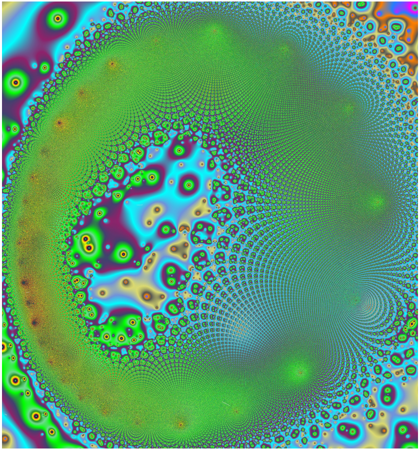
Rational Elliptic Surface without Section (2)



Shigehiro Ushiki Dec. 14, 2022

Abstract

There exist rational elliptic surfaces which don't admit sections.

In [DM](2022), possible multiple fibers for rational elliptic fibrations are described.

We construct concrete examples of rational elliptic surfaces, whose generic fibers are elliptic curves representing cohomology class $-3K$.

Contents

0. Introduction
1. Orbit data $(3,3,3)$*cyclic*
2. Configuration
3. Multiple fibation

0. Introduction

Elliptic surface

Let S be a complex manifold of complex dimension 2. Suppose there is an elliptic fibration onto $\mathbb{P}^1$:

$$\psi : S \to \mathbb{P}^1.$$

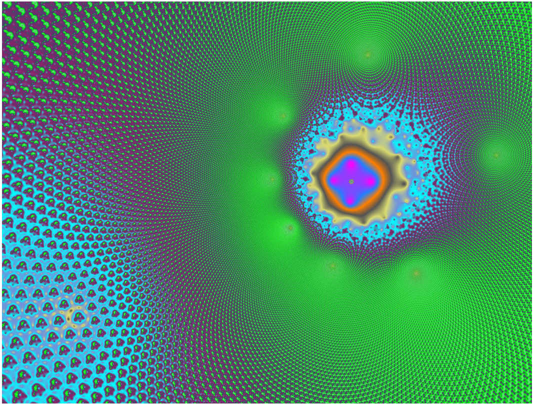
If there is a cross section

$$\sigma : \mathbb{P}^1 \to S, \qquad \psi \circ \sigma = id,$$

we can define Mordell-Weil group $MW(S)$ as the set of all sections.

However, there are elliptic surfaces which don't admit sections.

Picture of a section (QLc153t2B)



Theorem of Gizatullin

Let $F : S \to S$ be an automorphisms of rational surface S.

The **dynamical degree** λ_1 of F is defined as

$$\lambda_1 = \lim_{n\to\infty} ||(F^n)^*||^{1/n}.$$

THEOREM(Gizatullin [1980], Cantat [1999])
Assume $F \in \mathrm{Aut}(S)$, $\lambda_1 = 1$, and $\{||(F^n)^*||\}_{n\in\mathbb{N}}$ is unbounded. Then F preserves an elliptic fibration.

Elliptic fibration

PROPOSITION(Gizatullin[Gi],1980). Let S be a minimal rational elliptic surface. Then for m large enough, we have $\dim|-mK_S| \geq 1$. For m minimal with this property, $|-mK_S|$ is a pencil without base point whose generic fiber is a smooth and reduced elliptic curve.

REMARK(Grivaux[Gr], 2019). The elliptic fibration $S \to |-mK_S|^*$ doesn't have a rational section if $m \geq 2$. Indeed, the existence of multiple fibers ($m\mathcal{D}$) is an obstruction for the existence of a section.

Another elliptic surface

Consider a surface automorphism with invariant elliptic curve of modulus $\epsilon = \exp(\frac{\pi i}{3})$ for orbit data $(3, 3, 3)$, *cyclic*, choosing multiplier $\omega = \exp(\frac{2\pi i}{3})$.

The configuration of the singular fibers is $\mathrm{III}\ \mathrm{I}_1^9$.

