Exploration of complex Hénon dynamics

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Abstract

An interactive software for experimental study of complex Hénon dynamics is presented (in the talk). Julia sets and stable/unstable manifolds of saddle fixed points of the complex Hénon maps are visualized. These pictures gives some intuitive understanding of the dynamics. (Interactive graphics viewing is not recorded in this note.)

0. Introduction

In the early 80's of the last century, computer generated pictures of Julia sets and the Mandelbrot set opened a new way of research in complex dynamical systems. And they played an important role for the progress of complex dynamical systems theory. Computer graphics technology is highly developed in recent years, and now powerful computers are capable of visualizing higher dimensional objects. In order to make use of such computer facilities for a research of higher dimensional dynamical systems, we have to find what to visualize and how to visualize.

In this note, we report our first trials of the visualization of Julia sets and invariant manifolds of the complex Hénon map. Our computations are all numerical and do not have rigorous justifications. Periodic points are computed by the method proposed by Biham and Wenzel. Unstable manifolds and stable manifolds are computed by Poincaré's power series expansion formula. The Julia set, in this note, is to be understood as the smallest invariant closed set containing the saddle periodic points. Note that near parabolic points, the period of periodic points are large and hard to compute. Also, Biham and Wenzel's method fails to find many periodic pints when the parameters or periodic points are away from the real axis.

1. Classical strange attractor of Hénon

As is well known, Hénon's strange attractor lives in $\mathbb{R}^2$. In this note, the Hénon map $(x, y) \mapsto (X, Y)$ is defined by the following formula.

$$\begin{cases} X &= x^2 + c + by \\ Y &= x \end{cases}$$

Here, parameter c corresponds to $-a$ in the classical Hénon's family, and the coordinates x, y are rescaled so that we can compare the behavior of the dynamics with the one-dimensional Mandelbrot family of quadratic functions. For most parameters, there are two fixed points., which will be denoted as P and Q. Fixed point P corresponds to the beta fixed point (with external angle 0) for one dimensional quadratic map. The other fixed point, Q, corresponds to the alpha fixed point. In the following picture, periodic points of periods up to 19 are plotted. You may recognize the self-folded strange attractor is embedded as a subset. The picture is a little rotated in $\mathbb{C}$ to show that the "pruned branches" are emanating into the imaginary space.

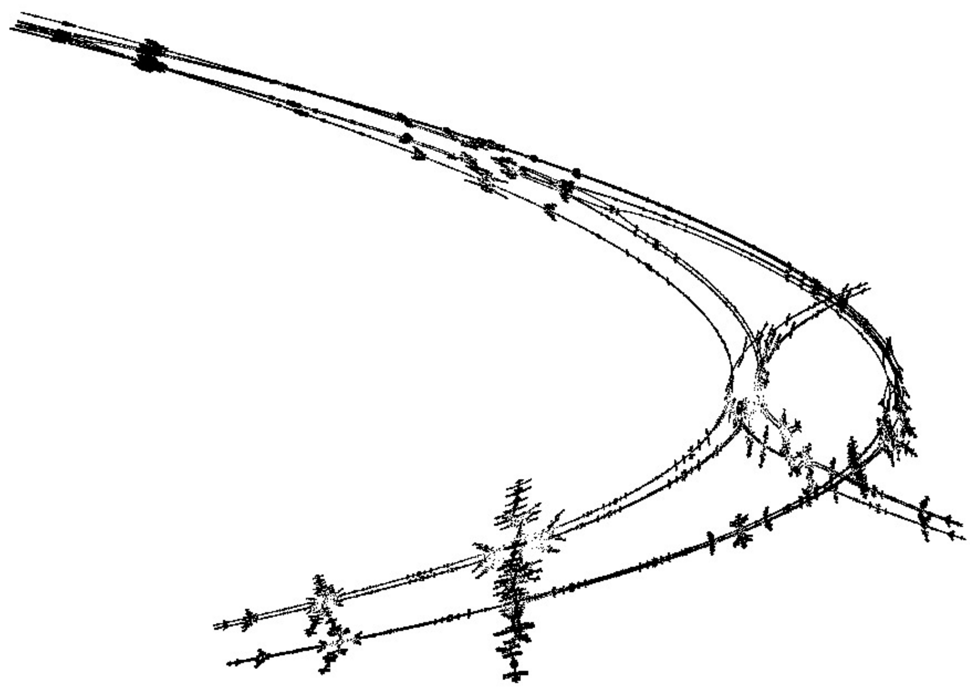


Fig.1

Observe "fish bone" like branches coming out from the turning points of the real strange attractor. There are components disjoint from the main component in the real axis. The existence of disjoint componets is more clearly observed in the following picture.

