yh_1(x, y)), y^p(1 + xh_2(x, y)))$$

where $k,\ p \geq 2$, and $h_1(x, y)$ and $h_2(x, y)$ are complex analytic near the origin. Then there exists a complex analytic change of coordinates $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ around the origin with

$$\Phi(0, 0) \;=\; (0, 0), \quad D\Phi_O = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

such that

$$\Phi \circ F \;=\; \Psi \circ \Phi$$

holds in a neighborhood of the origin, where $\Psi(x, y) = (x^k, y^p)$.

3. Super-Stable Manifold

Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be complex analytic in a neighborhood of the origin and suppose the origin is a fixed point of F , *i.e.*, $F(O) = O$. Fixed point is said to be of *super-saddle* type if one the eigenvalues of the Jacobian matrix at the fixed point is zero and the other eigenvalue, say b , is greater than one in modulus. We assume O is a super-saddle type fixed point with eigenvalues 0 and b ($|b| > 1$). We may assume that the Jacobian matrix of F at O is diagonal and of the form

$$DF_O = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}.$$

We may assume that the y-axis coinsides with the unstable manifold of O , and

$$F(0,y) = (0, by + O(y^2))$$

holds near the origin. This is always possible by taking an appropriate system of coordinates, since the unstable manifold corresponding to the eigenvalue b is a complex analytic curve passing by the fixed point of super-saddle type [4],[5]. Furthermore, we assume

$$F(x,0) = (x^2 + O(x^3), O(x^2))$$

on the x-axis. Finally, we assume the singular locus coinsides with the y-axis, *i.e.*, $\{\det(DF) = 0\} = \{0\} \times \mathbb{C}$ in the neighborhood of the origin.

THEOREM If $F : \mathbb{C}^2 \to \mathbb{C}^2$ satisfies the assumptions above, there exists a complex analytic embedding $\sigma : \mathbb{D} \to \mathbb{C}^2$ defined in a neighborhood of the origin of the unit disk $\mathbb{D} = \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$ satisfying $\sigma(0) = O$ and $D\sigma_0 = (1,0)$ such that $F \circ \sigma(\zeta) = \sigma(\zeta^2)$ holds in the neighborhood of the origin. This embedding is unique in the sense that if two such embeddings exist, they are identical in a neighborhood of the origin.

Let us call the image of this embedding the (local) *super-stable manifold.* The image $\sigma(\mathbb{D})$ consists of points which are attracted by O.

PROOF By a complex analytic change of coordinates with respect to the y-coordinate if necessary, we can assume F is linear on the y-axis, *i.e.*,

$$F(0,y) = (0, by).$$

By applying the Böttcher's theorem to the x-coordinate, we can assume

$$F(x,0) = (x^2, O(x^2)).$$

Then, $F(x,y)$ can be written in the form

$$F(x,y) = (F_1(x,y), F_2(x,y)) = (x^2(1 + g_1(x,y)), by + xg_2(x,y)),$$

where $g_1(x,y)$ and $g_2(x,y)$ are complex analytic functions defined in a neighborhood of the origin satisfying $g_1(x,0) = 0$, and $g_2(0,0) = 0$.

For $r_0 > 0$ and $u > 0$, let

$$\mathbb{D}_{r_0} = \{x \in \mathbb{C} \mid |x| < r_0\}, \quad \mathbb{D}_u = \{y \in \mathbb{C} \mid |y| < u\}.$$

We assume

$$0 < r_0 < \frac{1}{4}, \quad 0 < u < 1$$

and F is complex analytic in $\mathbb{D}_{r_0} \times \mathbb{D}_u$. Moreover, we assume $m_1,\ m_2,\ M_1,\ M_2$ are finite where

$$m_1 \;=\; \sup_{(x,y)\in\mathbb{D}_{r_0}\times\mathbb{D}_u} |g_1(x,y)|, \qquad m_2 \;=\; \sup_{(x,y)\in\mathbb{D}_{r_0}\times\mathbb{D}_u} |g_2(x,y)|,$$

$$M_1 \;=\; \sup_{(x,y)\in\mathbb{D}_{r_0}\times\mathbb{D}_u} |\frac{\partial g_1}{\partial y}(x,y)|, \qquad M_2 \;=\; \sup_{(x,y)\in\mathbb{D}_{r_0}\times\mathbb{D}_u} |\frac{\partial g_2}{\partial y}(x,y)|.$$