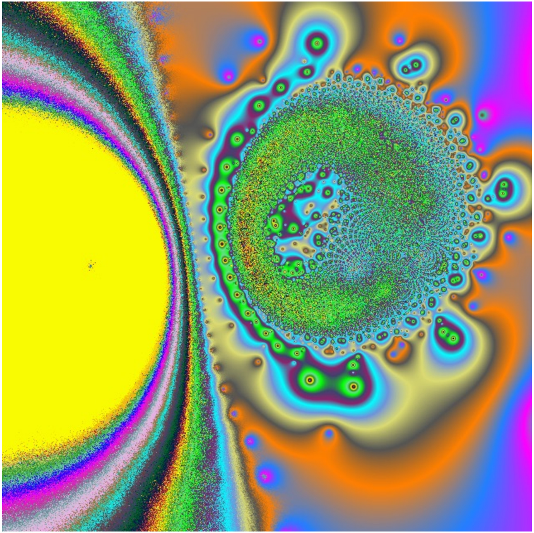
By choosing extra parameters, we find surface automorphisms with

$$\dim|-K| = 0, \quad \dim|-2K| = 0, \quad \dim|-3K| = 1.$$

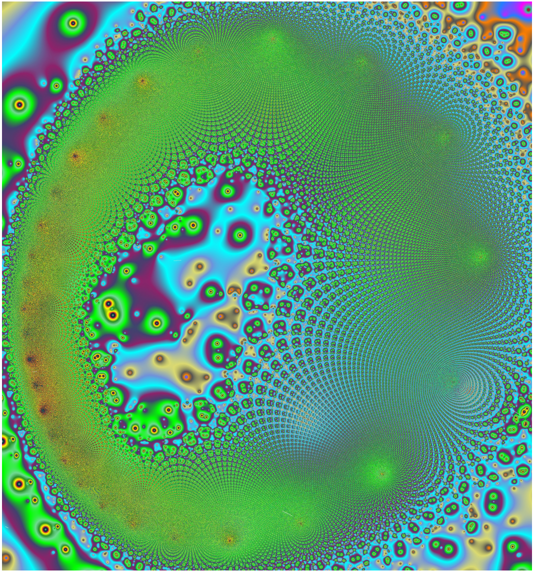
REM. This seems to be the case (a) of theorem 3.3 in [DM] with

$$m = n = 3, p = 0.$$

A (triple) section ? (EWc333b20B)



A (triple) section ? (EWc333b20B)



There exist ...

THEOREM. There exist automorphisms of elliptic surface, induced by quadratic Cremona transformations, such that the elliptic fibration don't admit sections.

Orbit data

1. Orbit data $(3, 3, 3)$, *cyclic*

From orbit data to Cremona transformation

Let $\epsilon = \exp(\pi i/3)$ and let $\Lambda_\epsilon = \mathbb{Z} + \epsilon\mathbb{Z}$.
Let us construct a surface automorphism with

$$\text{orbit data} : (3, 3, 3), \textit{cyclic},$$

$$X \cong \mathbb{C}/\Lambda_\epsilon,$$

and the multiplier for $f|_X$ is ϵ^2.

Suppose the translation of the inner dynamics is $b \in \mathbb{C}/\Lambda_\epsilon$.
And the inner dynamics $f|_X : t \mapsto \epsilon^2 t + b$.

The parametrization of elliptic curve $\{y^2 = 4x^3 - g_2 x - g_3\}$ is given by

$$p(t) = (\wp(t), \wp'(t)), \qquad t \in \mathbb{C}/\Lambda.$$

Parametrization

THEOREM(Diller, 2011) Let $X \subset \mathbb{P}^2$ be an irreducible cubic curve. Suppose we are given points $p(p_1^+), p(p_2^+), p(p_3^+) \in X_{reg}$, a multiplier $a \in \mathbb{C}^\times$, and a translation $b \in \mathbb{C}/\Lambda$. Then there exists at most one quadratic transformation f properly fixing X with $I(f) = \{p(p_1^+), p(p_2^+), p(p_3^+)\}$ and $f(p(t)) = p(at + b)$. This f exists if and only if the following hold.