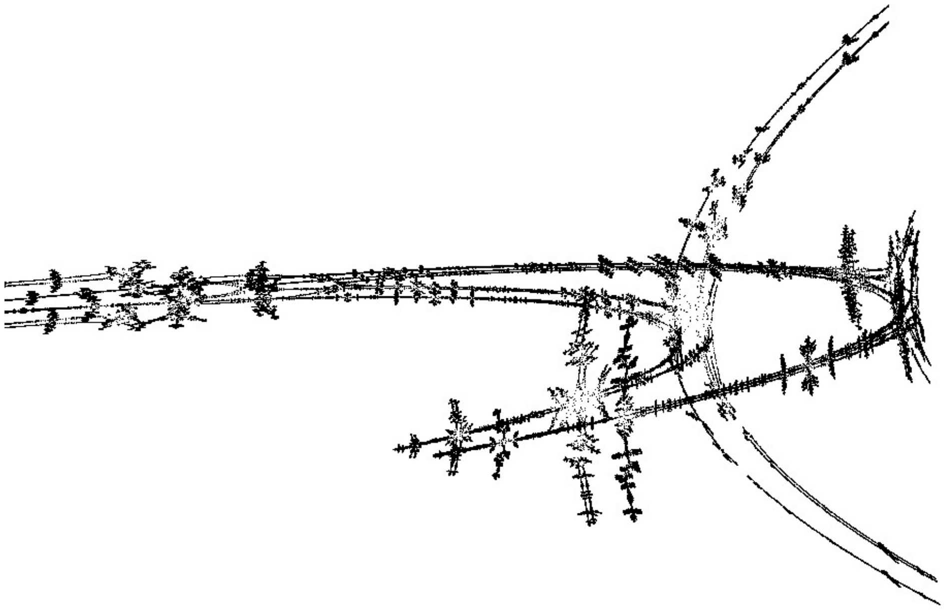


Fig.2

Observe that there is a gap, in the picture above(Fig.2), between the right upper components and the rest of the set. There is a critical point of the Green's function restricted to the unstable manifold of saddle point P. The unstable manifold of P is a complex analytic curve immersed in $\mathbb{C}^2$.

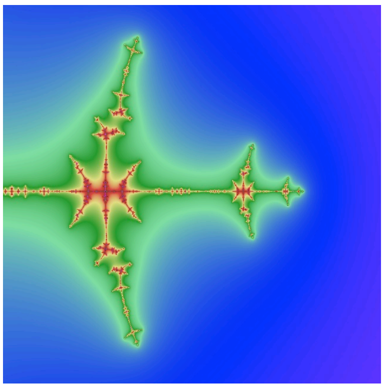


Fig.3

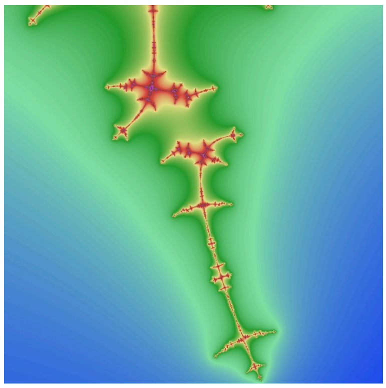


Fig.4

Fig.3 represents a square region of the unstable manifold of P, in the domain of definition of Poincaré's function $\varphi : \mathbb{C} \to \mathbb{C}^2$, colored according to the value of the Green's function. Fig. 4 is an enlargement of the lower part of Fig.3. In this picture, a "canal" is observed.

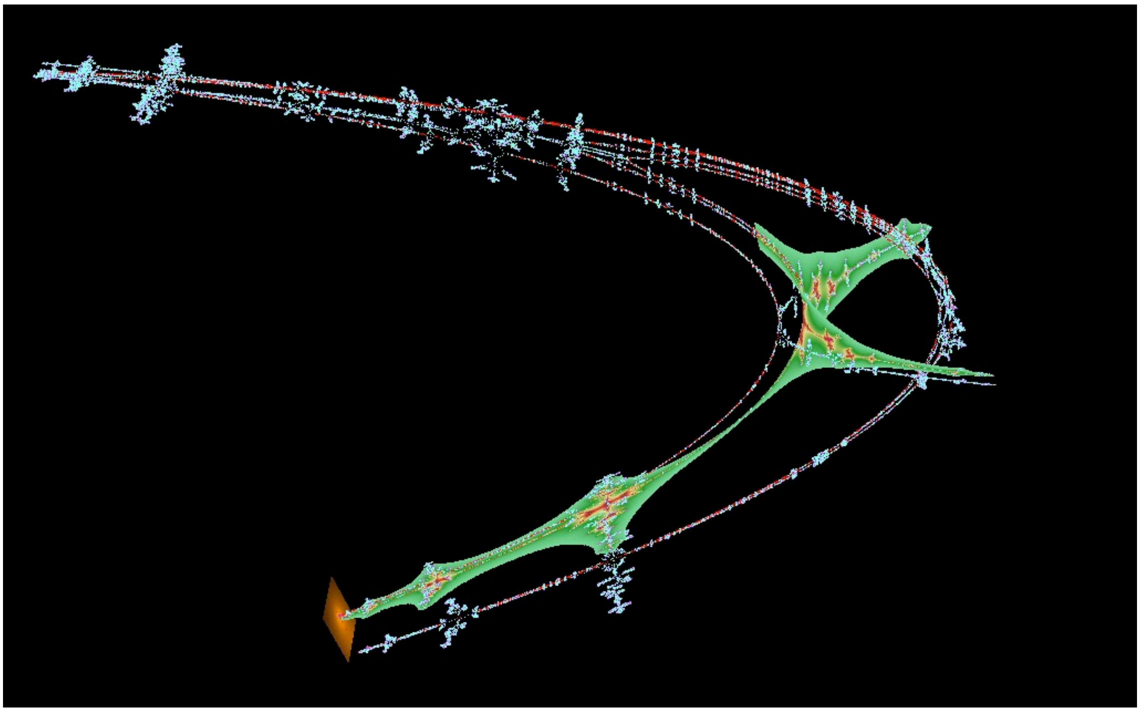


Fig.5

In Fig.5, stable manifold of P, square region represented in Fig.3, trimmed along a certain level of the value of Green's function is embedded in $\mathbb{C}^2$, together with the Julia set and a small square region of the stable manifold are shown. The saddle point P is located at the intersection of the invariant manifolds.

