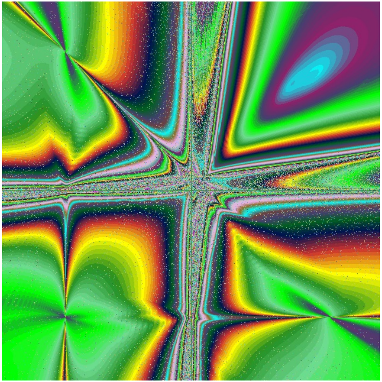
Twist dynamics of rational surface



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Abstract

Family of birational maps $f_a(x, y) = (y, \frac{y+a}{x})$ extends to a family of surface automorphisms $F_a : X_a \to X_a$.

Rational surface X_a is constructed by blowing-up the complex projective space $\mathbb{P}^2$ in 9 points.

If $a \neq 0, 1$, F_a is of infinite order. The degree of $F_a^{\circ n}$ grows quadratically. The entropy of these automorphisms are zero.

It is known that these automorphisms have invariant foliations (with singularities) with elliptic leaves.

In this note, we prove that there exists an F_a which has a quasi-periodic orbit dense in an elliptic curve.

Contents

1. Birational automorphism

Birational automorphism

Define a family of birational automorphisms $f_a : \mathbb{C}^2 \to \mathbb{C}^2$, for parameter $a \in \mathbb{C}$, by

$$f_a(x, y)_{\mathbb{C}^2} = (y, \frac{y+a}{x})_{\mathbb{C}^2}.$$

In homogeneous coordinates, with $(x, y)_{\mathbb{C}^2} \leftrightarrow [1 : x : y]$,

$$\tilde{f}_a[t : \tilde{x} : \tilde{y}] \;=\; [t\tilde{x} : \tilde{x}\tilde{y} : t(\tilde{y} + at)].$$

By lifting $\tilde{f}_a$ to a homogenious polynomial map $\tilde{\tilde{f}}_a : \mathbb{C}^3 \to \mathbb{C}^3$,

$$D\tilde{\tilde{f}}_a \;=\; \begin{pmatrix} \tilde{x} & t & 0 \\ 0 & \tilde{y} & \tilde{x} \\ \tilde{y} + 2at & 0 & t \end{pmatrix},$$

we have $\det D\tilde{\tilde{f}}_a \;=\; 2t\tilde{x}(\tilde{y} + at)$.

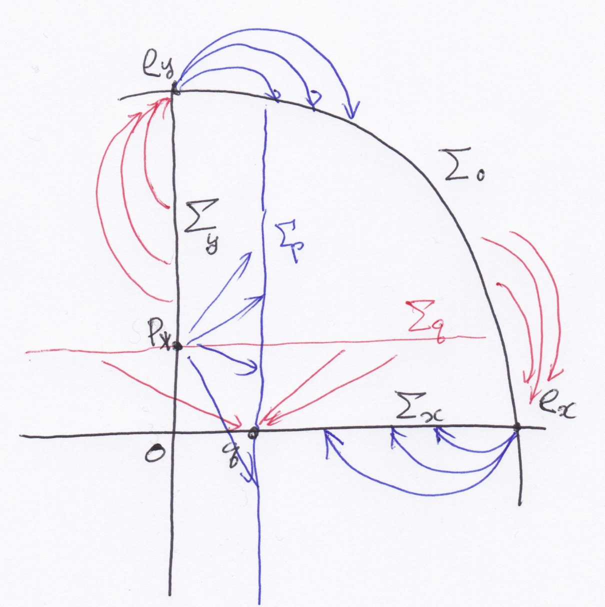
Critical locus and points of indeterminacy

The Jacobian det $D\tilde{\tilde{f}}_a = 2t\tilde{x}(\tilde{y} + at)$ vanishes in three exceptional curves :

$$\Sigma_0 = \{t = 0\}_{\mathbb{P}^2}, \quad \Sigma_y = \{\tilde{x} = 0\}_{\mathbb{P}^2}, \quad \Sigma_q = \{\tilde{y} + at = 0\}_{\mathbb{P}^2}.$$

$\tilde{f}_a[t : \tilde{x} : \tilde{y}] = [t\tilde{x} : \tilde{x}\tilde{y} : t(\tilde{y} + at)]$ has three points of indeterminacy :

$$e_x = [0 : 1 : 0], \quad e_y = [0 : 0 : 1], \quad p_* = [1 : 0 : -a].$$



e_y
Σ_0
Σ_y
Σ_p
p_*
Σ_q
e_x
Σ_x
0
q

2. Surface automorphism

Surface automorphism

THEOREM [Bedford,Kim, 2006]

If $a \neq 0, 1$, there is a complex manifold X_a obtained by blowing up $\mathbb{P}^2$ at 9 points, and a biholomorphic map $F_a : X_a \to X_a$ induced by f_a. The degree of $F_a^{\circ n}$ grows quadratically in n.

It is known that these automorphisms have invariant foliations by elliptic curves [Gizatullin 1980].

The entropy of these maps are all zero.

First blowup

Exceptional curves are mapped as :

$$\Sigma_y \Rightarrow e_y \Rightarrow \Sigma_0 \Rightarrow e_x \Rightarrow \Sigma_x,$$

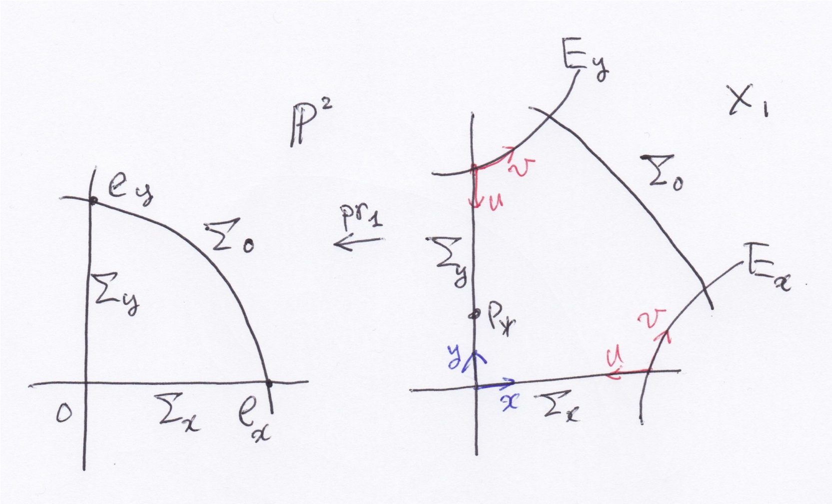
where $\Sigma_x = \{\tilde{y} = 0\}_{\mathbb{P}^2}$, and $\tilde{f}_a(\Sigma_x) = \Sigma_y$.

Blowup $\mathbb{P}^2$ in e_x and e_y to obtain a surface X_1 with projection $\mathrm{pr}_1 : X_1 \to \mathbb{P}^2$ and exceptional fibers

$$E_x = \mathrm{pr}_1^{-1}(e_x) \quad \text{and} \quad E_y = \mathrm{pr}_1^{-1}(e_y).$$

We have a cycle of curves of period 5.

$$\Sigma_y \Rightarrow E_y \Rightarrow \Sigma_0 \Rightarrow E_x \Rightarrow \Sigma_x \Rightarrow \Sigma_y.$$



$\mathbb{P}^2$
e_y
Σ_0
Σ_y
0
Σ_x
e_x
pr_1
E_y
X_1
v
u
Σ_0
Σ_y
E_x
P_γ
v
u
y
x
Σ_x

Coordinates in exceptional fibers

We need coordinates near the exceptional fibers. Let $(u, v)_{E_y}$ denote a local coordinate system of X_1 near E_y with $E_y = \{u = 0\}_{E_y}$ and

$$(u, v)_{E_y} \leftrightarrow [u : uv : 1].$$

Near Σ_y, f_a extends to a holomorphic map into X_1, except at p_*.

$$f_a(x, y)_{\mathbb{C}^2} = (\frac{x}{y + a}, y)_{E_y}.$$

So,

$$F_1(0, y)_{\mathbb{C}^2} = (0, y)_{E_y}.$$

Near E_y,

$$\tilde{f}_a(u,v)_{E_y} = \tilde{f}_a[u : uv : 1] = [uv : v : 1 + au].$$

So,

$$F_1(0,v)_{E_y} = [0 : v : 1] \in \Sigma_0.$$

Near Σ_0,

$$\tilde{f}_a[t : x : 1] = [t : 1 : \frac{t(1+at)}{x}].$$

With coordinates $(u,v)_{E_x} \leftrightarrow [u : 1 : uv]$, we have

$$F_1[t : x : 1] = (t, \frac{1+at}{x})_{E_x}, \quad \text{and} \quad F_1[0 : x : 1] = (0, \frac{1}{x})_{E_x}.$$