$p_1^+ + p_2^+ + p_3^+ \not\equiv 0$;

a is a multiplier for X_{reg};

$a(p_1^+ + p_2^+ + p_3^+) \equiv 3b$.

Finally, the points of indeterminacy for f^{-1} are given by $p_j^- = ap_j^+ - 2b$, $j = 1, 2, 3$.

Conditions for orbit data $(n_1, n_2, n_3), \sigma$ are as follows (mod Λ_ϵ).

$$p_{\sigma(j)}^+ \equiv f|_X^{n_j - 1}(p_j^-), \quad j = 1, 2, 3.$$

$X \cong \mathbb{C}/(\mathbb{Z}+\epsilon\mathbb{Z})$

Conditions for orbit data $(3,3,3), \sigma = (1,2,3)$, with multiplier $a = \epsilon^2$, and translation b, are as follows $(\bmod \Lambda_\epsilon)$.

$$p_1^+ + p_2^+ + p_3^+ \equiv -3\epsilon b \not\equiv 0,$$

$$p_1^- \equiv \epsilon^2 p_1^+ - 2b, \quad p_2^- \equiv \epsilon^2 p_2^+ - 2b, \quad p_3^- \equiv \epsilon^2 p_3^+ - 2b,$$

$$p_2^+ \equiv p_1^+ + 3\epsilon b, \quad p_3^+ \equiv p_2^+ + 3\epsilon b, \quad p_1^+ \equiv p_3^+ + 3\epsilon b.$$

From the last three equations, we get

$$9\epsilon b \equiv 0.$$

We put

$$b = \frac{1}{9}\beta, \qquad \beta \in \Lambda_\epsilon.$$

And from

$$p_1^+ + p_2^+ + p_3^+ \equiv 3p_1^+ + 9\epsilon b \equiv -3\epsilon b.$$

We get

$$3p_1^+ \equiv -3\epsilon b.$$

We put

$$3p_1^+ = -3\epsilon b + \alpha, \qquad \alpha \in \Lambda_\epsilon.$$

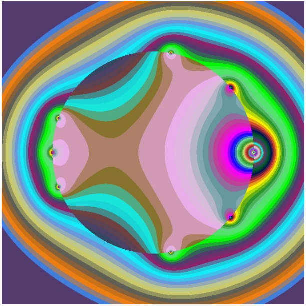
If $-3\epsilon b \not\equiv 0$, then we get solutions :

$$p_1^+ \equiv \frac{8\epsilon}{9}\beta + \frac{1}{3}\alpha, \quad p_2^+ \equiv \frac{2\epsilon}{9}\beta + \frac{1}{3}\alpha, \quad p_3^+ \equiv \frac{5\epsilon}{9}\beta + \frac{1}{3}\alpha,$$

$$p_1^- \equiv \frac{-1}{9}\beta + \frac{\epsilon^2}{3}\alpha, \quad p_2^- \equiv \frac{-4}{9}\beta + \frac{\epsilon^2}{3}\alpha, \quad p_3^- \equiv \frac{-7}{9}\beta + \frac{\epsilon^2}{3}\alpha.$$

If $\beta \in \Lambda_\epsilon \setminus 3\Lambda_\epsilon$, we have $-3\epsilon b \equiv -\frac{1}{3}\epsilon\beta \not\equiv 0$, and we can construct a surface automorphism.

Let $F_{\alpha,\beta} : S_{\alpha,\beta} \to S_{\alpha,\beta}$ denote our surface automorphism.

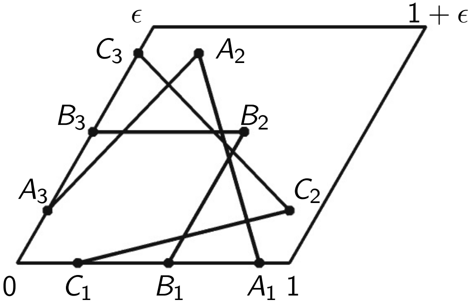


Remark. The characteristic polynomial for orbit data $(3, 3, 3)$, *cyclic* is

$$P(\lambda) = (\lambda - 1)^2(\lambda^2 - 1)(\lambda^6 + \lambda^3 + 1),$$

and ϵ^2 is not an eigenvalue.