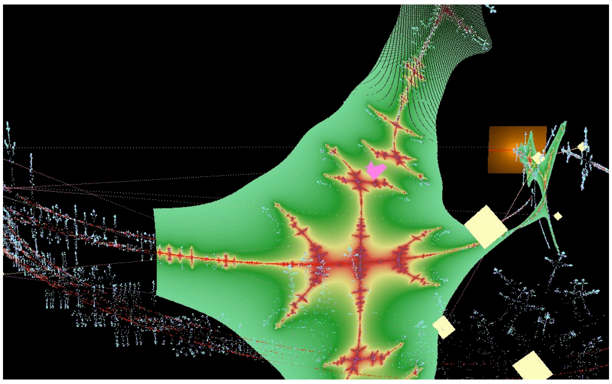


Fig.6

To see the behavior of the orbit of the critical point, take a point in the middle of the "canal" as in Fig.6.

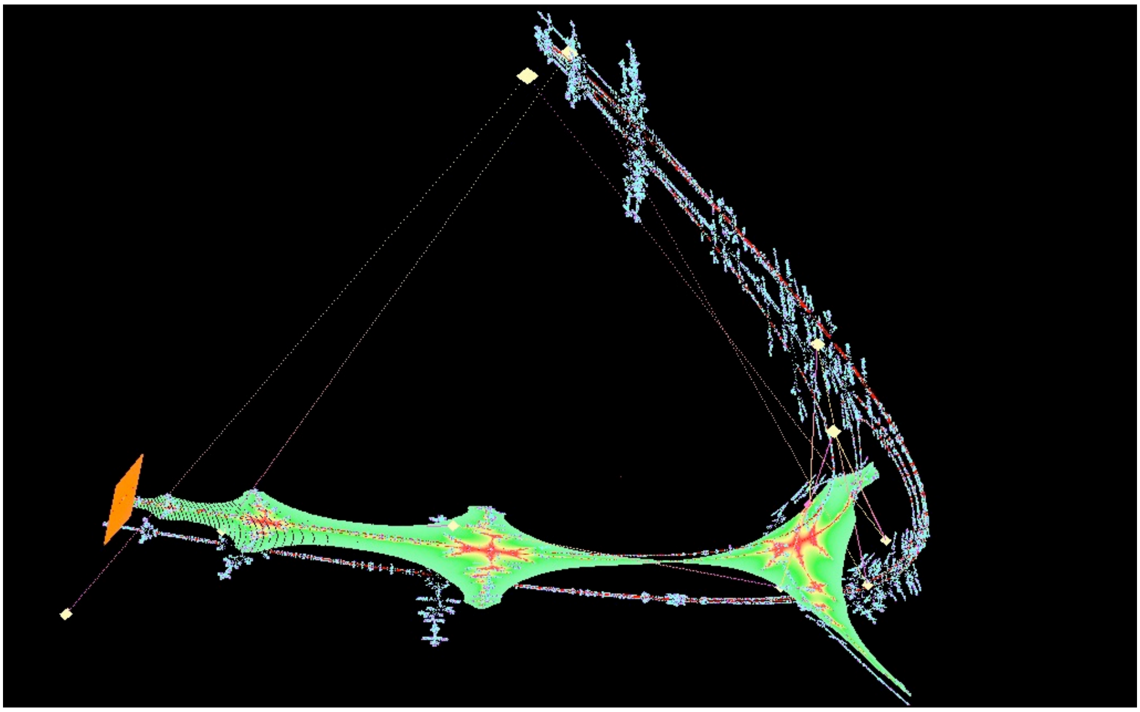


Fig.7

The tenth iterate of the critical point comes near the saddle point P, and escapes to the infinity.

