Twist dynamics of rational surface

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Family of birational maps \( f_a(x, y) = (y, \frac{y+a}{x}) \) extends to a family of surface automorphisms \( F_a : X_a \to X_a \).

Rational surface \( X_a \) is constructed by blowing-up the complex projective space \( \mathbb{P}^2 \) in 9 points.

If \( a \neq 0, 1 \), \( F_a \) is of infinite order. The degree of \( F_a^n \) grows quadratically. The entropy of these automorphisms are zero.

It is known that these automorphisms have invariant foliations (with singularities) with elliptic leaves.

In this note, we prove that there exists an \( F_a \) which has a quasi-periodic orbit dense in an elliptic curve.
Contents

1. Birational automorphism
2. Surface automorphism
3. Invariant function
4. Invariant curves
5. Elliptic curves
6. Translation in elliptic curve
7. Periods of elliptic curve
8. Dynamics near fixed points
9. Twist dynamics
10. Weierstraß $\wp$-function
1. Birational automorphism
Birational automorphism

Define a family of birational automorphisms \( f_a : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \), for parameter \( a \in \mathbb{C} \), by

\[
f_a(x, y)_{\mathbb{C}^2} = (y, \frac{y + a}{x})_{\mathbb{C}^2}.
\]

In homogeneous coordinates, with \((x, y)_{\mathbb{C}^2} \leftrightarrow [1 : x : y]\),

\[
\tilde{f}_a[t : \tilde{x} : \tilde{y}] = [t\tilde{x} : \tilde{x}\tilde{y} : t(\tilde{y} + at)].
\]

By lifting \( \tilde{f}_a \) to a homogenous polynomial map \( \tilde{f}_a : \mathbb{C}^3 \rightarrow \mathbb{C}^3 \),

\[
D\tilde{f}_a = \begin{pmatrix}
\tilde{x} & t & 0 \\
0 & \tilde{y} & \tilde{x} \\
\tilde{y} + 2at & 0 & t
\end{pmatrix},
\]

we have \( \det D\tilde{f}_a = 2t\tilde{x}(\tilde{y} + at) \).
Critical locus and points of indeterminacy

The Jacobian $\det D\tilde{f}_a = 2t\tilde{x}(\tilde{y} + at)$ vanishes in three exceptional curves:

$$\Sigma_0 = \{t = 0\}_{\mathbb{P}^2}, \quad \Sigma_y = \{\tilde{x} = 0\}_{\mathbb{P}^2}, \quad \Sigma_q = \{\tilde{y} + at = 0\}_{\mathbb{P}^2}.$$ 

$$\tilde{f}_a[t : \tilde{x} : \tilde{y}] = [t\tilde{x} : \tilde{x}\tilde{y} : t(\tilde{y} + at)]$$ has three points of indeterminacy:

$$e_x = [0 : 1 : 0], \quad e_y = [0 : 0 : 1], \quad p_* = [1 : 0 : -a].$$
2. Surface automorphism
**Theorem** [Bedford, Kim, 2006]

If $a \neq 0, 1$, there is a complex manifold $X_a$ obtained by blowing up $\mathbb{P}^2$ at 9 points, and a biholomorphic map $F_a : X_a \to X_a$ induced by $f_a$. The degree of $F_a^\circ n$ grows quadratically in $n$.

It is known that these automorphisms have invariant foliations by elliptic curves [Gizatullin 1980].

The entropy of these maps are all zero.
First blowup

Exceptional curves are mapped as:

\[ \Sigma_y \Rightarrow e_y \Rightarrow \Sigma_0 \Rightarrow e_x \Rightarrow \Sigma_x, \]

where \( \Sigma_x = \{ \tilde{y} = 0 \} \mathbb{P}^2 \), and \( \tilde{f}_a(\Sigma_x) = \Sigma_y \).

Blowup \( \mathbb{P}^2 \) in \( e_x \) and \( e_y \) to obtain a surface \( X_1 \) with projection \( \text{pr}_1 : X_1 \to \mathbb{P}^2 \) and exceptional fibers

\[ E_x = \text{pr}_1^{-1}(e_x) \quad \text{and} \quad E_y = \text{pr}_1^{-1}(e_y). \]

We have a cycle of curves of period 5.

\[ \Sigma_y \Rightarrow E_y \Rightarrow \Sigma_0 \Rightarrow E_x \Rightarrow \Sigma_x \Rightarrow \Sigma_y. \]
Coordinates in exceptional fibers

We need coordinates near the exceptional fibers. Let \((u, v)_{E_y}\) denote a local coordinate system of \(X_1\) near \(E_y\) with \(E_y = \{u = 0\}_{E_y}\) and

\[
(u, v)_{E_y} \leftrightarrow [u : uv : 1].
\]

Near \(\Sigma_y\), \(f_a\) extends to a holomorphic map into \(X_1\), except at \(p^*\).

\[
f_a(x, y)_{\mathbb{C}^2} = \left( \frac{x}{y + a}, y \right)_{E_y}.
\]

So,

\[
F_1(0, y)_{\mathbb{C}^2} = (0, y)_{E_y}.
\]
Near $E_y$,
\[
\tilde{f}_a(u, v)_{E_y} = \tilde{f}_a[u : uv : 1] = [uv : v : 1 + au].
\]

So,
\[
F_1(0, v)_{E_y} = [0 : v : 1] \in \Sigma_0.
\]

Near $\Sigma_0$,
\[
\tilde{f}_a[t : x : 1] = [t : 1 : \frac{t(1 + at)}{x}].
\]