Next, let $\beta = |b|$ and let

$$r = \min(r_0,\ \frac{1}{2(1+m_1)},\ \frac{(\beta-1)u}{2m_2}, \frac{\beta-1}{16uM_1},\ \frac{\beta-1}{4M_2}).$$

Then we have $F_1(x,y) \in \mathbb{D}_{\frac{r}{2}}$ for $(x,y) \in \mathbb{D}_r \times \mathbb{D}_u$. In fact,

$$|F_1(x,y)| \;=\; |x^2(1+g_1(x,y))| < r^2(1+m_1) < \frac{r}{2}.$$

Let

$$X \;=\; \{\varphi : \mathbb{D}_r \to \mathbb{D}_u\}$$

denote the space of complex analytic functions of $\mathbb{D}_r$ into $\mathbb{D}_u$ equiped with the supremum norm

$$\|\varphi\| \;=\; \sup_{x\in\mathbb{D}_r} |\varphi(x)|.$$

For a given $\varphi \in X$, consider a function equation with respect to ψ :

$$F_2(x,\psi(x)) \;=\; \varphi(F_1(x,\psi(x))).$$

As we shall prove in the following paragraphs, function $\psi \in X$ is uniquely determined as an implicit function. Define a "graph transform" $\Gamma : X \to X$ by $\Gamma(\varphi) = \psi$.

For a given $\varphi \in X$ and $x \in \mathbb{D}_r$, there exists a unique $y \in \mathbb{D}_u$ satisfying

$$F_2(x,y) \;=\; \varphi(F_1(x,y)).$$

This fact is proved as follows. Let

$$h(y) \;=\; F_2(x,y) - \varphi(F_1(x,y)).$$

As a modified Newton's method to solve equation $h(y) \;=\; 0$, consider an iterative procedure

$$N(y) \;=\; y - \frac{1}{b}h(y) \;=\; -\frac{1}{b}(xg_2(x,y) - \varphi(F_1(x,y))).$$

Then, as $\varphi(F_1(x,y)) \in \mathbb{D}_u$ if $y \in \mathbb{D}_u$, we see

$$|N(y)| \;\le\; \frac{1}{\beta}(|xg_2(x,y)| + |\varphi(F_1(x,y))|)$$

$$< \frac{1}{\beta}(rm_2+u) \;\le\; \frac{1}{\beta}(\frac{\beta-1}{2}u+u) = \frac{\beta+1}{2\beta}u < u$$

Hence $N(y) \in \mathbb{D}_{\frac{\beta+1}{2\beta}u} \subset \mathbb{D}_u$.

Next, let us verify that $N : \mathbb{D}_u \to \mathbb{D}_u$ is a contraction mapping. If $y_1, y_2 \in \mathbb{D}_u$, then

$$N(y_1) - N(y_2) = -\frac{1}{b}(x(g_2(x,y_1) - g_2(x,y_2)) + (\varphi(F_1(x,y_1)) - \varphi(F_1(x,y_2)))).$$

On the other hand, we have

$$|g_2(x,y_1) - g_2(x,y_2)| = |\int_{y_2}^{y_1} \frac{\partial g_2}{\partial y}(x,y)dy| \le M_2|y_1 - y_2|$$

and

$$|\varphi(F_1(x,y_1)) - \varphi(F_1(x,y_2))| = |\int_{y_2}^{y_1} \varphi'(F_1(x,y))\frac{\partial F_1}{\partial y}(x,y)dy|$$
$$\le \frac{4u}{r}r^2 M_1|y_1 - y_2| = 4urM_1|y_1 - y_2|.$$

Here, we used the fact that $|\varphi'(z)| \le \frac{4u}{r}$ for $z \in \mathbb{D}_{\frac{r}{2}}$. Therefore we obtain

$$|N(y_1) - N(y_2)| \le \frac{1}{\beta}(rM_2 + 4urM_1)|y_1 - y_2|$$

$$\le \frac{1}{\beta}(\frac{\beta-1}{4} + \frac{\beta-1}{4})|y_1 - y_2| = \frac{\beta-1}{2\beta}|y_1 - y_2| < \frac{1}{2}|y_1 - y_2|,$$

which shows that $N : \mathbb{D}_u \to \mathbb{D}_u$ is a contraction mapping. Hence, the unique fixed point of N yields the unique solution of $h(y) = 0$ in $\mathbb{D}_u$. By setting $y = \psi(x)$ for each $x \in \mathbb{D}_r$, we obtain a function $\psi : \mathbb{D}_r \to \mathbb{D}_u$. Note that $|\psi(x)| \le \frac{\beta+1}{2\beta}u < u$. As ψ is defined as an implicit function, ψ is complex analytic. Thus the existence and uniqueness of $\psi = \Gamma(\varphi)$ is proved.