Finally, $(u, v)_{E_x}$ is mapped as

$$\tilde{f}_a[u : 1 : uv] = [1 : v : u(v + a)].$$

So,

$$F_1(0, v)_{E_x} = [1 : v : 0] \leftrightarrow (v, 0)_{\mathbb{C}^2} \in \Sigma_x.$$

Second blowup

$F_1 : X_1 \to X_1$ has an exceptional curve (strict transform of)

$$\Sigma_q = \{\tilde{y} + at = 0\}_{\mathbb{P}^2}$$

and a point of indeterminacy

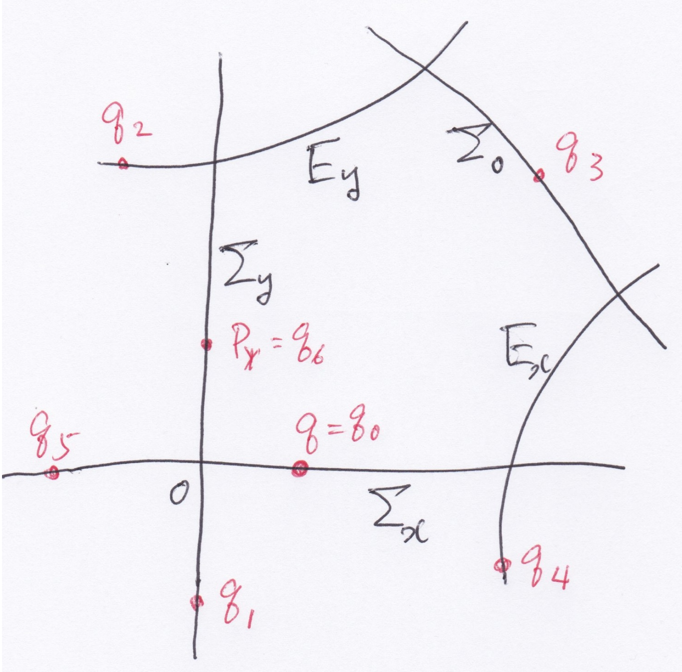
$$p_* = [1 : 0 : -a] \leftrightarrow (0, -a)_{\mathbb{C}^2}.$$

Exceptional curve Σ_q is mapped to

$$q = [1 : -a : 0] \leftrightarrow (-a, 0)_{\mathbb{C}^2}.$$

Set $q_k = F_1^{\circ k}(q)$ for $k = 0, 1, 2, \cdots, 6$. Then $q_6 = p_*$.

$$q_0 = q = (-a, 0)_{\mathbb{C}^2} \in \Sigma_q,$$
$$q_1 = (0, -1)_{\mathbb{C}^2} \in \Sigma_y,$$
$$q_2 = (0, -1)_{E_y} \in E_y,$$
$$q_3 = [0 : -1 : 1] \in \Sigma_0,$$
$$q_4 = (0, -1)_{E_x} \in E_x,$$
$$q_5 = (-1, 0)_{\mathbb{C}^2} \in \Sigma_x,$$
$$q_6 = p_* = (0, -a)_{\mathbb{C}^2} \in \Sigma_y.$$



q_2
E_y
Σ_0
q_3
Σ_y
$P_\gamma = q_6$
E_x
q_5
$q = q_0$
0
Σ_x
q_4
q_1

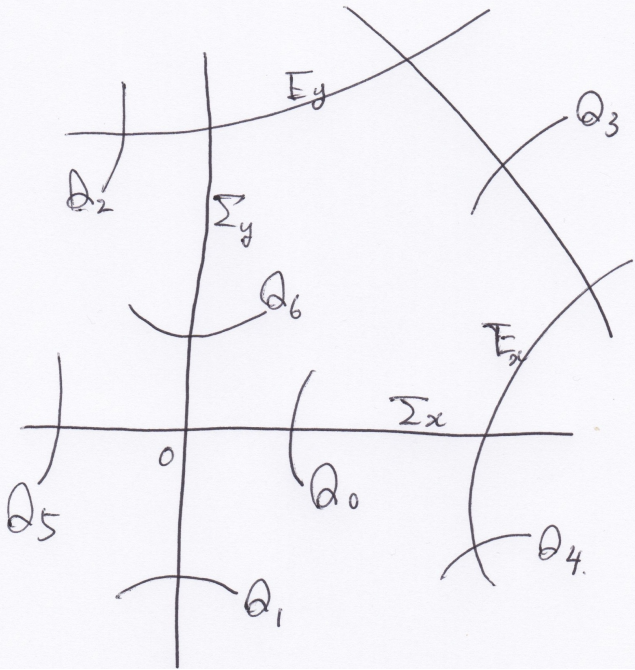
We blow up X_1 in 7 points $q_0, q_1, \ldots, q_6$, to obtain a surface X_a with projection

$$\mathrm{pr}_2 : X_a \to X_1,$$

and denote the exceptional fibers as

$$\mathrm{pr}_2^{-1}(q_k) = Q_k, \quad k = 0, 1, \cdots, 6.$$

$F_1 : X_1 \to X_1$ induces an automorphism $F_a : X_a \to X_a$.



E_y
Q_3
Q_2
Σ_y
Q_6
E_x
Σx
0
Q_0
Q_5
Q_4.
Q_1

Around exceptional fibers

Let $(\xi, \eta)_{Q_0}$ be a local coordinate of X_a near exceptional fiber Q_0, given by $(\xi, \eta)_{Q_0} \leftrightarrow [1 : -a + \xi : \xi\eta]$.

Near Σ_q,

$$F_a[1 : x : y] = [x : xy : y{+}a] = [1 : -a{+}(y{+}a) : \frac{y + a}{x}] \leftrightarrow (y{+}a, \frac{1}{x})_{Q_0}.$$

Near the exceptional fiber $Q_6 = \mathrm{pr}^{-1}(p_*)$, take local coordinates

$$(\xi, \eta)_{Q_6} \leftrightarrow [1 : \xi\eta : -a + \xi].$$

Recall the coordinates near E_y, given by $(u, v)_{E_y} \leftrightarrow [u : uv : 1]$.

$$F_a(\xi, \eta)_{Q_6} = (\eta, -a + \xi)_{E_y}.$$

3. Invariant function

Invariant function

It is known that rational function

$$r_a(x,y) = (1+\frac{1}{x})(1+\frac{1}{y})(x+y+a)$$

is invariant under f_a [Lyness 1942, 1945, 1961]. We understand this function as

$$r_a(x,y) = \frac{(x+1)(y+1)(x+y+a)}{xy}.$$

PROPOSITION. If $a \neq 0, 1$, then r_a lifts to a holomorphic map

$$\tilde{r}_a : X_a \to \hat{\mathbb{C}}.$$

Proof of proposition

Invariant function r_a has indeterminate points in $\mathbb{C}^2$.

$$q_0 = (-a, 0)_{\mathbb{C}^2}, \quad q_1 = (0, -1)_{\mathbb{C}^2},$$

$$q_5 = (-1, 0)_{\mathbb{C}^2}, \quad q_6 = p_* = (0, -a)_{\mathbb{C}^2}.$$

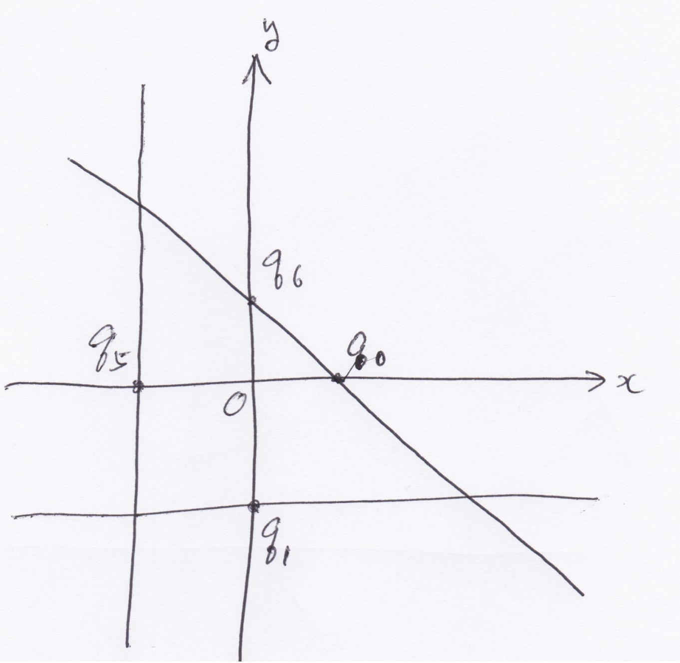
These points are blown up and $\tilde{r}_a$ is holomorphic near the exceptional fibers.

For example, in local coordinates $(\xi, \eta)_{Q_0} \leftrightarrow (-a + \xi, \xi\eta)_{\mathbb{C}^2}$, near Q_0

$$\tilde{r}_a(\xi, \eta)_{Q_0} = \frac{(-a + \xi + 1)(\xi\eta + 1)(\eta + 1)}{(-a + \xi)\eta}.$$

Especially,

$$\tilde{r}_a(0, \eta)_{Q_0} = \frac{(1 - a)(\eta + 1)}{-a\eta}.$$



y
q_6
q_5
q_0
x
O
q_1

In the line at infinity $\Sigma_0 \setminus \{[0:1:0],[0:0:1],[0:-1:1]\}$, $\tilde{r}_a[0:x:y] = \infty$.