With coordinates $(u, v)_{E_x} \leftrightarrow [u : 1 : uv]$, we have
\[
F_1[t : x : 1] = (t, \frac{1 + at}{x})_{E_x}, \quad \text{and} \quad F_1[0 : x : 1] = (0, \frac{1}{x})_{E_x}.
\]
Finally, \((u, v)_{E_x}\) is mapped as

\[ \tilde{f}_a[1 : u : uv] = [1 : v : u(v + a)]. \]

So,

\[ F_1(0, v)_{E_x} = [1 : v : 0] \leftrightarrow (v, 0)_{\mathbb{C}^2} \in \Sigma_x. \]
Second blowup

\[ F_1 : X_1 \to X_1 \] has an exceptional curve (strict transform of)

\[ \Sigma_q = \{ \tilde{y} + at = 0 \}_{\mathbb{P}^2} \]

and a point of indeterminacy

\[ p_* = [1 : 0 : -a] \leftrightarrow (0, -a)_{\mathbb{C}^2}. \]

Exceptional curve \( \Sigma_q \) is mapped to

\[ q = [1 : -a : 0] \leftrightarrow (-a, 0)_{\mathbb{C}^2}. \]

Set \( q_k = F_1^{\circ k}(q) \) for \( k = 0, 1, 2, \cdots, 6 \). Then \( q_6 = p_* \).
\[ q_0 = q = (-a, 0)_{\mathbb{C}^2} \in \Sigma_q, \]
\[ q_1 = (0, -1)_{\mathbb{C}^2} \in \Sigma_y, \]
\[ q_2 = (0, -1)_{E_y} \in E_y, \]
\[ q_3 = [0 : -1 : 1] \in \Sigma_0, \]
\[ q_4 = (0, -1)_{E_x} \in E_x, \]
\[ q_5 = (-1, 0)_{\mathbb{C}^2} \in \Sigma_x, \]
\[ q_6 = p_* = (0, -a)_{\mathbb{C}^2} \in \Sigma_y. \]
We blow up $X_1$ in 7 points $q_0, q_1, \ldots, q_6$, to obtain a surface $X_a$ with projection
\[ \text{pr}_2 : X_a \to X_1, \]
and denote the exceptional fibers as
\[ \text{pr}_2^{-1}(q_k) = Q_k, \quad k = 0, 1, \ldots, 6. \]

$F_1 : X_1 \to X_1$ induces an automorphism $F_a : X_a \to X_a$. 
Around exceptional fibers

Let \((\xi, \eta)_{Q_0}\) be a local coordinate of \(X_a\) near exceptional fiber \(Q_0\), given by \((\xi, \eta)_{Q_0} \leftrightarrow [1 : -a + \xi : \xi \eta]\).

Near \(\Sigma_q\),

\[F_a[1 : x : y] = [x : xy : y + a] = [1 : -a + (y + a) : \frac{y + a}{x}] \leftrightarrow (y + a, \frac{1}{x})_{Q_0}.\]

Near the exceptional fiber \(Q_6 = \text{pr}^{-1}(p_*)\), take local coordinates

\[(\xi, \eta)_{Q_6} \leftrightarrow [1 : \xi \eta : -a + \xi].\]

Recall the coordinates near \(E_y\), given by \((u, v)_{E_y} \leftrightarrow [u : uv : 1].\)

\[F_a(\xi, \eta)_{Q_6} = (\eta, -a + \xi)_{E_y}.\]
3. Invariant function
Invariant function

It is known that rational function

\[ r_a(x, y) = \left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right)(x + y + a) \]

is invariant under \( f_a \) [Lyness 1942, 1945, 1961]. We understand this function as

\[ r_a(x, y) = \frac{(x + 1)(y + 1)(x + y + a)}{xy}. \]

**Proposition.** If \( a \neq 0, 1 \), then \( r_a \) lifts to a holomorphic map

\[ \tilde{r}_a : X_a \to \hat{\mathbb{C}}. \]
Proof of proposition

Invariant function $r_a$ has indeterminate points in $\mathbb{C}^2$.

$$q_0 = (-a, 0)_{\mathbb{C}^2}, \quad q_1 = (0, -1)_{\mathbb{C}^2},$$

$$q_5 = (-1, 0)_{\mathbb{C}^2}, \quad q_6 = p_* = (0, -a)_{\mathbb{C}^2}.$$ These points are blown up and $\tilde{r}_a$ is holomorphic near the exceptional fibers.

For example, in local coordinates $(\xi, \eta)_{Q_0} \leftrightarrow (-a + \xi, \xi \eta)_{\mathbb{C}^2},$ near $Q_0$

$$\tilde{r}_a(\xi, \eta)_{Q_0} = \frac{(-a + \xi + 1)(\xi \eta + 1)(\eta + 1)}{(-a + \xi)\eta}.$$ Especially,

$$\tilde{r}_a(0, \eta)_{Q_0} = \frac{(1 - a)(\eta + 1)}{-a \eta}.$$
In the line at infinity $\Sigma_0 \setminus \{[0 : 1 : 0], [0 : 0 : 1], [0 : -1 : 1]\}$, 
\[ \tilde{r}_a[0 : x : y] = \infty. \]

Near exceptional fiber $E_y$ with coordinates 
$(u, v)_{E_y} \leftrightarrow [u : uv : 1], 
\tilde{r}_a (u, v)_{E_y} = \frac{(v + 1)(u + 1)(uv + 1 + au)}{uv}. 
So, \tilde{r}_a (0, v)_{E_y} = \infty$ unless $v = -1.$
Near $Q_2$, take local coordinates $(\xi, \eta)_{Q_2}$ by

$$(\xi, \eta)_{Q_2} \leftrightarrow (\xi, -1 + \xi \eta)_{E_y} \leftrightarrow [\xi : \xi(-1 + \xi \eta) : 1].$$