Next, let us prove that the graph transform $\Gamma : X \to X$ is a contraction mapping with respect to the norm

$$\|\varphi\| = \sup_{x \in \mathbb{D}_r} |\varphi(x)|.$$

For $\varphi_1, \varphi_2 \in X$, let $\psi_1 = \Gamma(\varphi_1)$ and $\psi_2 = \Gamma(\varphi_2)$. For $x \in \mathbb{D}_r$,

$$|F_2(x,\psi_1(x)) - F_2(x,\psi_2(x))| = |b(\psi_1(x) - \psi_2(x)) + x(g_2(x,\psi_1(x)) - g_2(x,\psi_2(x)))|$$
$$\ge |b(\psi_1(x) - \psi_2(x))| - |x\int_{\psi_2(x)}^{\psi_1(x)} \frac{\partial g_2}{\partial y}(x,y)dy|$$
$$\ge \beta|\psi_1(x) - \psi_2(x)| - |x|M_2|\psi_1(x) - \psi_2(x)|$$
$$\ge (\beta - rM_2)|\psi_1(x) - \psi_2(x)|.$$

On the other hand,

$$|\varphi_1(F_1(x,\psi_1(x))) - \varphi_2(F_1(x,\psi_2(x)))|$$
$$\le |\varphi_1(F_1(x,\psi_1(x))) - \varphi_2(F_1(x,\psi_1(x)))| + |\varphi_2(F_1(x,\psi_1(x))) - \varphi_2(F_1(x,\psi_2(x)))|$$
$$\le \|\varphi_1 - \varphi_2\| + |\int_{\psi_2(x)}^{\psi_1(x)} \varphi_2'(F_1(x,y))\frac{\partial F_1}{\partial y}(x,y)dy|$$
$$\le \|\varphi_1 - \varphi_2\| + 4urM_1|\psi_1(x) - \psi_2(x)|.$$

As $\psi_1 = \Gamma(\varphi_1)$ and $\psi_2 = \Gamma(\varphi_2)$, we have

$$F_2(x, \psi_1(x)) = \varphi_1(F_1(x, \psi_1(y))),$$

$$F_2(x, \psi_2(x)) = \varphi_2(F_1(x, \psi_2(y))).$$

Hence by combining the two inequalities above, we obtain

$$(\beta - rM_2 - 4urM_1)|\psi_1(x) - \psi_2(x)| \le \|\varphi_1 - \varphi_2\| \ .$$

By noting that

$$\beta - r(M_2 + 4uM_1) \ge \frac{1+\beta}{2} > 1 \ ,$$

and that these inequalities hold for all $x \in \mathbb{D}_r$, we conclude that

$$\|\psi_1 - \psi_2\| \ \le \ \frac{2}{1+\beta}\|\varphi_1 - \varphi_2\| \ .$$

Therefore, the graph transform $\Gamma : X \to X$ is a contraction mapping. As uniformly convergent sequence of complex analytic functions has a complex analytic limit function, the graph transform Γ has a unique fixed point in X . Let φ_0 denote the fixed point of Γ . From the conditions we posed on F , we have

$$\varphi_0(0) = 0, \quad \varphi_0'(0) = 0 \ .$$

Finally, we define the embedding $\sigma : \mathbb{D} \to \mathbb{C}^2$ as follows. Let $W = \{(x, \varphi_0(x)) \mid x \in \mathbb{D}_r\}$ be the graph of φ_0. W is a portion of a complex analytic curve passing by the origin O of $\mathbb{C}^2$, tangent to the x-axis at O, and invariant under F. We use the coordinate x as the local coordinate of W around the origin. An analytic map $\rho : W \to W$ is induced from F by $\rho(x) = F_1(x, \varphi_0(x))$. As is easily verified, we have

$$\rho(0) = 0, \quad \rho'(0) = 0, \quad \rho''(0) = 2 \ .$$

We can apply the Böttcher's theorem to $\rho : W \to W$. Thus, we obtain a complex analytic mapping $\sigma : \mathbb{D} \to W$ defined in a neighborhood of the origin with $\sigma(0) = 0$, $\sigma'(0) = 1$, which satisfy the function equation

$$\rho \circ \sigma(\zeta) = \sigma(\zeta^2).$$

As ρ is the restriction of F to W , this analytic map σ gives the embedding of the theorem.