Near exceptional fiber E_y with coordinates $(u,v)_{E_y} \leftrightarrow [u:uv:1]$,

$$\tilde{r}_a(u,v)_{E_y} = \frac{(v+1)(u+1)(uv+1+au)}{uv}.$$

So, $\tilde{r}_a(0,v)_{E_y} = \infty$ unless $v = -1$.

Near Q_2, take local coordinates $(\xi, \eta)_{Q_2}$ by

$$(\xi, \eta)_{Q_2} \leftrightarrow (\xi, -1 + \xi\eta)_{E_y} \leftrightarrow [\xi : \xi(-1 + \xi\eta) : 1].$$

Then

$$\tilde{r}_a(\xi, \eta)_{Q_2} = \frac{\eta(\xi + 1)(-\xi + \xi^2\eta + 1 + a\xi)}{-1 + \xi\eta},$$

and

$$\tilde{r}_a(0, \eta)_{Q_2} = -\eta.$$

Near other exceptional fibers, $\tilde{r}_a$ is defined similarly and gives a holomorphic map

$$\tilde{r}_a : X_a \to \hat{\mathbb{C}}.$$

PROPOSITION. $\tilde{r}_a$ is invariant under F_a.

$$\tilde{r}_a \circ F_a = \tilde{r}_a.$$

PROPOSITION. $\tilde{r}_a$ is non-constant in each of

$$\Sigma_0, E_x, E_y, Q_0, Q_1, Q_2, Q_3, Q_4, Q_5, Q_6.$$

PROPOSITION. Critical points of $\tilde{r}_a$ (or $\frac{1}{\tilde{r}_a}$) are fixed point or periodic point of F_a.

PROOF. Number of critical points is finite.

Critical points

In $\mathbb{C}^2$,

$$\frac{\partial r_a}{\partial x} = \frac{(y+1)(x^2-y-a)}{x^2y}, \quad \frac{\partial r_a}{\partial y} = \frac{(x+1)(y^2-x-a)}{xy^2},$$

and

$$\frac{\partial}{\partial x}(\frac{1}{r_a}) = \frac{-y(x^2-y-a)}{(y+1)(x+1)^2(x+y+a)^2},$$

$$\frac{\partial}{\partial y}(\frac{1}{r_a}) = \frac{-x(y^2-x-a)}{(x+1)(y+1)^2(x+y+a)^2}.$$

Critical points are

$$(-1,-1)_{\mathbb{C}^2}, (-1, 1-a)_{\mathbb{C}^2}, (1-a,-1)_{\mathbb{C}^2}, \qquad \text{(period 3)},$$

$$(\frac{1}{2} \pm \sqrt{a+\frac{1}{4}}, \frac{1}{2} \mp \sqrt{a+\frac{1}{4}})_{\mathbb{C}^2}, \qquad \text{(fixed points)},$$

$$(-\frac{1}{2} \pm \sqrt{a-\frac{3}{4}}, -\frac{1}{2} \mp \sqrt{a-\frac{3}{4}})_{\mathbb{C}^2}, \qquad \text{(period 2)},$$

$$(0,0)_{\mathbb{C}^2}, (0,0)_{E_y}, (0,\infty)_{E_y}, (0,\infty)_{E_x}, (0,0)_{E_x}, \qquad \text{(period 5)}.$$

Critical values

$$\begin{aligned}
\sigma_3 &= 0, && \text{(period 3 cycle)},\\
\sigma_5 &= \infty, && \text{(period 5 cycle)},\\
\sigma_2 &= a-1, && \text{(period 2 cycle)},\\
\sigma_{1\pm} &= a+5-\frac{1}{2a} \pm (4+\frac{1}{a})\sqrt{a+\frac{1}{4}}, && \text{(fixed oints)}.
\end{aligned}$$

The critical values of fixed point can be expressed as

$$\sigma_1 = \frac{(x_0+1)^3}{x_0}, \qquad x_0^2 = x_0 + a.$$

4. Invariant curves

Invariant curves

For $\sigma \in \hat{\mathbb{C}}$, let C_σ denote invariant cubic curve in X_a defined by

$$C_\sigma = \tilde{r_a}^{-1}(\sigma).$$

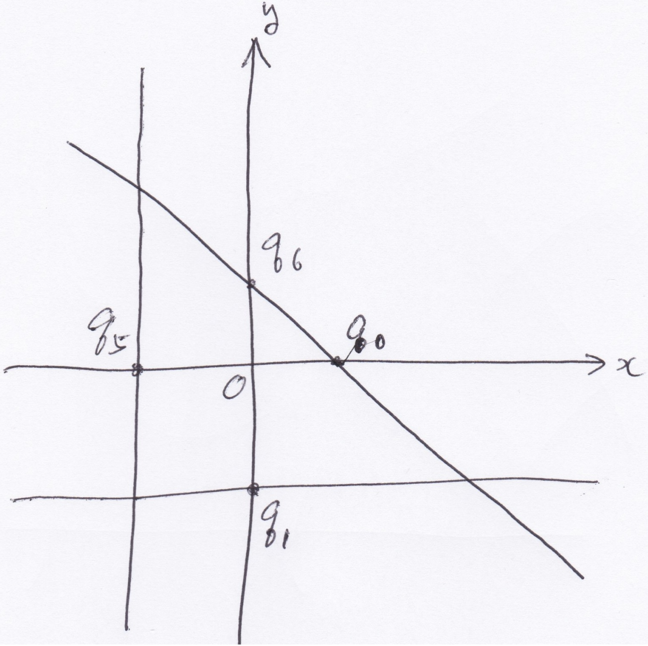
C_{σ_3} consists of three lines:

$$\{x + y + a = 0\}_{\mathbb{C}^2}, \quad \{y + 1 = 0\}_{\mathbb{C}^2}, \quad \{x + 1 = 0\}_{\mathbb{C}^2},$$

which are mapped cyclically.

$$(\zeta, -1)_{\mathbb{C}^2} \mapsto (-1, \frac{a-1}{\zeta})_{\mathbb{C}^2} \mapsto (\frac{a-1}{\zeta}, \frac{1-a}{\zeta} - a)_{\mathbb{C}^2} \mapsto (\frac{1-a}{\zeta} - a, -1)_{\mathbb{C}^2}.$$

Eigenvalues at periodic point in the curve (and of the 3-cycle) are $a - 1$ and $\frac{1}{a-1}$.



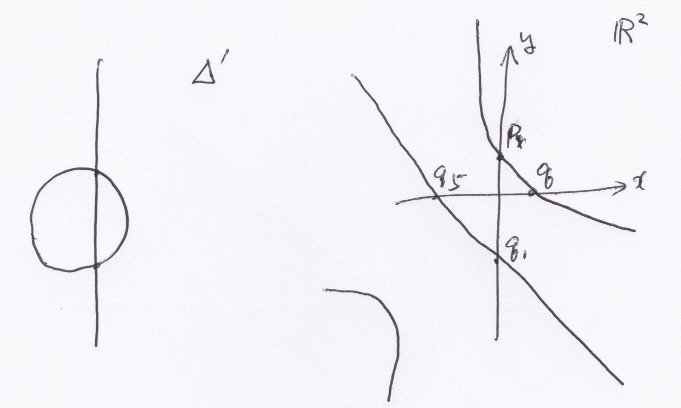
y
x
O
q6
q5
q0
q1

C_{σ_2} consists of a line and a quadric which are mapped to each other by F_a.

$$\{x+y+1=0\}_{\mathbb{C}^2}, \quad \{xy+x+y+a=0\}_{\mathbb{C}^2}.$$

$$(\zeta, -\zeta-1)_{\mathbb{C}^2} \mapsto (-1-\zeta, -1+\frac{a-1}{\zeta})_{\mathbb{C}^2} \mapsto (-1+\frac{a-1}{\zeta}, \frac{1-a}{\zeta})_{\mathbb{C}^2}.$$

Eigenvalues of the 2-cycle are $\frac{1}{1-a}(a-\frac{1}{2} \pm \sqrt{a-\frac{3}{4}})$.