Then

$$\tilde{r}_a(\xi, \eta)_{Q_2} = \frac{\eta(\xi + 1)(-\xi + \xi^2 \eta + 1 + a\xi)}{-1 + \xi \eta},$$

and

$$\tilde{r}_a(0, \eta)_{Q_2} = -\eta.$$

Near other exceptional fibers, $\tilde{r}_a$ is defined similarly and gives a holomorphic map

$$\tilde{r}_a : X_a \to \hat{\mathbb{C}}.$$
**Proposition.** \( \tilde{r}_a \) is invariant under \( F_a \).

\[
\tilde{r}_a \circ F_a = \tilde{r}_a.
\]

**Proposition.** \( \tilde{r}_a \) is non-constant in each of \( \Sigma_0, E_x, E_y, Q_0, Q_1, Q_2, Q_3, Q_4, Q_5, Q_6 \).

**Proposition.** Critical points of \( \tilde{r}_a \) (or \( \frac{1}{\tilde{r}_a} \)) are fixed point or periodic point of \( F_a \).

**Proof.** Number of critical points is finite.
Critical points

In $\mathbb{C}^2$, 

$$\frac{\partial r_a}{\partial x} = \frac{(y + 1)(x^2 - y - a)}{x^2 y}, \quad \frac{\partial r_a}{\partial y} = \frac{(x + 1)(y^2 - x - a)}{xy^2},$$

and 

$$\frac{\partial}{\partial x} \left( \frac{1}{r_a} \right) = \frac{-y(x^2 - y - a)}{(y + 1)(x + 1)^2(x + y + a)^2},$$

$$\frac{\partial}{\partial y} \left( \frac{1}{r_a} \right) = \frac{-x(y^2 - x - a)}{(x + 1)(y + 1)^2(x + y + a)^2}.$$
Critical points are

\((-1, -1)_{\mathbb{C}^2}, (-1, 1 - a)_{\mathbb{C}^2}, (1 - a, -1)_{\mathbb{C}^2}, \) (period 3),

\(\left(\frac{1}{2} \pm \sqrt{a + \frac{1}{4}}, \frac{1}{2} \mp \sqrt{a + \frac{1}{4}}\right)_{\mathbb{C}^2},\) (fixed points),

\(\left(-\frac{1}{2} \pm \sqrt{a - \frac{3}{4}}, -\frac{1}{2} \mp \sqrt{a - \frac{3}{4}}\right)_{\mathbb{C}^2},\) (period 2),

\((0, 0)_{\mathbb{C}^2}, (0, 0)_{E_y}, (0, \infty)_{E_y}, (0, \infty)_{E_x}, (0, 0)_{E_x},\) (period 5).
Critical values

\[ \sigma_3 = 0, \quad \text{(period 3 cycle)}, \]
\[ \sigma_5 = \infty, \quad \text{(period 5 cycle)}, \]
\[ \sigma_2 = a - 1, \quad \text{(period 2 cycle)}, \]
\[ \sigma_{1 \pm} = a + 5 - \frac{1}{2a} \pm (4 + \frac{1}{a}) \sqrt{a + \frac{1}{4}}, \quad \text{(fixed points)}. \]

The critical values of fixed point can be expressed as

\[ \sigma_1 = \frac{(x_0 + 1)^3}{x_0}, \quad x_0^2 = x_0 + a. \]
4. Invariant curves
Invariant curves

For \( \sigma \in \hat{\mathbb{C}} \), let \( C_\sigma \) denote invariant cubic curve in \( X_a \) defined by

\[
C_\sigma = \tilde{r}_a^{-1}(\sigma).
\]

\( C_{\sigma_3} \) consists of three lines:

\[
\{ x + y + a = 0 \}_{\mathbb{C}^2}, \quad \{ y + 1 = 0 \}_{\mathbb{C}^2}, \quad \{ x + 1 = 0 \}_{\mathbb{C}^2},
\]

which are mapped cyclically.

\[
(\zeta, -1)_{\mathbb{C}^2} \mapsto (-1, \frac{a - 1}{\zeta})_{\mathbb{C}^2} \mapsto \left( \frac{a - 1}{\zeta}, \frac{1 - a}{\zeta} - a \right)_{\mathbb{C}^2} \mapsto \left( \frac{1 - a}{\zeta} - a, -1 \right)_{\mathbb{C}^2}.
\]

Eigenvalues at periodic point in the curve (and of the 3-cycle) are \( a - 1 \) and \( \frac{1}{a - 1} \).
$C_{\sigma_2}$ consists of a line and a quadric which are mapped to each other by $F_a$.

\begin{align*}
\{ x + y + 1 = 0 \}_{\mathbb{C}^2}, \quad \{ xy + x + y + a = 0 \}_{\mathbb{C}^2}.
\end{align*}

\begin{align*}
(\zeta, -\zeta - 1)_{\mathbb{C}^2} \mapsto (-1 - \zeta, -1 + \frac{a - 1}{\zeta})_{\mathbb{C}^2} \mapsto (-1 + \frac{a - 1}{\zeta}, \frac{1 - a}{\zeta})_{\mathbb{C}^2}.
\end{align*}

Eigenvalues of the 2-cycle are \( \frac{1}{1-a}(a - \frac{1}{2} \pm \sqrt{a - \frac{3}{4}}) \).
$C_{\sigma_5}$ consists of five ”lines”