Base points for $\alpha = 0, \beta = 1$



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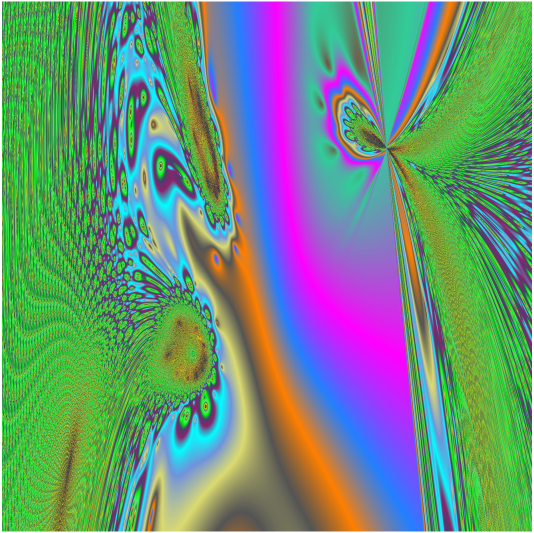


2. Configuration

Now, let $A_i \in H^2(S, \mathbb{Z})$ denote the cohomology class of the exceptional fiber $[\pi^{-1}(f^{i-1}(p(p_1^-)))]$, $i = 1, 2, 3$. Let $B_i = [\pi^{-1}(f^{i-1}(p(p_2^-)))]$, $i = 1, 2, 3$, and $C_i = [\pi^{-1}(f^{i-1}(p(p_3^-)))]$, $i = 1, 2, 3$.

Let $H \in H^2(S, \mathbb{Z})$ denote the class of a generic line $[\pi^{-1}(L)]$.
A basis of $H^2(S, \mathbb{Z})$ is given by classes

$$H, A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3, .$$

Automorphism $F^* : H^2(S, \mathbb{Z}) \to H^2(S, \mathbb{Z})$ acts as follows.

$$H \mapsto 2H - A_3 - B_3 - C_3,$$

$$A_3 \mapsto A_2 \mapsto A_1 \mapsto H - A_3 - B_3,$$

$$B_3 \mapsto B_2 \mapsto B_1 \mapsto H - B_3 - C_3,$$

$$C_3 \mapsto C_2 \mapsto C_1 \mapsto H - A_3 - C_3.$$

Periodic roots of positive degree

Let

$$\mathcal{X} = 3H - A_1 - A_2 - A_3 - B_1 - B_2 - B_3 - C_1 - C_2 - C_3$$

denote the class of anticanonical curve, represented by our invariant elliptic curve $X \cong \mathbb{C}/(\mathbb{Z} + \epsilon\mathbb{Z})$.

A class $\mathcal{R} \in H^2(S, \mathbb{Z})$ is said to be **a root of positive degree** if

$$\mathcal{R} \cdot \mathcal{X} = 0, \quad \mathcal{R}^2 = -2, \quad \mathcal{R} \cdot H > 0.$$

The characteristic polynomial for orbit data $(3, 3, 3)$, *cyclic* is

$$P(\lambda) = (\lambda - 1)^2(\lambda^2 - 1)(\lambda^6 + \lambda^3 + 1).$$

If there is a periodic root, the period is 1, 2, or 9.

Period 1 and 2

We have

$$\begin{aligned}\mathrm{Ker}(F^* - id) &= <\mathcal{X}>,\\ \mathrm{Ker}(F^{*2} - id) &= <\mathcal{L}, \mathcal{Q}> .\end{aligned}$$

where

$$\mathcal{L} = H - A_2 - B_2 - C_2,$$

$$\mathcal{Q} = 2H - A_1 - A_3 - B_1 - B_3 - C_1 - C_3.$$

We have

$$F^*\mathcal{L} = \mathcal{Q}, \quad F^*\mathcal{Q} = \mathcal{L},$$

$$\mathcal{L}^2 = \mathcal{Q}^2 = -2, \qquad \mathcal{L} \cdot \mathcal{Q} = 2,$$

$$\mathcal{L} + \mathcal{Q} = \mathcal{X}.$$

These are roots of positive degree.