Δ'
ℝ²
y
x
q₅
q₆
q₁

C_{σ_5} consists of five "lines"

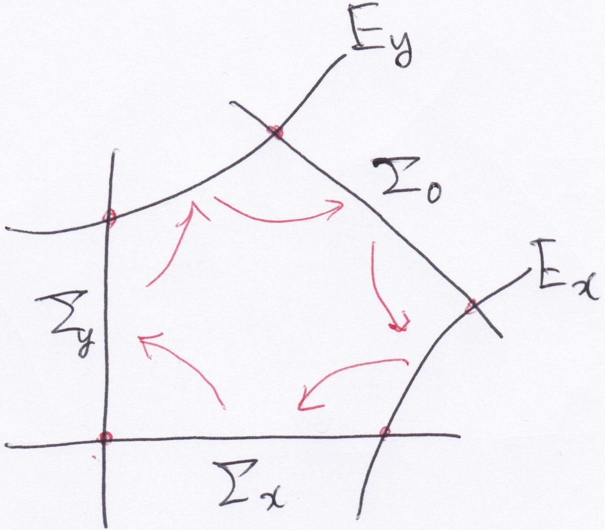
$$\Sigma_x, \Sigma_y, E_y, \Sigma_0, E_x.$$

which are mapped cyclically by F_a.

$$(\zeta, 0)_{\mathbb{C}^2} \mapsto (0, \frac{a}{\zeta})_{\mathbb{C}^2} \mapsto (0, \frac{a}{\zeta})_{E_y} \mapsto [0 : \frac{a}{\zeta} : 1] \mapsto$$

$$\mapsto (0, \frac{\zeta}{a})_{E_x} \mapsto (\frac{\zeta}{a}, 0)_{\mathbb{C}^2} \mapsto (0, \frac{a^2}{\zeta})_{\mathbb{C}^2} \mapsto \cdots .$$

Eigenvalues of 5-cycle are a^{-1} and a.



E_y
Σ_0
E_x
Σ_y
Σ_x

C_{σ_1} is a Riemann sphere with a node.
For fixed point $(x_0, x_0)_{\mathbb{C}^2}$, ($x_0^2 = x_0 + a$) , set $x = x_0 + u$, $y = y_0 + v$.

$$\{uv(u+v) + (x_0+1)(u^2 - \frac{1}{x_0}uv + v^2) = 0\}.$$

Eigenvalues of fixed point $(x_0, x_0)_{\mathbb{C}^2}$ are

$$\frac{1}{2x_0}(1 \pm \sqrt{1-4x_0}).$$

5. Elliptic curves

Elliptic curves

If $\sigma \in \hat{\mathbb{C}} \setminus \{0, \infty, a-1, \sigma_{1+}, \sigma_{1-}\}$, where $\sigma_{1\pm} = \frac{(x_\pm + 1)^3}{x_\pm}$ with $x_\pm = \frac{1}{2} \pm \sqrt{a + \frac{1}{4}}$, then C_σ is an elliptic curve defined by

$$P_\sigma(x, y) = (x+1)(y+1)(x+y+a) - \sigma xy = 0.$$

Remark

$P_\sigma(x, y) dx \wedge dy$ is an automorphic form of weight -1, and $\frac{dx \wedge dy}{P_\sigma(x,y)}$ is invariant under f_a^*.

$$P_\sigma(f_a(x, y)) = P_\sigma(x, y) \det Df_a(x, y).$$

$$f_a^* \left(\frac{dx \wedge dy}{P_\sigma(x, y)} \right) = \frac{dx \wedge dy}{P_\sigma(x, y)}.$$

The Jacobian of periodic point is always equal to 1.

Riemann surface

Let

$$s = \frac{x+y}{2}, \quad t = \frac{x-y}{2}.$$

Then $x + y = 2s, \;\; xy = s^2 - t^2$. From equation $P_\sigma(x, y) = 0$, we get

$$(*) \qquad (s - \frac{\sigma - a}{2})t^2 \;=\; s^3 + (2 - \frac{\sigma - a}{2})s^2 + (1 + a)s + \frac{a}{2}.$$

Note that when $s = \frac{\sigma - a}{2}$, the right hand side is equal to $\frac{1}{2}\sigma(\sigma - (a-1))$, which does not vanish since we assumed $\sigma \neq 0, a - 1$.

Equation $(*)$ defines a Riemann surface over s-plane.

Near Q_3

REMARK.

For $s = \frac{\sigma - a}{2}$, the elliptic curve C_σ passes through a point in Q_3.

$$Q_3 = \mathrm{pr}^{-1}(q_3), \qquad q_3 = [0 : -1 : 1] \in \Sigma_0.$$

Near Q_3, take a local coordinate $(\xi, \eta)_{Q_3}$ given by

$$(\xi, \eta)_{Q_3} \leftrightarrow [\xi : -1 : 1 + \xi\eta].$$

Note that $s = \frac{x+y}{2} = \frac{\eta}{2}$, if $\xi \neq 0$.

$$\tilde{r}_a(0, \eta)_{Q_3} = \eta + a = \sigma$$

gives $\eta = \sigma - a$.

From

$$\frac{\tilde{P}_\sigma(\xi,\eta)_{Q_3}}{\xi} = (-1+\xi)(1+\xi\eta+\xi)(\eta+a)+\sigma(1+\xi\eta) \;=\; 0,$$

we get a quadratic equation in ξ :

$$(\eta+a)(\eta+1)\xi^2+\eta(\sigma-a-\eta)\xi+\sigma-a-\eta \;=\; 0,$$

whose discriminant is

$$D \;=\; (\sigma-a-\eta)(\eta^2(\sigma-a-\eta)-4(\eta+a)(\eta+1)).$$

If $\eta=\sigma-a$ then $D=0$. If η is near $\sigma-a$ and $\eta\neq\sigma-a$, then $D\neq 0$ (because $\sigma\neq a-1$).
This shows $s=\frac{\sigma-a}{2}$ is a branch point of the Riemann surface.

Near Q_2 with local coordinates

$$(\xi, \eta)_{Q_2} \leftrightarrow (\xi, -1 + \xi\eta)_{E_y} \leftrightarrow [\xi : \xi(-1 + \xi\eta) : 1],$$

we get

$$\frac{\tilde{P}_\sigma(\xi, \eta)_{Q_2}}{\xi^2} = \sigma + \eta + (a - \sigma)\xi\eta + (a - 1)\xi^2\eta + \xi^2\eta^2 + \xi^3\eta^2.$$

By the implicit function theorem, C_σ is regular at $(0, -\sigma)_{Q_2}$.

Similarly, near Q_4, C_σ passes through $(0, -\sigma)_{Q_4}$ with local coordinates

$$(\xi, \eta)_{Q_4} \leftrightarrow (\xi, -1 + \xi\eta)_{E_x} \leftrightarrow [\xi : 1 : \xi(-1 + \xi\eta)].$$

Branch points

The Riemann surface defined by equation $(*)$ has 4 branch points.

$$s = \frac{\sigma - a}{2}, \quad \text{solutions of} \quad s^3 + (2 - \frac{\sigma - a}{2})s^2 + (1 + a)s + \frac{a}{2} = 0.$$

(In our case, these 4 points are distinct. By Riemann-Hurwitz formula, C_σ is a torus.)

Theorem(Abel,Jacobi) There exists a $\tau \in \mathbb{H}$, and a holomorphic isomorphism

$$AJ : \mathbb{C}/\Lambda_\tau \to C_\sigma,$$

where $\Lambda_\tau = \{n + m\tau \mid n, m \in \mathbb{Z}\}$.

6. Translation in elliptic curve

Translation in elliptic curves

For $\tau \in \mathbb{H}$, let $\mathbb{T}_\tau$ denote the torus $\mathbb{C}/\Lambda_\tau$.

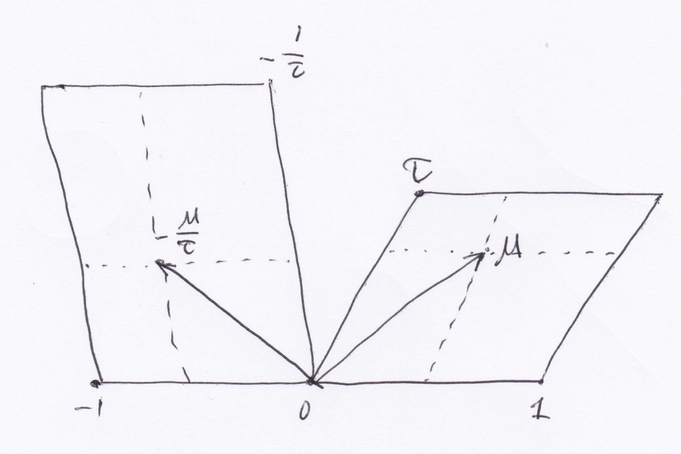
For $\mu \in \mathbb{C}$, **translation** by μ in $\mathbb{T}_\tau$ is defined as

$$T_{\tau,\mu} : \mathbb{T}_\tau \to \mathbb{T}_\tau, \qquad z \mapsto z + \mu \pmod{\Lambda_\tau}.$$

Suppose $\tau = \tau(\sigma)$ and $\mu = \mu(\sigma)$ depend holomorphically on σ. Let $\tau = u + iv$, $\mu = \xi + i\eta$. Define real functions $\varphi(\tau, \mu)$ and $\psi(\tau, \mu)$ by

$$\varphi(\tau, \mu) = \frac{\Im(\mu)}{\Im(\tau)} = \frac{\eta}{v},$$

$$\psi(\tau, \mu) = \frac{\Im(-\frac{\mu}{\tau})}{\Im(-\frac{1}{\tau})} = \frac{\Im(\mu\bar{\tau})}{\Im(\bar{\tau})} = \frac{u\eta - v\xi}{-v} = \xi - u\varphi(\tau, \mu).$$



$-\frac{1}{\tau}$
τ
$-\frac{\mu}{\tau}$
μ
-1
0
1

Let $\Phi : (x, y) \mapsto (\varphi(\tau, \mu), \psi(\tau, \mu))$ be defined as

$$(x, y) \mapsto \sigma = x + iy \mapsto (\tau(\sigma), \mu(\sigma)) \mapsto (\varphi(\tau, \mu), \psi(\tau, \mu)).$$

Φ is real analytic with respect to $(x, y) \in \mathbb{R}^2$.