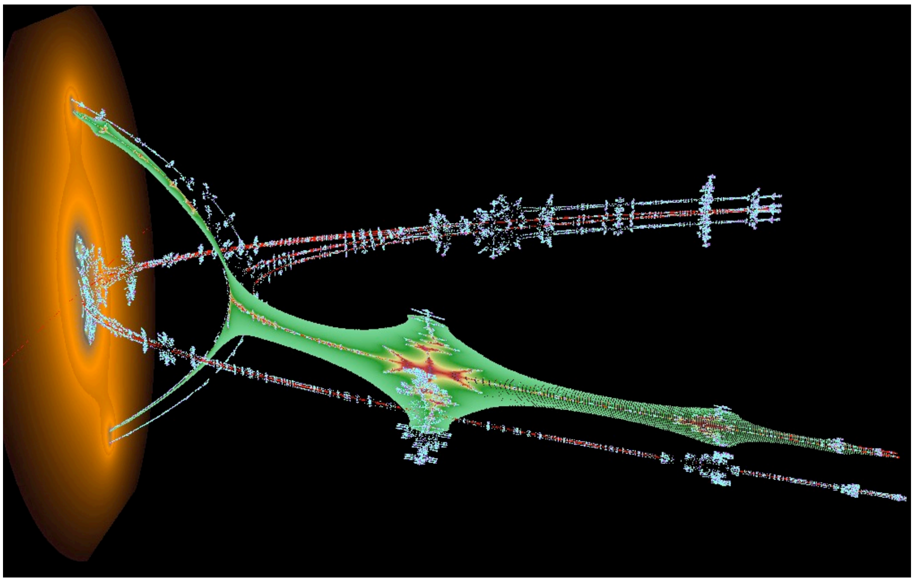


Fig.8

The behavior of critical point suggests that the stable manifold of P plays the role of a "separatorix". Fig. 8 shows that the stable manifold passes through the “end points” of the “pruned branches” near the turning location.

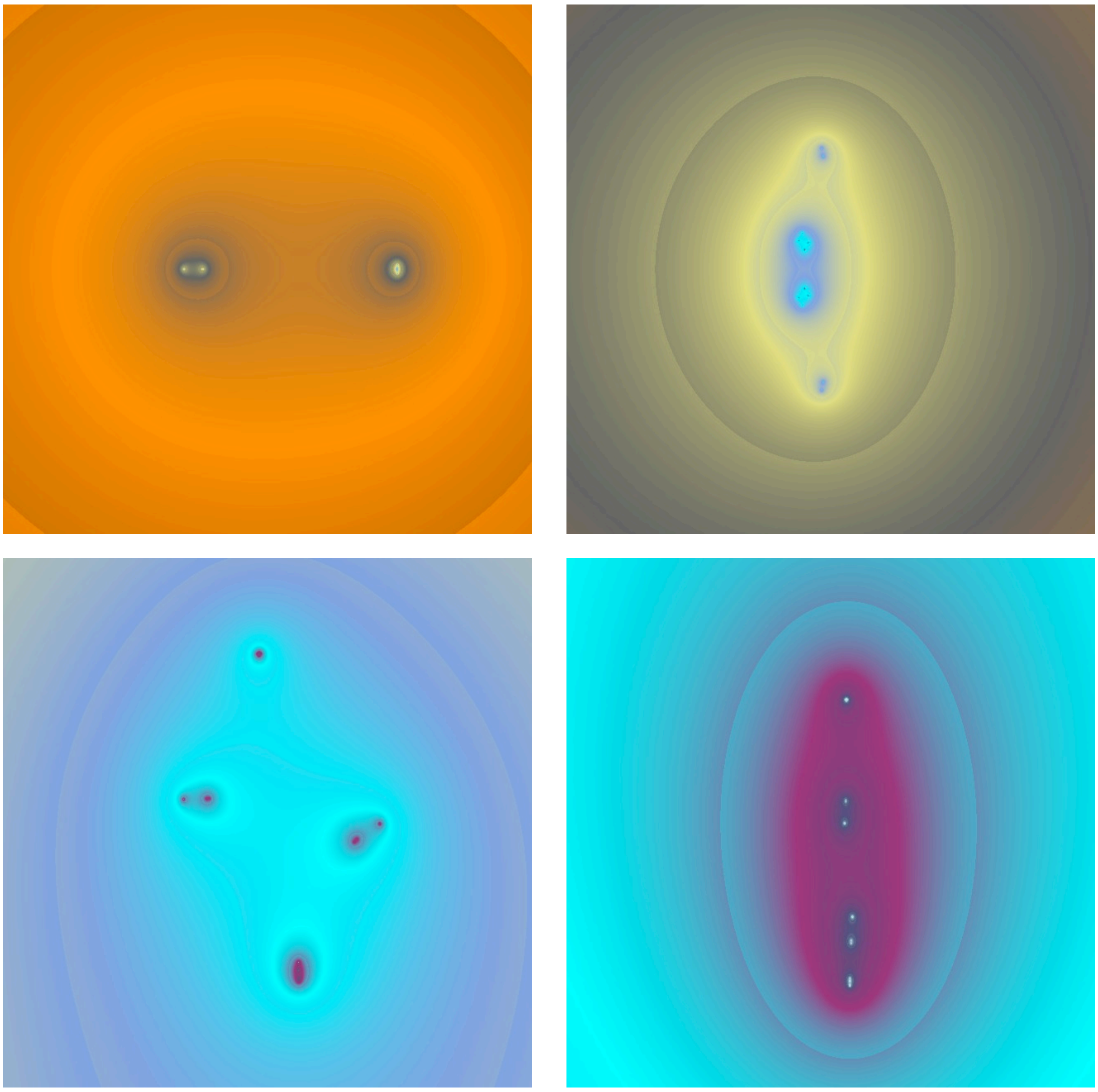


Fig.9

Fig.9 shows successive enlargements of the stable manifold of P. These pictures suggest that the intersection of the Julia set with the stable manifold of P consists is NOT self similar.

2. Homoclinic points and heteroclinic points

In the previous section, pictures of [un]stable manifolds are either some region in the domain of definition of Poincaré’s function, or the immersed image in $\mathbb{C}^2$ (projected to $\mathbb{R}^2$ in some way). In this section, we try to understand how they are immersed and intersect with each other.