Another periodic root of period 2

There exists another 2-cycle of roots of positive degree.

$$\mathcal{U} = \mathcal{L} + \mathcal{X},$$

$$\mathcal{V} = \mathcal{Q} + \mathcal{X},$$

with

$$F^*\mathcal{U} = \mathcal{V}, \quad F^*\mathcal{V} = \mathcal{U},$$

$$\mathcal{U}^2 = \mathcal{V}^2 = -2, \qquad \mathcal{U} \cdot \mathcal{V} = 2.$$

Moreover,

$$\mathcal{U} + \mathcal{V} = 3\mathcal{X}.$$

Singular fiber

If these roots are nodal and there exist curves representing these classes, they form a singular fiber of type I_2 or III.

To decide the type, recall the Lefschetz formula:

$$\sum_{f(p)=p} \operatorname{sign}(\det(Df_p - I)) = \sum_{i=0}^{\dim M} (-1)^i \operatorname{trace}(f_*|_{H_i(M,\mathbb{R})}).$$

To describe periodic cycles in terms of Lefschetz index, for $m \in \mathbb{N}$ and $k \in \mathbb{Z}$, let

$$\mathbf{m}(k) = \begin{cases} m & k \equiv 0 \pmod{m} \\ 0 & \text{otherwise} \end{cases} .$$

Periodic points

Recall the characteristic polynomial for orbit data $(3, 3, 3)$, *cyclic* :

$$P(\lambda) = (\lambda - 1)^2(\lambda^2 - 1)(\lambda^6 + \lambda^3 + 1).$$

The Lefschetz number $\Lambda(F^k)$ is expressed as

$$\Lambda(F^k) = \mathbf{1} + \mathbf{1} + \mathbf{1} + \mathbf{1} + \mathbf{2} - \mathbf{3} + \mathbf{9}.$$

The invariant elliptic curve $X \cong \mathbb{C}/(\mathbb{Z} + \epsilon\mathbb{Z})$, with inner dynamics $t \mapsto \epsilon^2 t + b$, has three fixed points. The inner dynamics is period three, and these periodic points are not counted in the Lefschetz number if $k \equiv 0 \pmod 3$.

So, the periodic points in X are counted as $\mathbf{1} + \mathbf{1} + \mathbf{1} - \mathbf{3}$. The cycle of period 9 comes from singular fibers I_1^9, obtained later.

The periodic points in the curves of period two are described by $\mathbf{1} + \mathbf{2}$, that is, the type of the singular fiber is III.

Roots of period 9

$$\mathrm{Ker}(F^{*9} - id) =< H - A_1 - A_2 - A_3,$$

$$A_1 - B_1, \quad A_2 - B_2, \quad A_3 - B_3,$$

$$A_1 - C_1, \quad A_2 - C_2, \quad A_3 - C_3 >$$

Their Picard projection of these roots do not vanish. These roots (and the roots in this subspace) are not nodal.

$$\widetilde{A_1 - B_1} \equiv \frac{1}{3}\beta,$$

$$H - \widetilde{A_1 - A_2} - A_3 \equiv \frac{-1 - \epsilon}{9}\beta,$$

$\cdots$ *e.t.c.*

3. Multiple fibration

Picard projection

For Rational surface, following commutative diagram holds.

$$
\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & \mathrm{Pic}(S) & \xrightarrow{c_1} & H^2(S,\mathbb{Z}) & \longrightarrow & 0, \\
 & & & & \downarrow r & & \downarrow \iota^* & & \\
0 & \to & \mathrm{Pic}_0(X) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{\deg} & H^2(X,\mathbb{Z}) & \longrightarrow & 0.
\end{array}
$$