PROPOSITION A.

$$\det D\Phi = -v(\varphi_x^2 + \varphi_y^2) = -\frac{v}{|\tau|^2}(\psi_x^2 + \psi_y^2).$$

PROOF. By Cauchy-Riemann formula,

$$u_x = v_y, \quad u_y = -v_x, \quad \xi_x = \eta_y, \quad \xi_y = -\eta_x.$$

As $\psi = \xi - u\phi$ and $\varphi = \frac{\eta}{v}$,

$$\psi_x = \xi_x - u_x\varphi - u\varphi_x = \eta_y - v_y\frac{\eta}{v} - u\varphi_x$$

$$= v\frac{v\eta_y - v_y\eta}{v^2} - u\varphi_x = v\varphi_y - u\varphi_x.$$

$$\psi_y = \xi_y - u_y\varphi - u\varphi_y = -\eta_x + v_x\frac{\eta}{v} - u\varphi_y$$

$$= v\frac{-v\eta_x + v_x\eta}{v^2} - u\varphi_y = -v\varphi_x - u\varphi_y.$$

Hence,

$$\det\begin{pmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{pmatrix} = \det\begin{pmatrix} \varphi_x & \varphi_y \\ v\varphi_y - u\varphi_x & -v\varphi_x - u\varphi_y \end{pmatrix}$$

$$= \det\begin{pmatrix} \varphi_x & \varphi_y \\ v\varphi_y & -v\varphi_x \end{pmatrix} = -v(\varphi_x^2 + \varphi_y^2).$$

And by

$$\varphi_x = \frac{1}{u^2+v^2}(-u\psi_x - v\psi_y), \quad \varphi_y = \frac{1}{u^2+v^2}(v\psi_x - u\psi_y),$$

we have

$$\det\begin{pmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{pmatrix} = -\frac{v}{u^2+v^2}(\psi_x^2+\psi_y^2).$$

This completes the proof of Proposition A.

7. Periods of elliptic curve

Periods of elliptic curve

If s_0, s_1, s_2, s_3 are branch points of Riemann surface C_σ over s-plane, then elliptic integral

$$\int^s \frac{ds}{\sqrt{h(s)}}$$

defines a coordinate in a torus $\mathbb{C}/\Lambda$, where

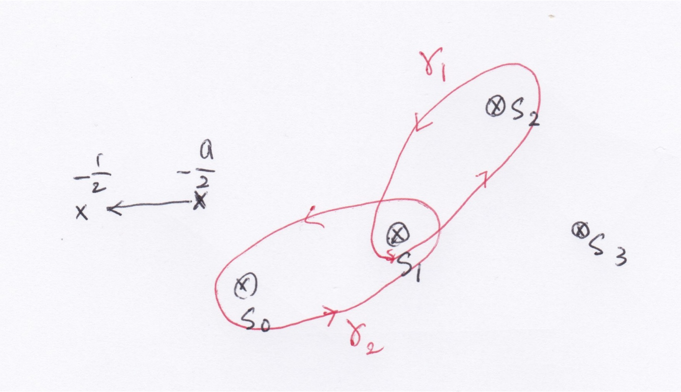
$$h(s) = (s - s_0)(s - s_1)(s - s_2)(s - s_3),$$

and

$$\Lambda = \{n\omega_1 + m\omega_2 \mid n, m \in \mathbb{Z}\}$$

with

$$\omega_1 = \int_{\gamma_1} \frac{ds}{\sqrt{h(s)}}, \quad \omega_2 = \int_{\gamma_2} \frac{ds}{\sqrt{h(s)}}.$$



$-\frac{1}{2}$
$-\frac{a}{2}$
γ_1
s_2
s_1
s_3
s_0
γ_2

As $\pi(q_0) = -\frac{a}{2}$, and $\pi(q_1) = -\frac{1}{2}$, (by an appropriate choice of branch)

$$\mu(\sigma) = \frac{1}{\omega_1} \int_{-\frac{a}{2}}^{-\frac{1}{2}} \frac{ds}{\sqrt{h(s)}}$$

gives the translation vector in torus $\mathbb{T}_\tau$ with

$$\tau(\sigma) = \frac{\omega_2}{\omega_1}.$$

Observe that τ and μ are (locally) holomorphic in σ.

Take a real value a, with $-\frac{1}{4} < a < 0$, as an example. We fix a and suppress suffix a, in the followings.

The surface automorphism

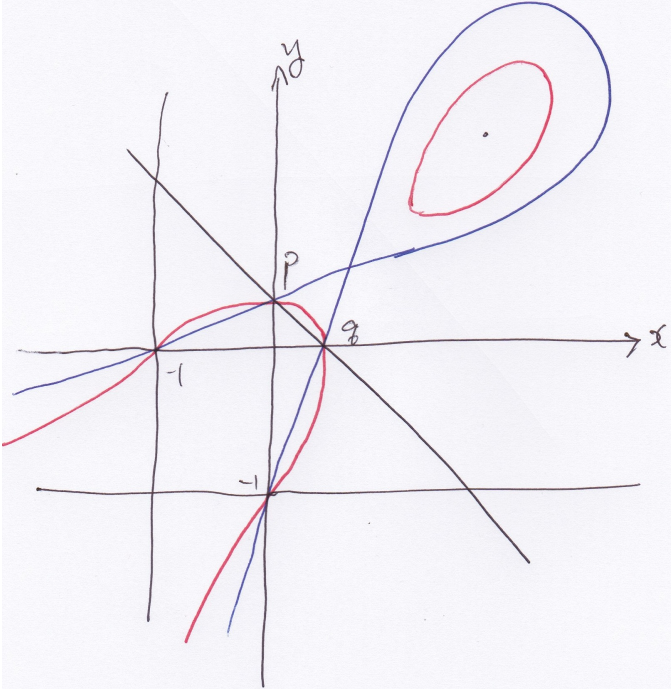
$$f = f_a : (x, y) \mapsto (y, \frac{y+a}{x})$$

has real rational invariant function

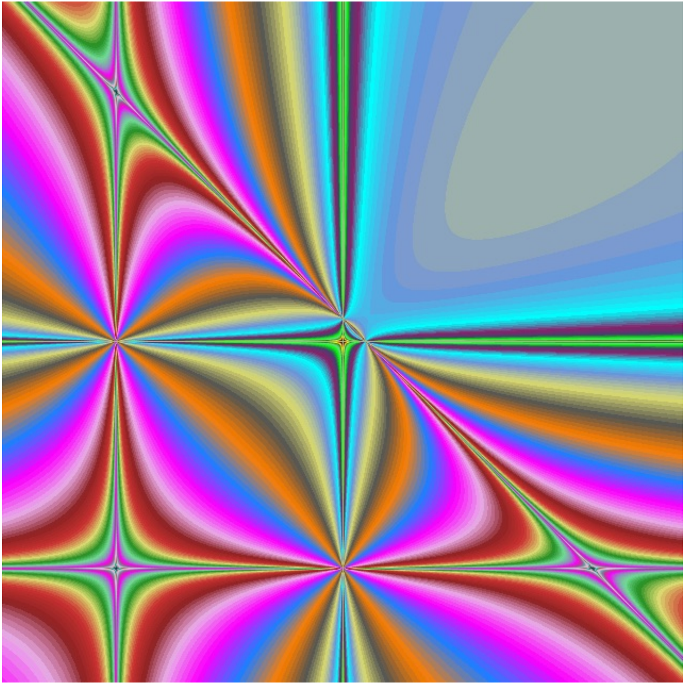
$$r(x, y) = r_a(x, y) = \frac{(x+1)(y+1)(x+y+a)}{xy}.$$