$\Sigma_x, \Sigma_y, E_y, \Sigma_0, E_x$.

which are mapped cyclically by $F_a$.

\[
(\zeta, 0)_{\mathbb{C}^2} \mapsto (0, \frac{a}{\zeta})_{\mathbb{C}^2} \mapsto (0, \frac{a}{\zeta})_{E_y} \mapsto [0 : \frac{a}{\zeta} : 1] \mapsto \\
\mapsto (0, \frac{\zeta}{a})_{E_x} \mapsto (\frac{\zeta}{a}, 0)_{\mathbb{C}^2} \mapsto (0, \frac{a^2}{\zeta})_{\mathbb{C}^2} \mapsto \cdots .
\]

Eigenvalues of 5-cycle are $a^{-1}$ and $a$. 
$C_{\sigma_1}$ is a Riemann sphere with a node.

For fixed point $(x_0, x_0)_{\mathbb{C}^2}, \ (x_0^2 = x_0 + a)$, set $x = x_0 + u, \ y = y_0 + v$.

\[
\begin{align*}
\{ uv(u + v) + (x_0 + 1)(u^2 - \frac{1}{x_0}uv + v^2) &= 0 \}
\end{align*}
\]

Eigenvalues of fixed point $(x_0, x_0)_{\mathbb{C}^2}$ are

\[
\frac{1}{2x_0}(1 \pm \sqrt{1 - 4x_0}).
\]
5. Elliptic curves
Elliptic curves

If \( \sigma \in \hat{\mathbb{C}} \setminus \{0, \infty, a - 1, \sigma_{1+}, \sigma_{1-}\} \), where \( \sigma_{1\pm} = \frac{(x_{\pm} + 1)^3}{x_{\pm}} \) with \( x_{\pm} = \frac{1}{2} \pm \sqrt{a + \frac{1}{4}} \), then \( C_\sigma \) is an elliptic curve defined by

\[
P_\sigma(x, y) = (x + 1)(y + 1)(x + y + a) - \sigma xy = 0.
\]

**Remark**

\( P_\sigma(x, y)dx \wedge dy \) is an automorphic form of weight \(-1\), and \( \frac{dx \wedge dy}{P_\sigma(x,y)} \) is invariant under \( f_a^* \).

\[
P_\sigma(f_a(x, y)) = P_\sigma(x, y) \det Df_a(x, y).
\]

\[
f_a^* \left( \frac{dx \wedge dy}{P_\sigma(x, y)} \right) = \frac{dx \wedge dy}{P_\sigma(x, y)}.
\]

The Jacobian of periodic point is always equal to 1.
Let
\[ s = \frac{x + y}{2}, \quad t = \frac{x - y}{2}. \]

Then \( x + y = 2s, \quad xy = s^2 - t^2. \) From equation \( P_\sigma(x, y) = 0, \) we get

\[ (*) \quad (s - \frac{\sigma - a}{2})t^2 = s^3 + (2 - \frac{\sigma - a}{2})s^2 + (1 + a)s + \frac{a}{2}. \]

Note that when \( s = \frac{\sigma - a}{2}, \) the right hand side is equal to \( \frac{1}{2}\sigma(\sigma - (a - 1)), \) which does not vanish since we assumed \( \sigma \neq 0, a - 1. \)

Equation \( (*) \) defines a Riemann surface over \( s \)-plane.
Near $Q_3$

Remark.
For $s = \frac{\sigma - a}{2}$, the elliptic curve $C_{\sigma}$ passes through a point in $Q_3$.

$$Q_3 = \text{pr}^{-1}(q_3), \quad q_3 = [0 : -1 : 1] \in \Sigma_0.$$  

Near $Q_3$, take a local coordinate $(\xi, \eta)_{Q_3}$ given by

$$(\xi, \eta)_{Q_3} \leftrightarrow [\xi : -1 : 1 + \xi \eta].$$

Note that $s = \frac{x+y}{2} = \frac{\eta}{2}$, if $\xi \neq 0$.

$$\tilde{r}_a(0, \eta)_{Q_3} = \eta + a = \sigma$$

which gives $\eta = \sigma - a$. 
From
\[ \frac{\tilde{P}_\sigma(\xi, \eta) Q_3}{\xi} = (-1 + \xi)(1 + \xi \eta + \xi)(\eta + a) + \sigma(1 + \xi \eta) = 0, \]
we get a quadratic equation in \( \xi \):
\[ (\eta + a)(\eta + 1)\xi^2 + \eta(\sigma - a - \eta)\xi + \sigma - a - \eta = 0, \]
whose discriminant is
\[ D = (\sigma - a - \eta)(\eta^2(\sigma - a - \eta) - 4(\eta + a)(\eta + 1)). \]
If \( \eta = \sigma - a \) then \( D = 0 \). If \( \eta \) is near \( \sigma - a \) and \( \eta \neq \sigma - a \), then \( D \neq 0 \) (because \( \sigma \neq a - 1 \)).
This shows \( s = \frac{\sigma - a}{2} \) is a branch point of the Riemann surface.
Near $Q_2$ with local coordinates

$$(\xi, \eta)_{Q_2} \leftrightarrow (\xi, -1 + \xi \eta)_{E_y} \leftrightarrow [\xi : \xi(-1 + \xi \eta) : 1],$$

we get

$$\tilde{P}_{\sigma}(\xi, \eta)_{Q_2} = \frac{\sigma + \eta + (a - \sigma)\xi \eta + (a - 1)\xi^2 \eta + \xi^2 \eta^2 + \xi^3 \eta^2}{\xi^2}.$$

