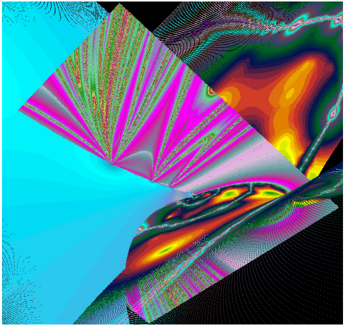
Julia set of a surface automorphism of positive entropy — an example of $J^* \neq J$



Shigehiro Ushiki, Kyoto

December 8, 2018

Abstract

We consider a surface automorphism $f : X \to X$ of positive entropy.

Rational surface X of our example is constructed by blowing-up the complex projective space $\mathbb{P}^2$ in a finite number of points.

In this note, we prove that in our example, the Julia set, the closure of the set of saddle points, and the support of the invariant measure of maximal entropy are different.

Contents

1. Julia set

Fatou set

Let $f : X \to X$ be an automorphism of a compact complex manifold X.

A point $p \in X$ is a point of the **forward Fatou set** F_f^+ if there exists an open neighborhood U of p on which the sequence $\{f^n\}_{n\in\mathbb{N}}$ forms a normal family of holomorphic mappings from U to X.

Define the **backward Fatou set** F_f^- and the **Fatou set** F_f by

$$F_f^- = F_{f^{-1}}^+, \qquad F_f = F_f^+ \cap F_f^- .$$

Julia set

Define the **forward Julia set** J_f^+, the **backward Julia set** J_f^-, and the **Julia set** J_f as follows.

$$J_f^+ = X \setminus F_f^+, \quad J_f^- = X \setminus F_f^-, \text{ and } \; J_f = J_f^+ \cap J_f^-.$$

Let J_f^* denote the closure of the set of saddle periodic points.

Clearly, $J_f^* \subset J_f$.

Loxodromic automorphism

Let f be an automorphism of a compact Kähler surface X.
Let $H^{1,1}(X,\mathbb{R}) = H^{1,1}(X,\mathbb{C}) \cap H^2(X,\mathbb{R})$.
Then, $f^* : H^{1,1}(X,\mathbb{R}) \to H^{1,1}(X,\mathbb{R})$ is an automorphism preserving the intersection pairing.

Define the **dynamical degree** λ_f by

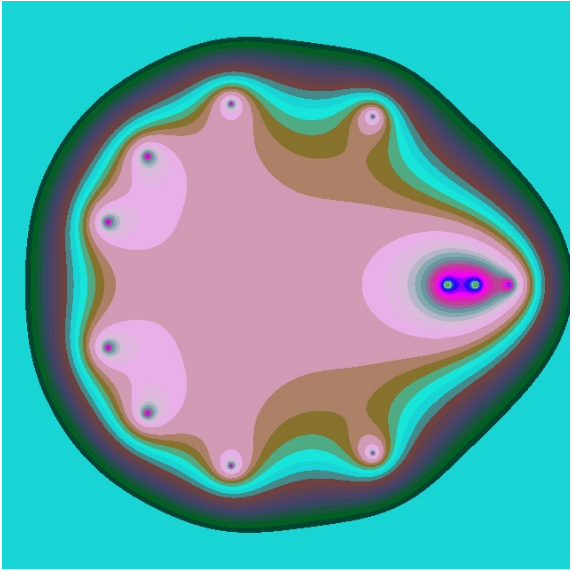
$$\lambda_f = \lim_{n\to\infty} ||f^{*n}||^{\frac{1}{n}}.$$

THEOREM. If $\lambda_f > 1$, then λ_f is an eigenvalue of f^* with multiplicity 1, and it is the unique eigenvalue with modulus > 1.

If $\lambda_f > 1$, then λ_f^{-1} is an eigenvalue, too. Other eigenvalues are of modulus 1.

f is said to be **loxodromic** if $\lambda_f > 1$.

Eigenvalues of a loxodromic automorphism



Invariant currents and invariant measures

Let f be a loxodromic automorphism of a compact Kähler surface X.

THEOREM(Cantat 2001, Dinh-Sibony 2005). There exist positive, closed currents T_f^+ and T_f^- with invariance property

$$f^* T_f^+ = \lambda_f T_f^+ \quad \text{and} \quad f^* T_f^- = \lambda_f^{-1} T_f^- .$$

We obtain an invariant measure $\mu_f = T_f^+ \wedge T_f^-$.