The branch points of the Riemann surface $C_\sigma = r^{-1}(\sigma)$ are the solutions of $h_\sigma(s) = 0$, where

$$h_\sigma(s) = (s - \frac{\sigma - a}{2})(s^3 + (2 - \frac{\sigma - a}{2})s^2 + (1+a)s + \frac{a}{2}).$$



y
x
p
q
-1
-1



From $h_\sigma(s) = 0$, we have

$$\sigma = 2s + a$$

and

$$\sigma = \frac{1}{s^2}(s+1)^2(2s+a).$$

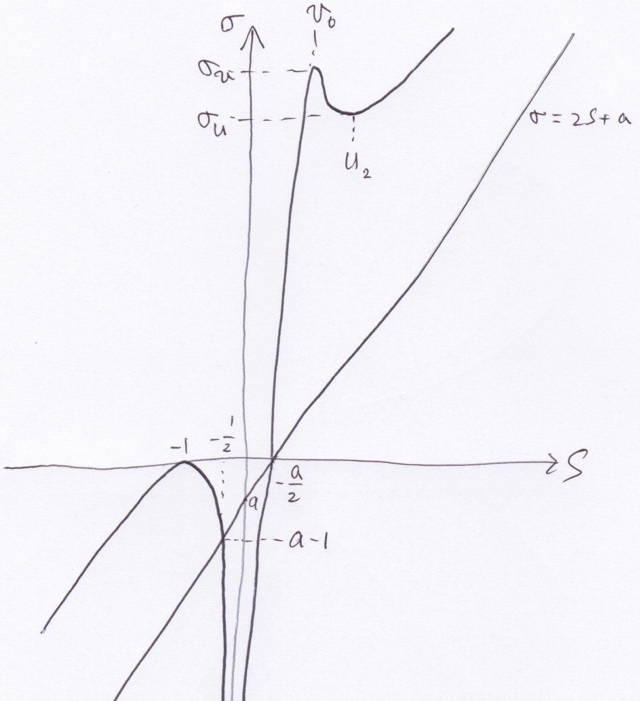
From $\frac{\partial \sigma}{\partial s} = 0$, we have

$$s = -1, \qquad s = \frac{1}{2} \pm \sqrt{a + \frac{1}{4}}.$$

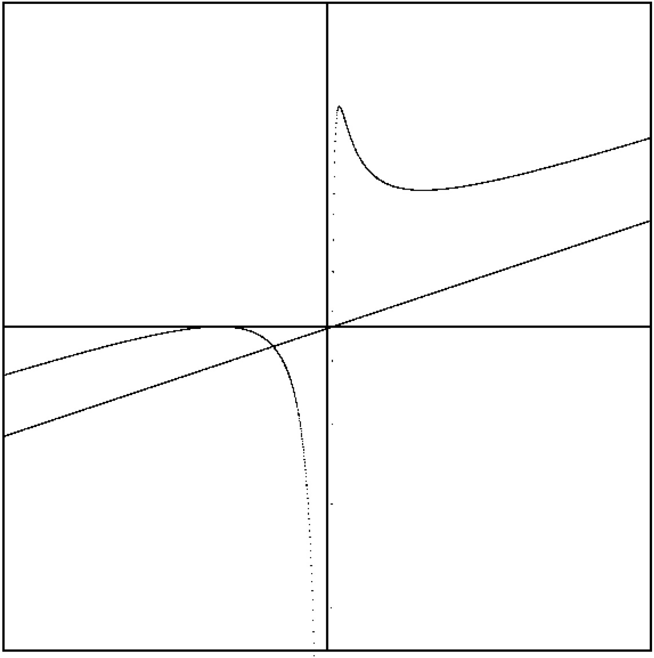
Recall that $-\frac{1}{4} < a < 0$. Let

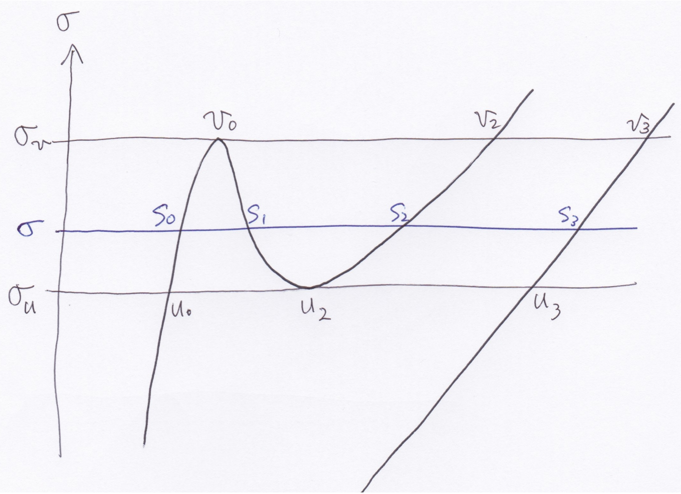
$$u_2 = \frac{1}{2} + \sqrt{a + \frac{1}{4}}, \qquad v_0 = \frac{1}{2} - \sqrt{a + \frac{1}{4}}.$$

Then, $0 < -\frac{a}{2} < v_0 < \frac{1}{2} < u_2$.



σ
v_0
σ_v
σ_u
u_2
σ = 2S + a
−1
$-\frac{1}{2}$
S
$-\frac{a}{2}$
a
a − 1




σ
σv
v0
v2
v3
σ
s0
s1
s2
s3
σu
u0
u2
u3

Let

$$\sigma_u = \sigma(u_2) = \frac{(u_2+1)^3}{u_2}, \qquad \sigma_v = \sigma(v_0) = \frac{(v_0+1)^3}{v_0}.$$

Let $u_0 < u_2 < u_3$ be the solutions of $h_{\sigma_u}(s) = 0$.

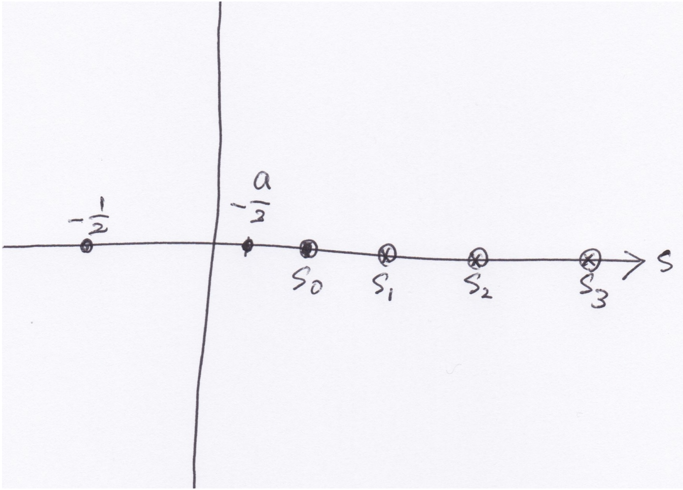
$$u_0 = \frac{\sigma_u - a}{2} - 2 - 2u_2, \quad u_3 = \frac{\sigma_u - a}{2}.$$

Let $v_0 < v_2 < v_3$ be the solutions of $h_{\sigma_v}(s) = 0$.

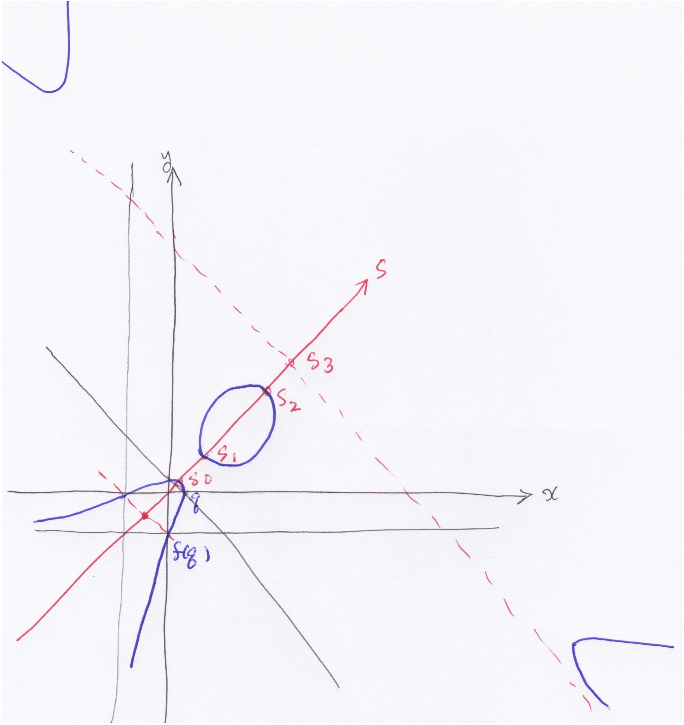
$$v_2 = \frac{\sigma_v - a}{2} - 2 - 2v_0, \qquad v_3 = \frac{\sigma_v - a}{2}.$$

Suppose σ be $\sigma_u < \sigma < \sigma_v$, and let s_0, s_1, s_2, s_3 be the solutions for $h_\sigma(s) = 0$ with

$$u_0 < s_0 < v_0 < s_1 < u_2 < s_2 < v_2, \qquad u_3 < s_3 < v_3, \qquad s_2 < s_3.$$



$-\frac{1}{2}$
$-\frac{a}{2}$
s_0
s_1
s_2
s_3
s



y
S
x
s_3
s_2
s_1
s_0
q
$f(q)$

We have

$$h_\sigma(s) = (s - s_0)(s - s_1)(s - s_2)(s - s_3).$$

Elliptic integral

$$\int_{s_1}^{s} \frac{ds}{\sqrt{h_\sigma(s)}}$$

gives a coordinate in the elliptic curve. Periods are

$$\omega_1 = 2\int_{s_1}^{s_2} \frac{ds}{\sqrt{h_\sigma(s)}}, \qquad \omega_2 = 2\int_{s_1}^{s_0} \frac{ds}{i\sqrt{-h_\sigma(s)}}.$$

Period ω_1 is real and positive. Period ω_2 is pure imaginary with positive imaginary part.