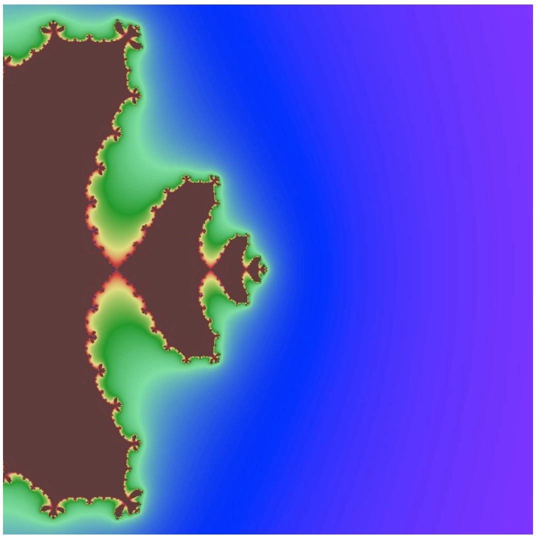


Fig.10

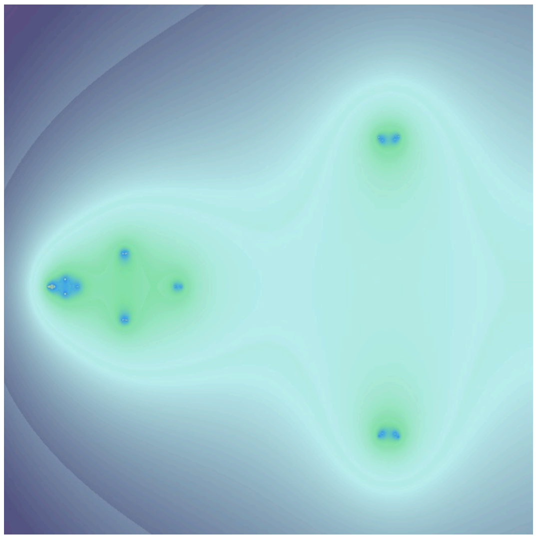


Fig.11

The unstable manifold of P, and the stable manifold of Q are shown in the above.

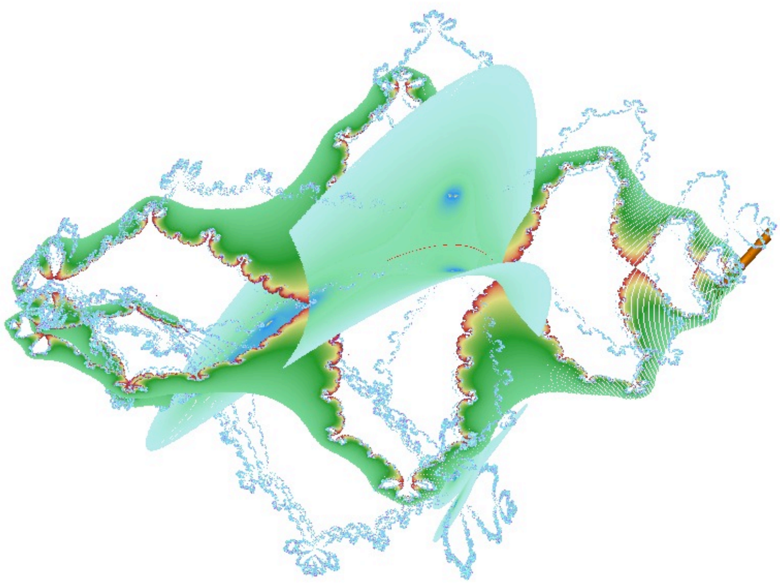


Fig.12

In this case (c=-0.7, b=0.3), in Fig.10, which represents a square region in the domain of definition of the Poincaré's function, the pinched points in the real axis are the points on the stable manifold of the other saddle point Q. Fig.11 shows the stable manifold of Q. The points in the real axis of this Cantor-like set contains the intersection points with the unstable manifold, i.e. heteroclinic points.

In Fig.12, these two curves are viewed. Observe tat the stable manifold of Q intersects at the pinched point of the unstable manifold of P. As these curves are in $\mathbb{C}^2$, they appear to intersect along a real curve, the intersection is (numerically) transversal.