In our case, $X \cong \mathbb{C}/(\mathbb{Z}+\epsilon\mathbb{Z})$ is an elliptic cubic curve,

$$\mathrm{Pic}_0(X) \simeq \mathbb{C}/\Lambda_\epsilon.$$

For $\mathcal{P} \in H^2(S,\mathbb{Z})$, with $\iota^*(\mathcal{P}) = 0$, we denote

$$\widetilde{\mathcal{P}} = r \circ c_1^{-1}(\mathcal{P}) \in \mathrm{Pic}_0(X).$$

We say $\widetilde{\mathcal{P}}$ is the **Picard projection** of $\mathcal{P}$.

Nodality

If $\mathcal{P} \in H^2(S, Z)$ is a cohomology class of a (strict transform of a curve $C \subset \mathbb{P}^2$, then

$$\iota^*(\mathcal{P}) = 0 \text{ and } r \circ c_1^{-1}(\mathcal{P}) = 0.$$

With our choice of Picard coordinates, we have the following fact.

THEOREM. $3d$ (not necessarily distinct) points $p_1, \cdots, p_{3d} \in X_{reg}$ comprise the intersection of X with a curve of degree d if and only if

each irreducible $V \subset X$ contains $d \cdot \deg V$ of the points; and
$\sum p_j \sim 0$.

Nodal periodic roots

For our automorphism $F_{\alpha,\beta} : S_{\alpha,\beta} \to S_{\alpha,\beta}$, the Picard projections of periodic roots of positive degree can be computed as follows $\pmod{\Lambda_\epsilon}$.

$$b = \frac{1}{9}\beta, \qquad \alpha \in \Lambda_\epsilon, \quad \beta \in \Lambda_\epsilon \setminus 3\Lambda_\epsilon,$$

$$\widetilde{\mathcal{L}} \equiv \widetilde{\mathcal{Q}} \equiv \frac{1+\epsilon}{3}\beta, \quad \widetilde{\mathcal{X}} \equiv \frac{2+2\epsilon}{3}\beta,$$

$$\widetilde{\mathcal{U}} \equiv \widetilde{\mathcal{V}} \equiv 3\widetilde{\mathcal{X}} \equiv 0.$$

So, we conclude that if $\frac{1+\epsilon}{3}\beta \equiv 0$ then the singular fiber of type III is a cubic curve consisting of a conic and a tangent line.

And if $\frac{1+\epsilon}{3}\beta \not\equiv 0$, then singular fiber of type III comprises a quartic curve and a quintic curve intersecting at a point. In this case $\mathcal{X}$ cannot be the class of generic fibers.

Configuration of sungular fibers

We conclude that the configuration of singular fibers in our case is :