By Abel-Jacobi theorem, this elliptic integral defines an isomorphism

$$C_\sigma \to \mathbb{C}/\Lambda,$$

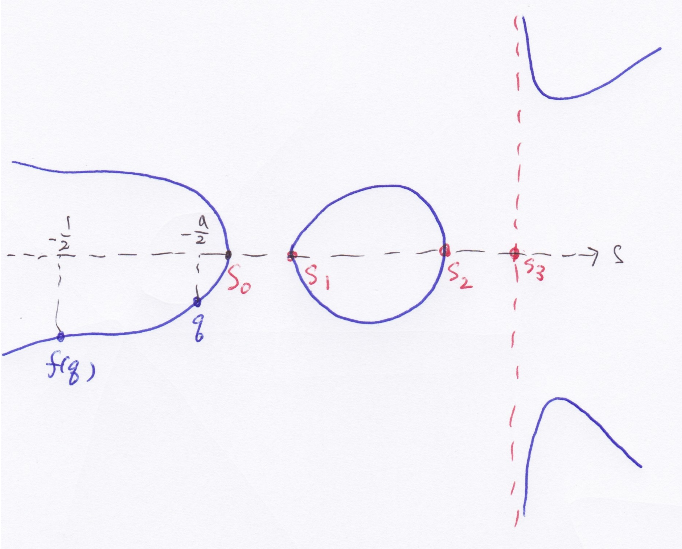
where $\Lambda = \{n\omega_1 + m\omega_2 \mid n, m \in \mathbb{Z}\}$.

As $q = (-a, 0)$, $f(q) = (0, -1)$, and $\pi(q) = -\frac{a}{2}$, $\pi(f(q)) = -\frac{1}{2}$, translation vector μ_1 is given by

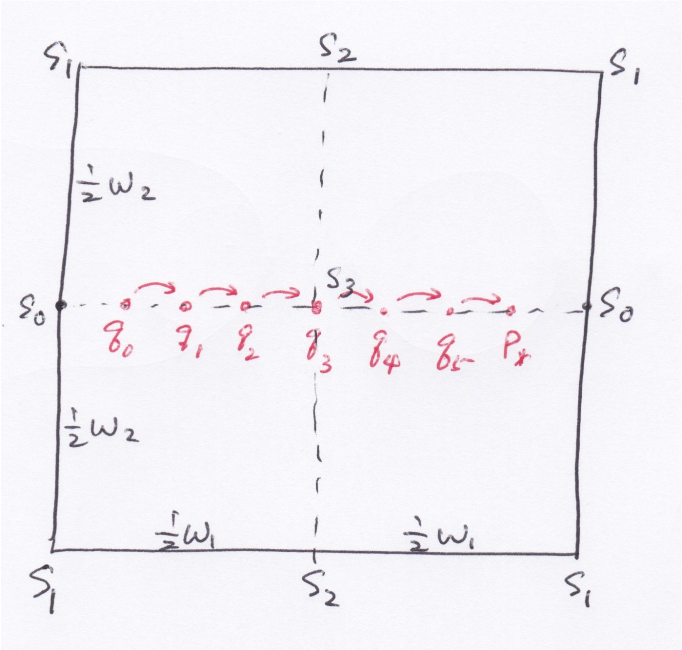
$$\mu_1 = \int_{-\frac{a}{2}}^{-\frac{1}{2}} \frac{ds}{-\sqrt{h_\sigma(s)}}.$$

We have

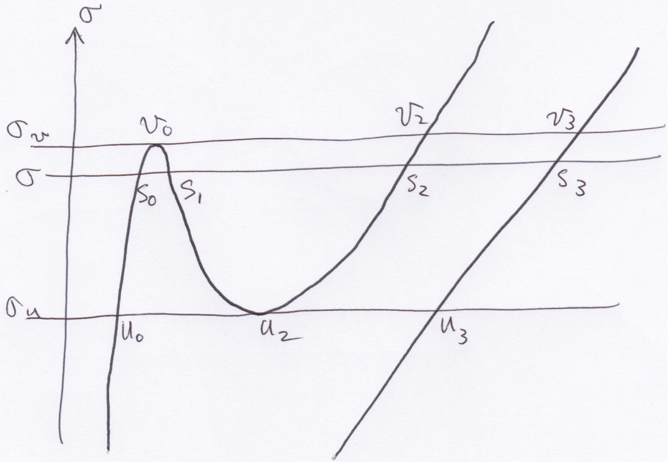
$$\tau = \tau(\sigma) = \frac{\omega_2}{\omega_1} \in \mathbb{H}, \qquad \mu = \mu(\sigma) = \frac{\mu_1}{\omega_1} \in \mathbb{C}.$$



$-\frac{1}{2}$
$-\frac{a}{2}$
s_0
s_1
s_2
s_3
s
q
$f(q)$



S2
S1
S1
1/2 w2
S3
S0
S0
q0
q1
q2
q3
q4
q5
P*
1/2 w2
1/2 w1
1/2 w1
S1
S2
S1



σ
σ_v
σ
σ_u
v_0
s_0
s_1
u_0
u_2
v_2
s_2
u_3
v_3
s_3

8. Dynamics near fixed points

Dynamics near fixed points

Under the assumption $-\frac{1}{4} < a < 0$, we have the following proposition. By abuse of notation, $\psi(\sigma) = \psi(\tau(\sigma), \mu(\sigma))$, $\psi(\tau, \mu) = \frac{\Im(-\frac{\mu}{\tau})}{\Im(-\frac{1}{\tau})}$.

PROPOSITION B.

$$\lim_{\sigma \nearrow \sigma_v} \psi(\sigma) = 0, \qquad \lim_{\sigma \searrow \sigma_u} \psi(\sigma) > 0.$$

PROOF. As $-\frac{1}{4} < a < 0$, and $\sigma_u < \sigma < \sigma_v$, we have

$$-\frac{a}{2} < u_0 < s_0 < v_0 < s_1 < u_2 < s_2 < v_2 < v_3, \quad u_2 < u_3 < s_3 < v_3.$$

Note that $u_0, v_0, u_2, v_2, u_3, v_3$ depend only on a and do not depend on σ.

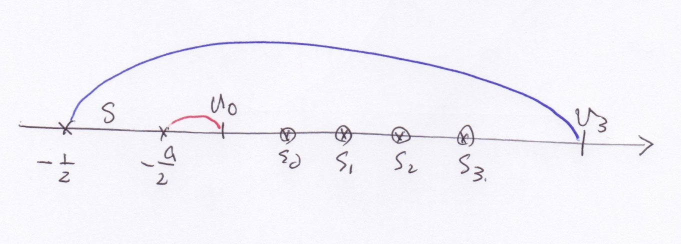
Recall $h_\sigma(s) = (s - s_0)(s - s_1)(s - s_2)(s - s_3)$.
First, we show that the translation vector

$$\mu_1(\sigma) = \int_{-\frac{a}{2}}^{-\frac{1}{2}} \frac{ds}{-\sqrt{h_\sigma(s)}}$$

is uniformly bounded for $\sigma_u < \sigma < \sigma_v$.