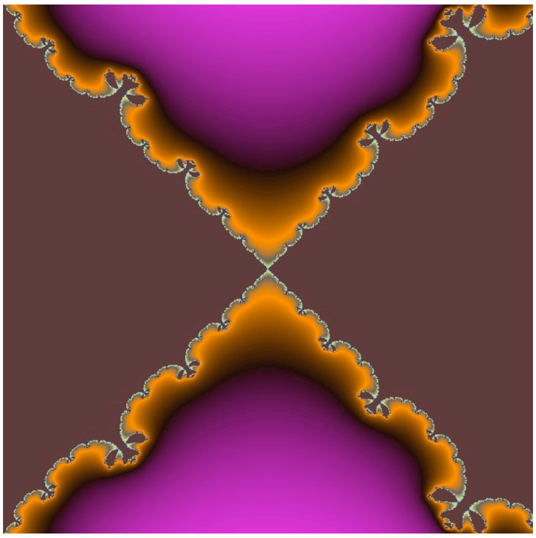


Fig.13

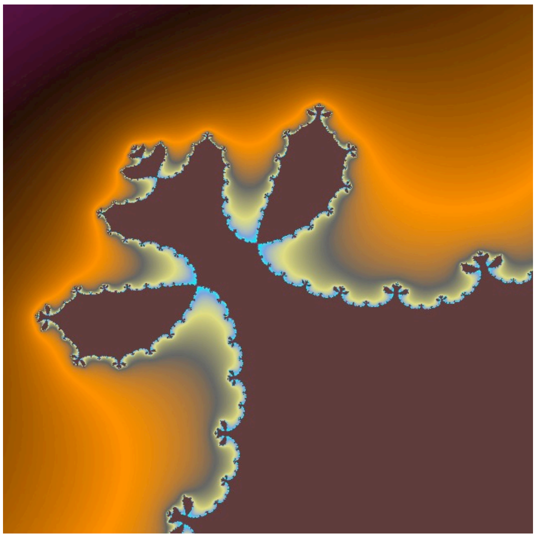


Fig.14

Fig.13 shows the unstable manifold of Q, and Fig.14 shows an enlargement of right upper part. The unstable manifold picture in the domain of definition of the Poincaré's function is similar to itself with respect to the origin by the multiplication by the eigenvalue. However, small portion of it is not necessarily similar to the whole picture. Note that this picture is different from that of the unstable manifold of P, although further enlargement reveals some detail similar to that of P, and vise-versa.

In Fig.15, the unstable manifold and the stable manifold of Q are plotted. they intersect at pinching point of the Julia set in the unstable manifold. The intersection point in Fig.10 is away from the real axis. These intersection points are homoclinic points. Numerically, the intersection is transversal. Therefore, there must be a horseshoe. The invariant set in the horseshoe is hidden in the Julia set and hard to recognize the Cantor set structure.

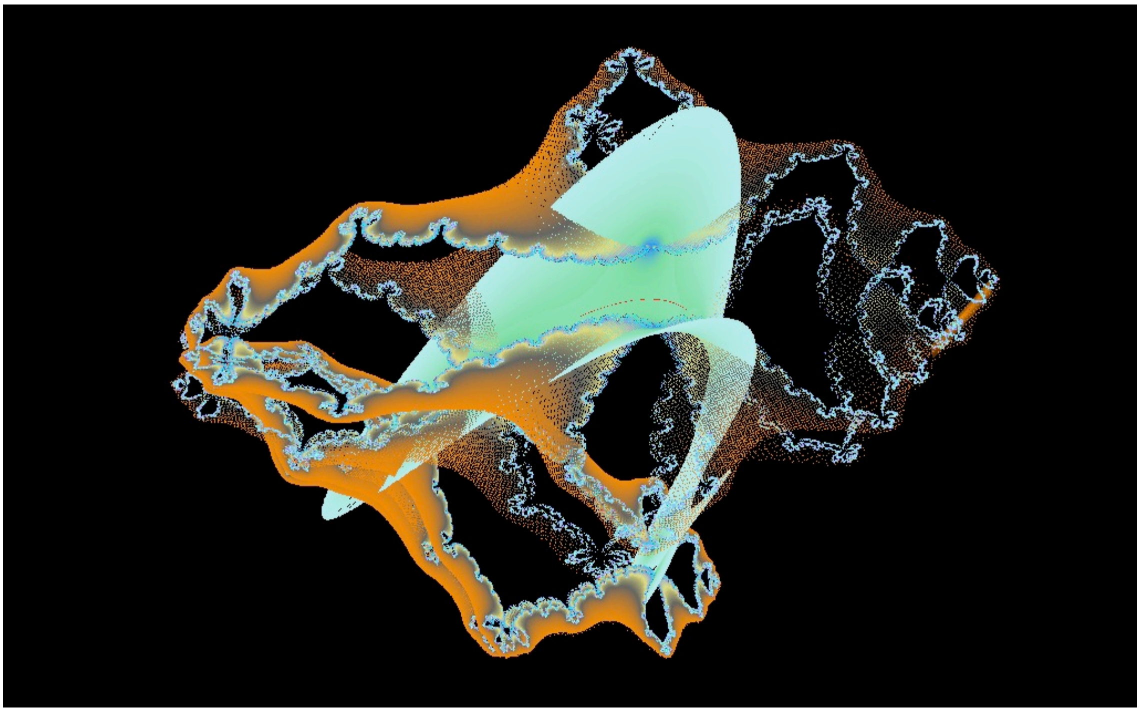


Fig.15

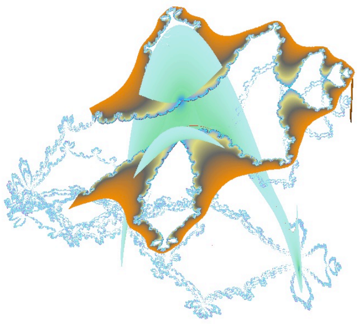


Fig.16

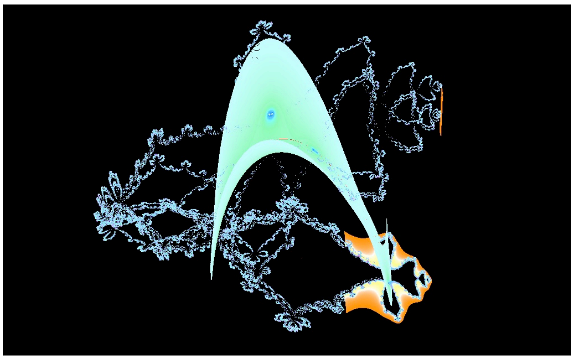


Fig17

Fig.16 and Fig.17 shows pictures with small portion of unstable manifold of Q and the stable manifold of Q.