By the implicit function theorem, $C_\sigma$ is regular at $(0, -\sigma)_{Q_2}$. Similarly, near $Q_4$, $C_\sigma$ passes through $(0, -\sigma)_{Q_4}$ with local coordinates

$$(\xi, \eta)_{Q_4} \leftrightarrow (\xi, -1 + \xi \eta)_{E_x} \leftrightarrow [\xi : 1 : \xi(-1 + \xi \eta)].$$
Branch points

The Riemann surface defined by equation (*) has 4 branch points.

\[ s = \frac{\sigma - a}{2}, \text{ solutions of } \quad s^3 + \left(2 - \frac{\sigma - a}{2}\right)s^2 + (1 + a)s + \frac{a}{2} = 0. \]

(In our case, these 4 points are distinct. By Riemann-Hurwiz formula, \( C_\sigma \) is a torus.)

**Theorem** (Abel, Jacobi) There exists a \( \tau \in \mathbb{H} \), and a holomorphic isomorphism

\[ AJ : \mathbb{C}/\Lambda_\tau \to C_\sigma, \]

where \( \Lambda_\tau = \{ n + m\tau \mid n, m \in \mathbb{Z} \} \).
6. Translation in elliptic curve
Translation in elliptic curves

For $\tau \in \mathbb{H}$, let $T_\tau$ denote the torus $\mathbb{C}/\Lambda_\tau$.

For $\mu \in \mathbb{C}$, translation by $\mu$ in $T_\tau$ is defined as

$$T_{\tau,\mu} : T_\tau \rightarrow T_\tau, \quad z \mapsto z + \mu \pmod{\Lambda_\tau}.$$

Suppose $\tau = \tau(\sigma)$ and $\mu = \mu(\sigma)$ depend holomorphically on $\sigma$. Let $\tau = u + iv$, $\mu = \xi + i\eta$. Define real functions $\varphi(\tau, \mu)$ and $\psi(\tau, \mu)$ by

$$\varphi(\tau, \mu) = \frac{\Im(\mu)}{\Im(\tau)} = \frac{\eta}{v},$$

$$\psi(\tau, \mu) = \frac{\Im(-\frac{\mu}{\tau})}{\Im(-\frac{1}{\tau})} = \frac{\Im(\mu\bar{\tau})}{\Im(\bar{\tau})} = \frac{u\eta - v\xi}{-v} = \xi - u\varphi(\tau, \mu).$$
Let $\Phi : (x, y) \mapsto (\varphi(\tau, \mu), \psi(\tau, \mu))$ be defined as

$$(x, y) \mapsto \sigma = x + iy \mapsto (\tau(\sigma), \mu(\sigma)) \mapsto (\varphi(\tau, \mu), \psi(\tau, \mu)).$$

$\Phi$ is real analytic with respect to $(x, y) \in \mathbb{R}^2$.

**Proposition A.**

$$\det D\Phi = -v(\varphi^2_x + \varphi^2_y) = -\frac{v}{|\tau|^2}(\psi^2_x + \psi^2_y).$$

**Proof.** By Cauchy-Riemann formula,

$$u_x = v_y, \quad u_y = -v_x, \quad \xi_x = \eta_y, \quad \xi_y = -\eta_x.$$
As \( \psi = \xi - u\phi \) and \( \varphi = \frac{\eta}{v} \),

\[
\psi_x = \xi_x - u_x \varphi - u \varphi_x = \eta_y - v_y \frac{\eta}{v} - u \varphi_x
\]

\[
= v \frac{v \eta_y - v_y \eta}{v^2} - u \varphi_x = v \varphi_y - u \varphi_x.
\]

\[
\psi_y = \xi_y - u_y \varphi - u \varphi_y = -\eta_x + v_x \frac{\eta}{v} - u \varphi_y
\]

\[
= v \frac{-v \eta_x + v_x \eta}{v^2} - u \varphi_y = -v \varphi_x - u \varphi_y.
\]

Hence,

\[
\det \begin{pmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{pmatrix} = \det \begin{pmatrix} \varphi_x & \varphi_y \\ v \varphi_y - u \varphi_x & -v \varphi_x - u \varphi_y \end{pmatrix}
\]

\[
= \det \begin{pmatrix} \varphi_x & \varphi_y \\ v \varphi_y & -v \varphi_x \end{pmatrix} = -v (\varphi_x^2 + \varphi_y^2).
\]
And by

\[
\varphi_x = \frac{1}{u^2 + \nu^2}(-u\psi_x - \nu\psi_y), \quad \varphi_y = \frac{1}{u^2 + \nu^2}(\nu\psi_x - u\psi_y),
\]

we have

\[
\det \begin{pmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{pmatrix} = -\frac{\nu}{u^2 + \nu^2}(\psi_x^2 + \psi_y^2).
\]

This completes the proof of Proposition A.
7. Periods of elliptic curve
Periods of elliptic curve