Clearly, if $(x, y) \in W$, then

$$\lim_{n \to \infty} F^{\circ n}(x, y) = O.$$

For $(x, y) \in \mathbb{D}_r \times \mathbb{D}_u$, we have

$$|F_2(x, y) - \varphi_0(F_1(x, y))|$$

$$\ge |F_2(x, y) - F_2(x, \varphi_0(x))| - |F_2(x, \varphi_0(x)) - \varphi_0(F_1(x, y))|$$

$$\geq (\beta - rM_2)|y - \varphi_0(x)| - |\varphi_0(F_1(x,\varphi_0(x))) - \varphi_0(F_1(x,y))|$$
$$\geq (\beta - rM_2 - 4urM_1)|y - \varphi_0(x)| \geq \frac{1+\beta}{2}|y - \varphi_0(x)| \ .$$

Therefore, the "distance from W", measured by $|y-\varphi_0(x)|$, grows geometrically. Hence, if $y \neq \varphi_0(x)$, then $|F_2^{\circ n}(x,y)| \geq u$ for some $n > 0$. Thus, the (local) super-stable manifold W is the local stable set of the super-saddle point O . The global super-stable set is defined as the union of the preimages of W.

Appendix Numerical Experiments

Assume $F : \mathbb{C}^2 \to \mathbb{C}^2$ is a globally defined complex analytic mapping and satisfies the conditions of section 2. Let $A(O)$ denote the basin of attraction of the super-attractive fixed point, *i.e.*,

$$A(O) \ = \ \{(x,y) \in \mathbb{C}^2 \mid \lim_{n\to\infty} F^{\circ n}(x,y) \ = \ O\}.$$

Then, if possible, by extending analytically the generalized Böttcher's function, Φ , of section 2, we would obtain a complex analytic mapping

$$\Phi \ : \ A(O) \ \to \ \mathbb{D} \times \mathbb{D}.$$

Numerical experiments strongly suggest the cases of $A(O)$ with a "fractal" boundary. All pictures of this appendix represent rectangular regions in the complex line $\{(x, 0.5)\}$.

Example 1. Figures 1 to 3 show sections of the attractive basin of complex dynamical system defined by

$$F(x,y) \ = \ (x^2 - 4x^2y, y^2 + 2.7\sqrt{-1}xy^2) \ .$$

The x-coordinates of the rectangular regions are as follows.
Fig. 1. : $-1.2 \leq \Re(x) \leq 1.2, \quad -0.96 \leq \Im(x) \leq 0.96$
Fig. 2. : $-0.77 \leq \Re(x) \leq -0.67, \quad 0.46 \leq \Im(x) \leq 0.54$
Fig. 3. : $-0.732 \leq \Re(x) \leq -0.7276, \quad 0.4898 \leq \Im(x) \leq 0.4962$

Note that the x-axis and the y-axis are invariant under F and the Julia sets are unit cicles of the axes. The number of iterations needed before the orbit falls in a small ball centered at O is used for the color coding.

Example 2. Figures 4 to 6 are for

$$F(x,y) \ = \ (x^2 - 4x^2y + x^3, y^2 + 2.7\sqrt{-1}xy^2).$$

The rectangular regions represented by the pictures are as follows.
Fig. 4. : $-0.9 \leq \Re(x) \leq 1.5, \quad -0.96 \leq \Im(x) \leq 0.96$
Fig. 5. : $0.2 \leq \Re(x) \leq 0.4, \quad 0.62 \leq \Im(x) \leq 0.78$
Fig. 6. : $1.161 \leq \Re(x) \leq 1.187, \quad -0.05 \leq \Im(x) \leq -0.28$

Example 3. Figures 7 to 9 are for

$$F(x,y) \ = \ (x^2 - 4x^2y + x^3, y^2 + 2.7\sqrt{-1}xy^2 - y^3).$$

The regions are as follows.
Fig. 7. : $-0.9 \leq \Re(x) \leq 1.5, \quad -0.96 \leq \Im(x) \leq 0.96$

Fig. 8. : $0.375 \leq \Re(x) \leq 0.385, \quad 0.64 \leq \Im(x) \leq 0.648$
Fig. 9. : $1.22 \leq \Re(x) \leq 1.34, \quad 0.195 \leq \Im(x) \leq 0.285$

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Fig.1

Fig.2

Fig.3

Fig.4

Fig.5

Fig.6

Fig.7

Fig.8

Fig.9