If $-\frac{1}{2} < s < -\frac{a}{2}$, then

$$0 < (u_0 + \frac{a}{2})^4 < h_\sigma(s) < (v_3 + \frac{1}{2})^4.$$



Hence,

$$0 < \frac{\frac{1}{2} - \frac{a}{2}}{(v_3 + \frac{1}{2})^2} \leq \int_{-\frac{1}{2}}^{-\frac{a}{2}} \frac{ds}{\sqrt{h_\sigma(s)}} \leq \frac{\frac{1}{2} - \frac{a}{2}}{(u_0 + \frac{a}{2})^2} < \infty.$$

Next, consider the case $\sigma \nearrow \sigma_v$. We have

$$\lim_{\sigma \nearrow \sigma_v} s_0 = \lim_{\sigma \nearrow \sigma_v} s_1 = v_0, \qquad \lim_{\sigma \nearrow \sigma_v} s_2 = v_2, \qquad \lim_{\sigma \nearrow \sigma_v} s_3 = v_3.$$

Let us show that

$$\lim_{\sigma \nearrow \sigma_v} \omega_1 = +\infty.$$

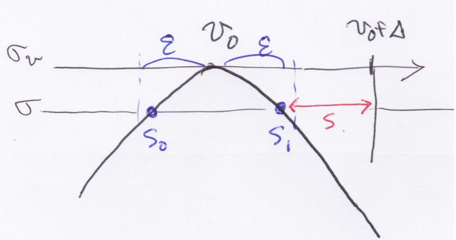
Take $0 < \Delta < \min(v_0 - u_0, u_2 - v_0)$, and fix Δ.
Then take $0 < \varepsilon < \Delta$.
Take σ sufficiently near σ_v so that

$$|s_0 - v_0| < \varepsilon \quad \text{and} \quad |s_1 - v_0| < \varepsilon.$$

Then for $s_1 < s \leq v_0 + \Delta$, we have

$$0 < s - s_0 < s - s_1 + 2\varepsilon, \quad 0 < s - s_1 \leq \Delta,$$

$$0 < s_2 - s \leq v_2 - v_0, \quad 0 < s_3 - s \leq v_3 - v_0.$$



From these inequalities, we have

$$0 < h_\sigma(s) \le (s - s_1 + 2\varepsilon)(s - s_1)(v_2 - v_0)(v_3 - v_0),$$

and

$$\int_{s_1}^{v_0+\Delta} \frac{ds}{\sqrt{h_\sigma(s)}} \ge \frac{1}{\sqrt{(v_2 - v_0)(v_3 - v_0)}} \int_{s_1}^{v_0+\Delta} \frac{ds}{\sqrt{(s - s_1 + 2\varepsilon)(s - s_1)}}$$

$$= \frac{1}{\sqrt{(v_2 - v_0)(v_3 - v_0)}} \int_0^{v_0+\Delta-s_1} \frac{dt}{\sqrt{t(t + 2\varepsilon)}}.$$

Set $\Delta_1 = v_0 + \Delta - s_1$ and

$$I_\varepsilon = \int_0^{\Delta_1} \frac{dt}{\sqrt{t(t + 2\varepsilon)}} = \int_1^{1+\frac{\Delta_1}{\varepsilon}} \frac{du}{\sqrt{u^2 - 1}} = \operatorname{arccosh}(1 + \frac{\Delta_1}{\varepsilon}).$$

Hence,

$$\lim_{\varepsilon \searrow 0} \int_{s_1}^{s_2} \frac{ds}{\sqrt{h_\sigma(s)}} \geq \lim_{\varepsilon \searrow 0} \int_{s_1}^{v_0+\Delta} \frac{ds}{\sqrt{h_\sigma(s)}}$$

$$\geq \frac{1}{\sqrt{(v_2 - v_0)(v_3 - v_0)}} \lim_{\varepsilon \searrow 0} I_\varepsilon = +\infty.$$

So,

$$\lim_{\sigma \nearrow \sigma_v} \omega_1(\sigma) = +\infty, \quad \text{and} \quad \lim_{\sigma \nearrow \sigma_v} \psi(\sigma) = 0.$$

Note that ω_2 remains pure imaginary and bounded in this case, since

$$\lim_{\sigma \nearrow \sigma_v} s_2 = v_2, \qquad \lim_{\sigma \nearrow \sigma_v} s_3 = v_3,$$

and v_2,v_3 are simple roots of h_{σ_v}.

Next, consider the case $\sigma \searrow \sigma_u$.
Let us show that

$$\lim_{\sigma \searrow \sigma_u} \omega_1(\sigma) = \frac{2\pi}{\sqrt{(u_2 - u_0)(u_3 - u_2)}} > 0.$$

Assume, for $\varepsilon > 0$, that

$$\max\{|s_0 - u_0|, |s_1 - u_2|, |s_2 - u_2|, |s_3 - u_3|\} < \varepsilon.$$

Then, for $s_1 < s < s_2$,

$$u_2 - u_0 - 2\varepsilon < s - s_0 < u_2 - u_0 + 2\varepsilon,$$

$$u_3 - u_2 - 2\varepsilon < s_3 - s < u_3 - u_2 + 2\varepsilon.$$

So,

$$(u_2 - u_0 - 2\varepsilon)(u_3 - u_2 - 2\varepsilon)(s - s_1)(s_2 - s) < h_\sigma(s)$$

$$< (u_2 - u_0 + 2\varepsilon)(u_3 - u_2 + 2\varepsilon)(s - s_1)(s_2 - s).$$

And as

$$\int_{s_1}^{s_2} \frac{ds}{\sqrt{(s - s_1)(s_2 - s)}} = \int_{-1}^{1} \frac{dt}{\sqrt{1 - t^2}} = \pi,$$

we have

$$\frac{\pi}{\sqrt{(u_2 - u_0 + 2\varepsilon)(u_3 - u_2 + 2\varepsilon)}} \leq \int_{s_1}^{s_2} \frac{ds}{\sqrt{h_\sigma(s)}}$$

$$\leq \frac{\pi}{\sqrt{(u_2 - u_0 - 2\varepsilon)(u_3 - u_2 - 2\varepsilon)}}.$$

This shows

$$\lim_{\sigma \searrow \sigma_u} \omega_1(\sigma) = 2 \lim_{\sigma \searrow \sigma_u} \int_{s_1}^{s_2} \frac{ds}{\sqrt{h_\sigma(s)}} = \frac{2\pi}{\sqrt{(u_2 - u_0)(u_3 - u_2)}}.$$

Hence

$$\lim_{\sigma \searrow \sigma_u} \psi(\sigma) = \lim_{\sigma \searrow \sigma_u} \frac{\mu_1(\sigma)}{\omega_1(\sigma)} > 0$$

is a finite positive value. This completes the proof of Proposition B.

9. Twist dynamics

Twist dynamics

THEOREM. There exists a parameter a, such that surface automorphism $F_a : X_a \to X_a$ has an orbit which is dense in an elliptic curve in X_a.

PROOF. Take $a \in \mathbb{R}$, $-\frac{1}{4} < a < 0$. By proposition B, real analytic function $\psi(\sigma)$ is non-constant. There is a point σ_0 with $\psi_x^2(\sigma_0) + \psi_y^2(\sigma_0) \neq 0$.

By proposition A, in a neighborhood of σ_0, there exists a value with irrational

$$\varphi(\sigma), \quad \psi(\sigma), \text{ and } \frac{\varphi(\sigma)}{\psi(\sigma)}.$$

10. Weierstraß $\wp$-function

Weierstraß $\wp$-function

For each $a \in \mathbb{C} \setminus \{0,1\}$, and $\sigma \in \hat{\mathbb{C}} \setminus \{0, \infty, a-1, \sigma_{1+}, \sigma_{1-}\}$, there exists a birational map $(s,t) \mapsto (X,Y)$, such that elliptic curve C_σ :

$$t^2(2s + a - \sigma) = (s+1)^2(2s+a) - \sigma s^2$$

is transformed into the Weierstraß normal form :

$$Y^2 = 4X^3 - g_2 X - g_3.$$

Let $\rho = \frac{\sigma - a}{2}$ and $u = s - \rho$. Then

$$t^2 u = u^3 + (2\rho + 2)u^2 + (\rho^2 + 4\rho + a + 1)u + 2\rho^2 + (a+1)\rho + \frac{a}{2}.$$

By setting

$$b_1 = 2\rho + 2, \quad 3b_2 = \rho^2 + 4\rho + a + 1, \quad b_3 = 2\rho^2 + (a+1)\rho + \frac{a}{2},$$

we get

$$t^2 u = u^3 + b_1 u^2 + 3b_2 u + b_3.$$

If $b_3 = 0$, then $\sigma = 0$ or $\sigma = a - 1$. So, under our assumptions, $b_3 \neq 0$.

Set $u = \frac{b_3}{X - b_2}$ to have

$$t^2 \frac{b_3}{X - b_2} = \frac{b_3^3}{(X - b_2)^3} + \frac{b_1 b_3^2}{(X - b_2)^2} + \frac{3b_2 b_3}{X - b_2} + b_3.$$

Multiply $\frac{(X-b_2)^3}{b_3}$ to both sides.

We get

$$t^2(X-b_2)^2 = X^3-(3b_2^2-b_1b_3)X-(b_2^3+3b_2^2+b_1b_2b_3-b_3^2).$$

Let $Y=2t(X-b_2)$, and set $g_2=4(3b_2^2-b_1b_3)$, $g_3=4(b_2^3+3b_2^2+b_1b_2b_3-b_3^2)$, to obtain

$$Y^2=4X^3-g_2X-g_3.$$

Birational transform $(s,t)\leftrightarrow(X,Y)$ is as follows.

$$X=\frac{b_3}{s-\frac{\sigma-a}{2}},\qquad Y=2t(\frac{b_3}{s-\frac{\sigma-a}{2}}-b_2),$$

$$s=\frac{b_3}{X}+\frac{\sigma-a}{2},\qquad t=\frac{Y}{2(X-b_2)}.$$

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