If \( s_0, s_1, s_2, s_3 \) are branch points of Riemann surface \( C_\sigma \) over \( s \)-plane, then elliptic integral

\[
\int^s ds \frac{1}{\sqrt{h(s)}}
\]

defines a coordinate in a torus \( \mathbb{C}/\Lambda \), where

\[
h(s) = (s - s_0)(s - s_1)(s - s_2)(s - s_3),
\]

and

\[
\Lambda = \{ n\omega_1 + m\omega_2 \mid n, m \in \mathbb{Z} \}
\]

with

\[
\omega_1 = \int_{\gamma_1} \frac{ds}{\sqrt{h(s)}}, \quad \omega_2 = \int_{\gamma_2} \frac{ds}{\sqrt{h(s)}}.
\]
As \( \pi(q_0) = -\frac{a}{2} \), and \( \pi(q_1) = -\frac{1}{2} \), (by an appropriate choice of branch)

\[
\mu(\sigma) = \frac{1}{\omega_1} \int_{-\frac{a}{2}}^{-\frac{1}{2}} \frac{ds}{\sqrt{h(s)}}
\]

gives the translation vector in torus \( \mathbb{T}_\tau \) with

\[
\tau(\sigma) = \frac{\omega_2}{\omega_1}.
\]

Observe that \( \tau \) and \( \mu \) are (locally) holomorphic in \( \sigma \).
Take a real value $a$, with $-\frac{1}{4} < a < 0$, as an example. We fix $a$ and suppress suffix $a$, in the followings.

The surface automorphism

$$f = f_a : (x, y) \mapsto (y, \frac{y + a}{x})$$

has real rational invariant function

$$r(x, y) = r_a(x, y) = \frac{(x + 1)(y + 1)(x + y + a)}{xy}.$$ 

The branch points of the Riemann surface $C_\sigma = r^{-1}(\sigma)$ are the solutions of $h_\sigma(s) = 0$, where

$$h_\sigma(s) = (s - \frac{\sigma - a}{2})(s^3 + (2 - \frac{\sigma - a}{2})s^2 + (1 + a)s + \frac{a}{2}).$$
From $h_\sigma(s) = 0$, we have

$$\sigma = 2s + a$$

and

$$\sigma = \frac{1}{s^2}(s + 1)^2(2s + a).$$

From $\frac{\partial \sigma}{\partial s} = 0$, we have

$$s = -1, \quad s = \frac{1}{2} \pm \sqrt{a + \frac{1}{4}}.$$

Recall that $-\frac{1}{4} < a < 0$. Let

$$u_2 = \frac{1}{2} + \sqrt{a + \frac{1}{4}}, \quad v_0 = \frac{1}{2} - \sqrt{a + \frac{1}{4}}.$$

Then, $0 < -\frac{a}{2} < v_0 < \frac{1}{2} < u_2$. 
Let
\[ \sigma_u = \sigma(u_2) = \frac{(u_2 + 1)^3}{u_2}, \quad \sigma_v = \sigma(v_0) = \frac{(v_0 + 1)^3}{v_0}. \]

Let \( u_0 < u_2 < u_3 \) be the solutions of \( h_{\sigma_u}(s) = 0 \).
\[ u_0 = \frac{\sigma_u - a}{2} - 2 - 2u_2, \quad u_3 = \frac{\sigma_u - a}{2}. \]

Let \( v_0 < v_2 < v_3 \) be the solutions of \( h_{\sigma_v}(s) = 0 \).
\[ v_2 = \frac{\sigma_v - a}{2} - 2 - 2v_0, \quad v_3 = \frac{\sigma_v - a}{2}. \]

Suppose \( \sigma \) be \( \sigma_u < \sigma < \sigma_v \), and let \( s_0, s_1, s_2, s_3 \) be the solutions for \( h_{\sigma}(s) = 0 \) with
\[ u_0 < s_0 < v_0 < s_1 < u_2 < s_2 < v_2, \quad u_3 < s_3 < v_3, \quad s_2 < s_3. \]
We have

\[ h_\sigma(s) = (s - s_0)(s - s_1)(s - s_2)(s - s_3). \]

Elliptic integral

\[ \int_{s_1}^{s} \frac{ds}{\sqrt{h_\sigma(s)}} \]

gives a coordinate in the elliptic curve. Periods are

\[ \omega_1 = 2 \int_{s_1}^{s_2} \frac{ds}{\sqrt{h_\sigma(s)}}, \quad \omega_2 = 2 \int_{s_1}^{s_0} \frac{ds}{i\sqrt{-h_\sigma(s)}}. \]

Period \( \omega_1 \) is real and positive. Period \( \omega_2 \) is pure imaginary with positive imaginary part.
By Abel-Jacobi theorem, this elliptic integral defines an isomorphism
\[ C_\sigma \to \mathbb{C}/\Lambda, \]
where \( \Lambda = \{ n\omega_1 + m\omega_2 \mid n, m \in \mathbb{Z} \} \).

As \( q = (-a, 0), \ f(q) = (0, -1), \) and \( \pi(q) = -\frac{a}{2}, \)
\( \pi(f(q)) = -\frac{1}{2}, \) translation vector \( \mu_1 \) is given by

\[ \mu_1 = \int_{-\frac{a}{2}}^{-\frac{1}{2}} \frac{ds}{-\sqrt{h_\sigma(s)}}. \]

We have

\[ \tau = \tau(\sigma) = \frac{\omega_2}{\omega_1} \in \mathbb{H}, \quad \mu = \mu(\sigma) = \frac{\mu_1}{\omega_1} \in \mathbb{C}. \]
8. Dynamics near fixed points
Dynamics near fixed points

Under the assumption \(-\frac{1}{4} < a < 0\), we have the following proposition. By abuse of notation, \(\psi(\sigma) = \psi(\tau(\sigma), \mu(\sigma))\),
\[
\psi(\tau, \mu) = \frac{\Im\left(-\frac{\mu}{\tau}\right)}{\Im\left(-\frac{1}{\tau}\right)}.
\]

**Proposition B.**

\[
\lim_{\sigma \nearrow \sigma_v} \psi(\sigma) = 0, \quad \lim_{\sigma \searrow \sigma_u} \psi(\sigma) > 0.
\]

**Proof.** As \(-\frac{1}{4} < a < 0\), and \(\sigma_u < \sigma < \sigma_v\), we have
\[
-\frac{a}{2} < u_0 < s_0 < v_0 < s_1 < u_2 < s_2 < v_2 < v_3, \quad u_2 < u_3 < s_3 < v_3.
\]

Note that \(u_0, v_0, u_2, v_2, u_3, v_3\) depend only on \(a\) and do not depend on \(\sigma\).
Recall $h_\sigma(s) = (s - s_0)(s - s_1)(s - s_2)(s - s_3)$.