$$\mathrm{III}\ \mathrm{I}_1^9.$$

And the generic fiber represents the class

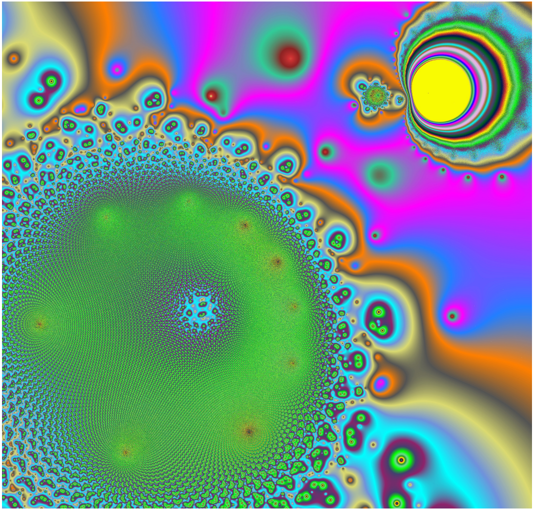
$$3\mathcal{X}.$$

Persson's list of configurations

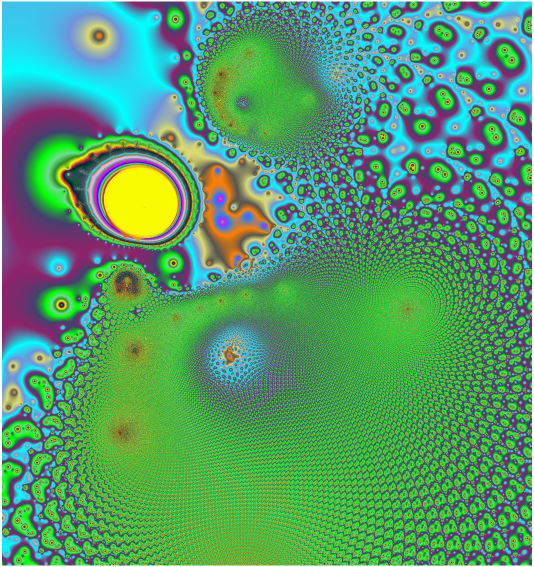
In the list of configurations of singular fibers given by Persson([P],1990), those containing I_9 or I_1^9 are :

$$\mathrm{III}\ \mathrm{I}_1^9, \quad \mathrm{I}_9\ \mathrm{I}_1^3, \quad \mathrm{I}_3\ \mathrm{I}_1^9.$$

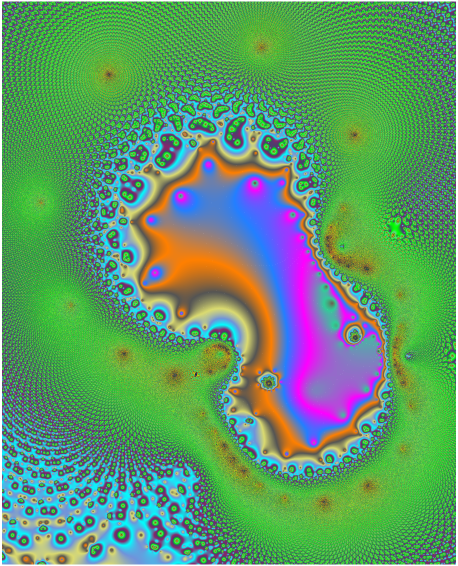
Multiple section (?) (EWc333b10B)



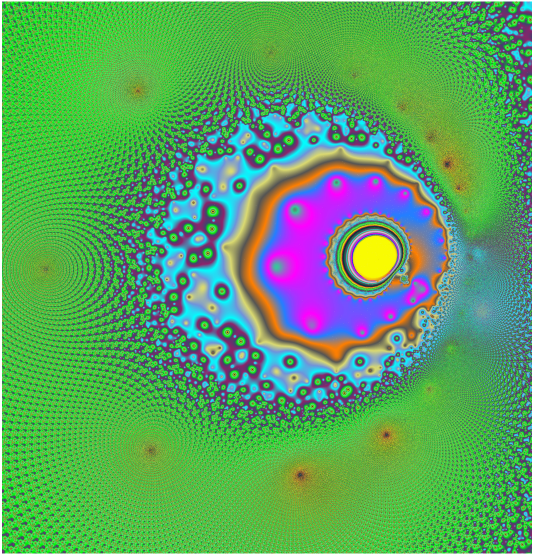
Another multiple section (?) (EWc333b10C)



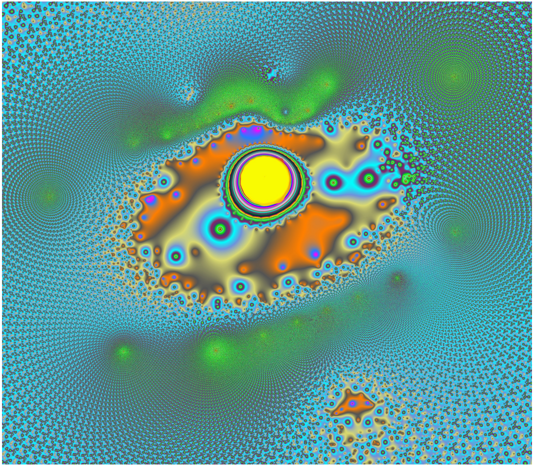
Diagonal slice (EWc333b10D)