Although objects living in $\mathbb{C}^2$ are quite hard to understand, we can try to visualize them with the help of computer graphics. The color version of this note will be uploaded on the author's web page:

`http://www.math.h.kyoto-u.ac.jp/~ushiki/index.html`

with interactive graphics software. Followings are some interesting pictures.

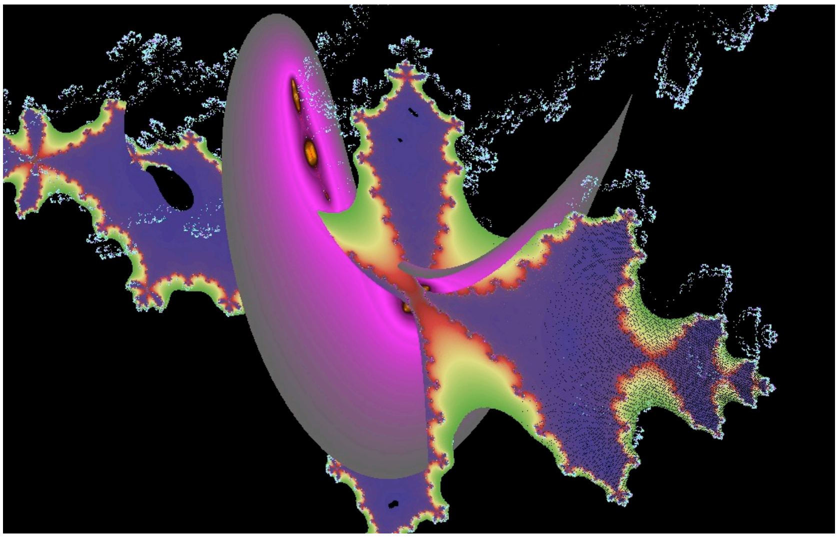


Fig.18

In this picture, the stable manifold of Q (which is located at a pinched location of the JUlia set) intersects with the unstable manifold of P in two points. The intersection is close to a heheroclinic tangency.

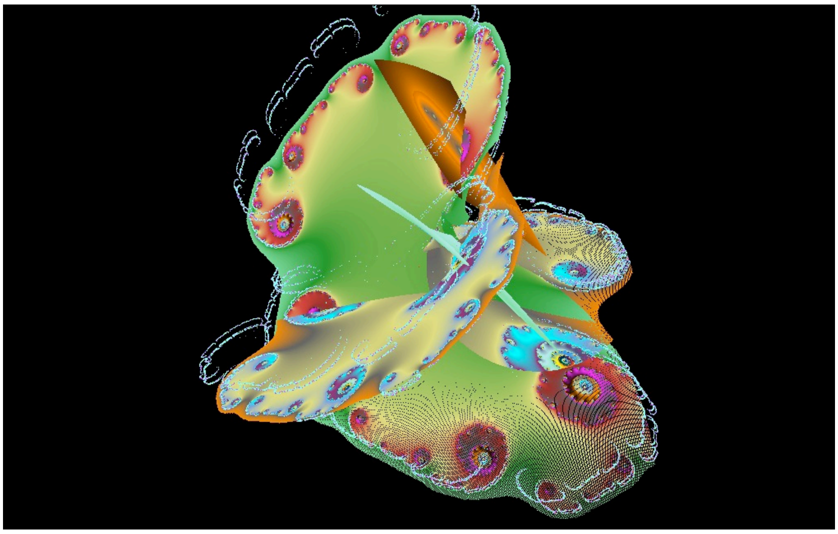


Fig.19

This picture shows a near-parabolic situation. Spiraling fixed points and solenoidal cauliflower are observed.

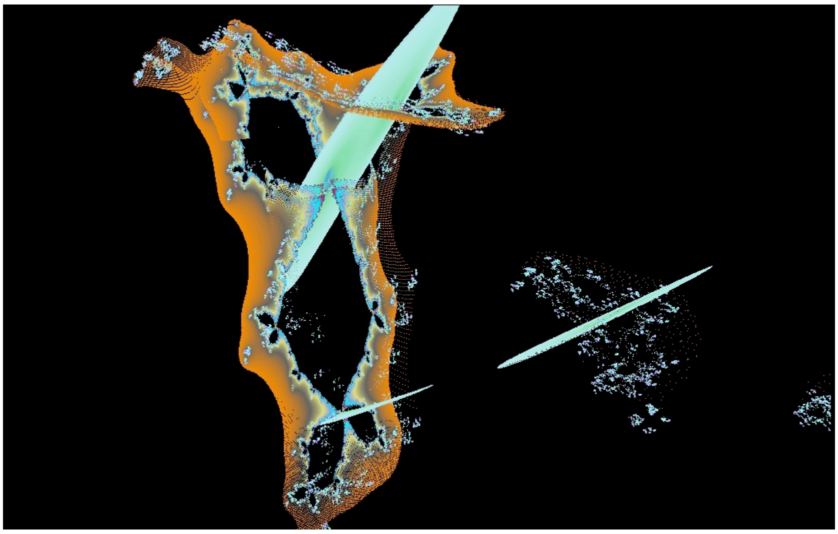


Fig.20

For some parameter, there is a case with rabbit-like Julia set. Some part of the stable and unstable manifolds of Q intersect in Q(the alpha fixed point of the rabbit).