First, we show that the translation vector

$$\mu_1(\sigma) = \int_{-\frac{\sigma}{2}}^{\frac{1}{2}} \frac{ds}{-\sqrt{h_\sigma(s)}}$$

is uniformly bounded for $\sigma_u < \sigma < \sigma_v$. 
If $-\frac{1}{2} < s < -\frac{a}{2}$, then

$$0 < (u_0 + \frac{a}{2})^4 < h_\sigma(s) < (v_3 + \frac{1}{2})^4.$$  

Hence,

$$0 < \frac{\frac{1}{2} - \frac{a}{2}}{(v_3 + \frac{1}{2})^2} \leq \int_{-\frac{1}{2}}^{-\frac{a}{2}} \frac{ds}{\sqrt{h_\sigma(s)}} \leq \frac{\frac{1}{2} - \frac{a}{2}}{(u_0 + \frac{a}{2})^2} < \infty.$$
Next, consider the case $\sigma \rightarrow \sigma_v$. We have

\[
\lim_{\sigma \rightarrow \sigma_v} s_0 = \lim_{\sigma \rightarrow \sigma_v} s_1 = v_0, \quad \lim_{\sigma \rightarrow \sigma_v} s_2 = v_2, \quad \lim_{\sigma \rightarrow \sigma_v} s_3 = v_3.
\]

Let us show that

\[
\lim_{\sigma \rightarrow \sigma_v} \omega_1 = +\infty.
\]
Take $0 < \Delta < \min(v_0 - u_0, u_2 - v_0)$, and fix $\Delta$.
Then take $0 < \varepsilon < \Delta$.
Take $\sigma$ sufficiently near $\sigma_v$ so that

$$|s_0 - v_0| < \varepsilon \text{ and } |s_1 - v_0| < \varepsilon.$$

Then for $s_1 < s \leq v_0 + \Delta$, we have

$$0 < s - s_0 < s - s_1 + 2\varepsilon, \quad 0 < s - s_1 \leq \Delta,$$

$$0 < s_2 - s \leq v_2 - v_0, \quad 0 < s_3 - s \leq v_3 - v_0.$$
From these inequalities, we have

\[ 0 < h_\sigma(s) \leq (s - s_1 + 2\varepsilon)(s - s_1)(v_2 - v_0)(v_3 - v_0), \]

and

\[
\int_{s_1}^{v_0+\Delta} \frac{ds}{\sqrt{h_\sigma(s)}} \geq \frac{1}{\sqrt{(v_2 - v_0)(v_3 - v_0)}} \int_{s_1}^{v_0+\Delta} \frac{ds}{\sqrt{(s - s_1 + 2\varepsilon)(s - s_1)}} \]

\[
= \frac{1}{\sqrt{(v_2 - v_0)(v_3 - v_0)}} \int_{0}^{v_0+\Delta-s_1} \frac{dt}{\sqrt{t(t + 2\varepsilon)}}. \]

Set \( \Delta_1 = v_0 + \Delta - s_1 \) and

\[ I_\varepsilon = \int_{0}^{\Delta_1} \frac{dt}{\sqrt{t(t + 2\varepsilon)}} = \int_{1}^{1+\frac{\Delta_1}{\varepsilon}} \frac{du}{\sqrt{u^2 - 1}} = \arccosh(1 + \frac{\Delta_1}{\varepsilon}). \]
Hence,

\[
\lim_{\varepsilon \downarrow 0} \int_{s_1}^{s_2} \frac{ds}{\sqrt{h_{\sigma}(s)}} \geq \lim_{\varepsilon \downarrow 0} \int_{s_1}^{v_0+\Delta} \frac{ds}{\sqrt{h_{\sigma}(s)}} \geq \frac{1}{\sqrt{(v_2 - v_0)(v_3 - v_0)}} \lim_{\varepsilon \downarrow 0} l_{\varepsilon} = +\infty.
\]

So,

\[
\lim_{\sigma \uparrow \sigma_v} \omega_1(\sigma) = +\infty, \text{ and } \lim_{\sigma \uparrow \sigma_v} \psi(\sigma) = 0.
\]

Note that \( \omega_2 \) remains pure imaginary and bounded in this case, since

\[
\lim_{\sigma \uparrow \sigma_v} s_2 = v_2, \quad \lim_{\sigma \uparrow \sigma_v} s_3 = v_3,
\]

and \( v_2, v_3 \) are simple roots of \( h_{\sigma_v} \).
Next, consider the case $\sigma \searrow \sigma_u$.