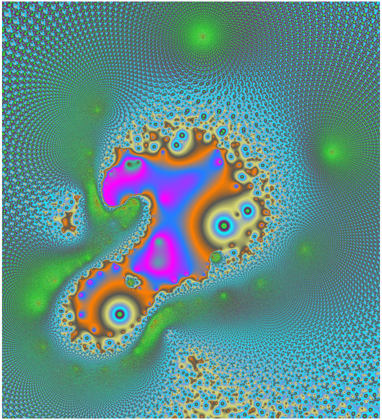
Multiple section (?) (EWc333b12B)



Another multiple section (?) (EWc333b12C)



Diagonal slice (EWc333b12D)

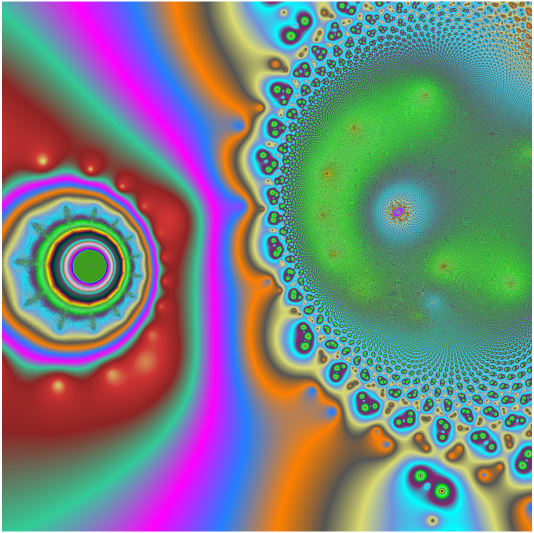


Thank you !

EWc333a00b10W

F_v	J	d	r	e	p
$3\mathrm{I}_0$	0	9	8	0	$3\mathbf{1} - \mathbf{3}$
III	1	9	8	3	$\mathbf{1} + \mathbf{2}$
I_1^9	∞	9×1	0	9×1	$\mathbf{9}$

Picture of a section (EWc333a00b11B)



Multiple fiber

THEOREM. (Dolgachev-Martin,[DM]2022) Let $f : X \to B$ be a genus one surface with jacobian $J(f) : J(X) \to B$ and let $\mathrm{Aut}_f(X)$ be the group of automorphisms of X preserving f. Assume that f is cohomologically flat. Then there is a homomorphism $\varphi : \mathrm{Aut}_f(X) \to \mathrm{Aut}_{J(f)}(J(f))$ satisfying the following properties, where $g \in \mathrm{Aut}_f(X)$:
(1) Both g and $\varphi(g)$ induce the same automorphism of B.
(2) $\mathrm{Ker}(\varphi) \cong \mathrm{MW}(J(f))$.
(3) $\varphi(g)$ preserves the zero section of $J(f) : J(X) \to B$.
(4) If g acts trivially on $\mathrm{Num}(X)$, then $\varphi(g)$ acts trivially on $\mathrm{Num}(J(X))$.
(5) Let mF_0 be a fiber of f of multiplicity m and let $(J_0^\sharp)^0$ be the identity component of the smooth part $J_0^\sharp$ of the corresponding fiber J_0 of $J(f)$, then either $\varphi(g)$ acts trivially on $(J_0^\sharp)^0$ or one of the following holds, where $n = \mathrm{ord}(\varphi(g)|_{(J_0^\sharp)^0})$:

- (a) F_0 is smooth, $m = n = 3, p \neq 3$.
- (b) F_0 is smooth, $m = 2, n \in \{2, 4\}, p \neq 2$.
- (c) F_0 is smooth and ordinary, $m = n = p = 2$.
- (d) F_0 is an irreducible nodal curve, $m = n = 2, p \neq 2$.
- (e) F_0 is of type $\tilde{A}_1, m = n = 2, p \neq 2$.

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