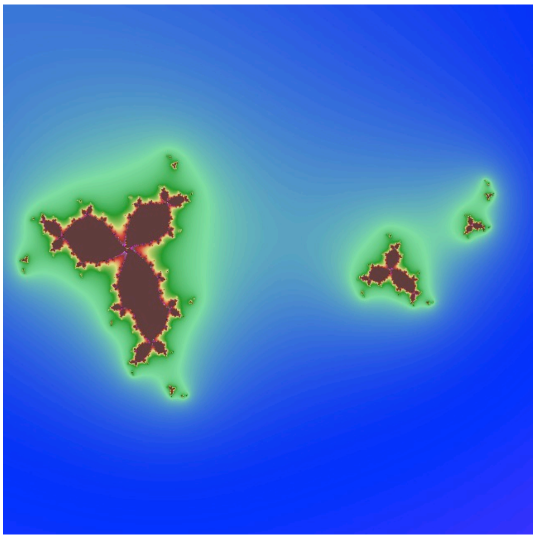


Fig.21

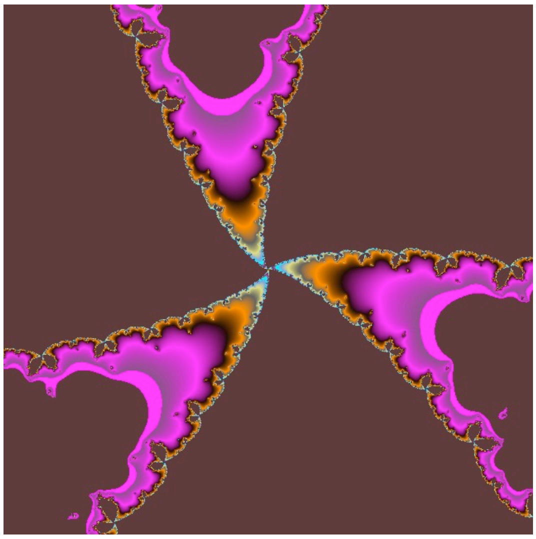
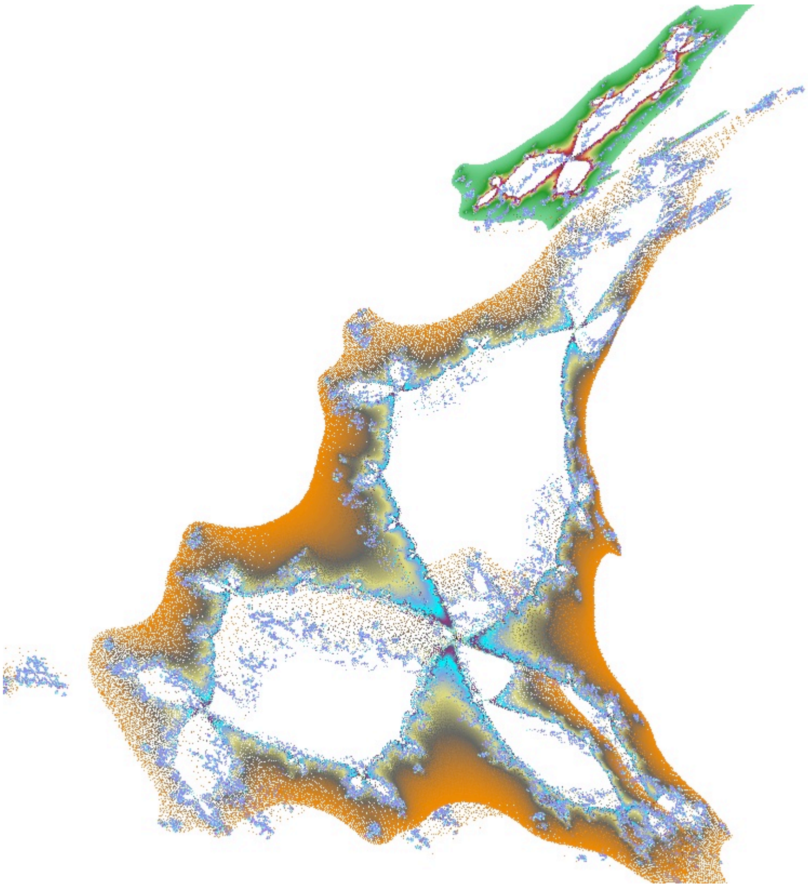


Fig.22

Fig.21 shows a part of the unstable manifold of P, and Fig.22 shows a part of the unstable manifold of Q. They are embedded in $\mathbb{C}^2$ as in the following picture.



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