Let us show that

$$\lim_{\sigma \searrow \sigma_u} \omega_1(\sigma) = \frac{2\pi}{\sqrt{(u_2 - u_0)(u_3 - u_2)}} > 0.$$ 

Assume, for $\varepsilon > 0$, that

$$\max\{|s_0 - u_0|, |s_1 - u_2|, |s_2 - u_2|, |s_3 - u_3|\} < \varepsilon.$$ 

Then, for $s_1 < s < s_2$,

$$u_2 - u_0 - 2\varepsilon < s - s_0 < u_2 - u_0 + 2\varepsilon,$$

$$u_3 - u_2 - 2\varepsilon < s_3 - s < u_3 - u_2 + 2\varepsilon.$$
So,

\[(u_2 - u_0 - 2\varepsilon)(u_3 - u_2 - 2\varepsilon)(s - s_1)(s_2 - s) < h_\sigma(s)\]

\[< (u_2 - u_0 + 2\varepsilon)(u_3 - u_2 + 2\varepsilon)(s - s_1)(s_2 - s).\]

And as

\[
\int_{s_1}^{s_2} \frac{ds}{\sqrt{(s - s_1)(s_2 - s)}} = \int_{-1}^{1} \frac{dt}{\sqrt{1 - t^2}} = \pi,
\]

we have

\[
\frac{\pi}{\sqrt{(u_2 - u_0 + 2\varepsilon)(u_3 - u_2 + 2\varepsilon)}} \leq \int_{s_1}^{s_2} \frac{ds}{\sqrt{h_\sigma(s)}} \leq \frac{\pi}{\sqrt{(u_2 - u_0 - 2\varepsilon)(u_3 - u_2 - 2\varepsilon)}}.
\]
This shows

\[
\lim_{\sigma \searrow \sigma_u} \omega_1(\sigma) = 2 \lim_{\sigma \searrow \sigma_u} \int_{s_1}^{s_2} \frac{ds}{\sqrt{h_\sigma(s)}} = \frac{2\pi}{\sqrt{(u_2 - u_0)(u_3 - u_2)}}.
\]

Hence

\[
\lim_{\sigma \searrow \sigma_u} \psi(\sigma) = \lim_{\sigma \searrow \sigma_u} \frac{\mu_1(\sigma)}{\omega_1(\sigma)} > 0
\]

is a finite positive value. This completes the proof of Proposition B.
9. Twist dynamics
Theorem. There exists a parameter $a$, such that surface automorphism $F_a : X_a \rightarrow X_a$ has an orbit which is dense in an elliptic curve in $X_a$.

Proof. Take $a \in \mathbb{R}$, $-\frac{1}{4} < a < 0$. By proposition B, real analytic function $\psi(\sigma)$ is non-constant. There is a point $\sigma_0$ with $\psi_x^2(\sigma_0) + \psi_y^2(\sigma_0) \neq 0$.

By proposition A, in a neighborhood of $\sigma_0$, there exists a value with irrational

$\varphi(\sigma), \quad \psi(\sigma), \quad \text{and} \quad \frac{\varphi(\sigma)}{\psi(\sigma)}.$
10. Weierstraß $℘$-function
Weierstraß $\wp$-function

For each $a \in \mathbb{C} \setminus \{0, 1\}$, and $\sigma \in \hat{\mathbb{C}} \setminus \{0, \infty, a - 1, \sigma_{1+}, \sigma_{1-}\}$, there exists a birational map $(s, t) \mapsto (X, Y)$, such that elliptic curve $C_{\sigma}$:

$$t^2(2s + a - \sigma) = (s + 1)^2(2s + a) - \sigma s^2$$

is transformed into the Weierstraß normal form:

$$Y^2 = 4X^3 - g_2X - g_3.$$

Let $\rho = \frac{\sigma - a}{2}$ and $u = s - \rho$. Then

$$t^2u = u^3 + (2\rho + 2)u^2 + (\rho^2 + 4\rho + a + 1)u + 2\rho^2 + (a + 1)\rho + \frac{a}{2}.$$
By setting
\[ b_1 = 2\rho + 2, \quad 3b_2 = \rho^2 + 4\rho + a + 1, \quad b_3 = 2\rho^2 + (a + 1)\rho + \frac{a}{2}, \]
we get
\[ t^2 u = u^3 + b_1 u^2 + 3b_2 u + b_3. \]
If \( b_3 = 0 \), then \( \sigma = 0 \) or \( \sigma = a - 1 \). So, under our assumptions, \( b_3 \neq 0 \).

Set \( u = \frac{b_3}{X - b_2} \) to have
\[ t^2 \frac{b_3}{X - b_2} = \frac{b_3^3}{(X - b_2)^3} + \frac{b_1 b_3^2}{(X - b_2)^2} + \frac{3b_2 b_3}{X - b_2} + b_3. \]
Multiply \( \frac{(X-b_2)^3}{b_3} \) to both sides.
We get
\[ t^2(X - b_2)^2 = X^3 - (3b_2^2 - b_1b_3)X - (b_2^3 + 3b_2^2 + b_1b_2b_3 - b_3^2). \]

Let \( Y = 2t(X - b_2) \), and set \( g_2 = 4(3b_2^2 - b_1b_3) \), \( g_3 = 4(b_2^3 + 3b_2^2 + b_1b_2b_3 - b_3^2) \), to obtain
\[ Y^2 = 4X^3 - g_2X - g_3. \]

Birational transform \((s, t) \leftrightarrow (X, Y)\) is as follows.

\[
X = \frac{b_3}{s - \frac{\sigma - a}{2}}, \quad Y = 2t\left(\frac{b_3}{s - \frac{\sigma - a}{2}} - b_2\right),
\]
\[
s = \frac{b_3}{X} + \frac{\sigma - a}{2}, \quad t = \frac{Y}{2(X - b_2)}.
\]