THEOREM(Bedford-Lyubich-Smilie 1993,Cantat 2003). Let $\Lambda(f,k)$ denote the set of saddle periodic points of f of period k. Then

$$\frac{1}{\lambda_f^k} \sum_{p \in \Lambda(f,k)} \delta_p$$

converges to μ_f as k goes to ∞.

Ahlfors current

Let $\xi : \mathbb{C} \to X$ be an entire curve.

$$A(r) = \int_{t=0}^{r} \int_{\theta=0}^{2\pi} ||\xi'(te^{i\theta})||_{\kappa}^{2}\, t\, d\theta\, dt,$$

$$N(r) = \frac{\int_0^r [\xi(\mathbb{D}_t)] \frac{dt}{t}}{\int_0^r A(t) \frac{dt}{t}}.$$

THEOREM. Let X be a compact Kähler surface with Kähler form κ. Let $\xi : \mathbb{C} \to X$ be an entire curve. There exist sequences of radii (r_n) going to ∞ such that $(N(r_n))$ converge toward a closed positive current T. If $\xi(\mathbb{C})$ is not contained in a compact curve, then $[T]$ intersects all classes of curves positively, and $\langle [T], [T] \rangle \geq 0$.

Main Theorem

If f is a loxodromic automorphism of a compact Kähler surface X, then

$$\text{supp}(\mu_f) \subset J_f^* \subset J_f.$$

THEOREM(U.). There exists a loxodromic automorphism f of a compact Kähler surface X such that

$$\text{supp}(\mu_f) \neq J_f^* \neq J_f.$$

REMARK.
If f is a Hénon map, then $\text{supp}(\mu_f) = J_f^* \subset J_f$.
If f is a hyperbolic Hénon map, then $\text{supp}(\mu_f) = J_f^* = J_f$.

2. Surface automorphism

Surface automorphism (Bedford-Kim, McMullen)

Define birational automorphism $\varphi_{a,b} : \mathbb{C}^2 \to \mathbb{C}^2$, for parameter $a, b \in \mathbb{C}$, by

$$\varphi_{a,b} : (x, y) \mapsto (y, \frac{y+a}{x+b} + b).$$

$\varphi_{a,b}$ has an indeterminacy point $p_* = (-b, -a)$.
Its "image" is line $\{x = -a\}$.
$\varphi_{a,b}$ has a pole along line $\{x = -b\}$.
Its "image" is point $e_y = [0 : 0 : 1] \in \mathbb{P}^2$.
$\varphi_{a,b}$ is critical along line $\{y = -a\}$.
Its image is $q_* = (-a, b)$.

Surface automorphism (V_n family)

Define

$$V_n = \{(a,b) \in \mathbb{C}^2 \mid \varphi^j_{a,b}(q_*) \neq p_* \text{ for } 0 \leq j < n, \text{ and } \varphi^n_{a,b}(q_*) = p_*\}.$$

THEOREM(Bedford-Kim 2006).

$\varphi_{a,b}$ is birationally conjugate to an automorphism of a compact complex surface, $X_{a,b}$, if and only if $(a,b) \in V_n$ for some $n \geq 0$. The surface $X_{a,b}$ is obtained by blowing up the projective plane $\mathbb{P}^2$ at the $n+3$ points

$$e_x = [0:1:0], \quad e_y = [0:0:1], \text{ and } \varphi^j_{a,b}(q_*), \ 0 \leq j \leq n.$$

Surface automorphism (Blow up)

Let E_x, E_y, Q_j denote the exceptional fibers above points e_x, e_y, $\varphi^j_{a,b}(q_*)$, respectively, of blow up $\pi : X_{a,b} \to \mathbb{P}^2$.

Exceptional curves are mapped as :

$$\{x = -b\} \Rightarrow E_y \Rightarrow \text{(Line at infinity)} \Rightarrow E_x \Rightarrow \{y = b\},$$

$$\{y = -a\} \Rightarrow Q_0 \Rightarrow Q_1 \Rightarrow \cdots \Rightarrow Q_n \Rightarrow \{x = -a\}.$$

$\varphi_{a,b}$ induces a surface automorphism $f_{a,b} : X_{a,b} \to X_{a,b}$.

Surface automorphism (Cohomology)

The cohomology group $H^2(X_{a,b})$ is generated by the class of generic line H, and the classes of exceptional fibers E_x, E_y, $Q_0, \cdots, Q_n$.

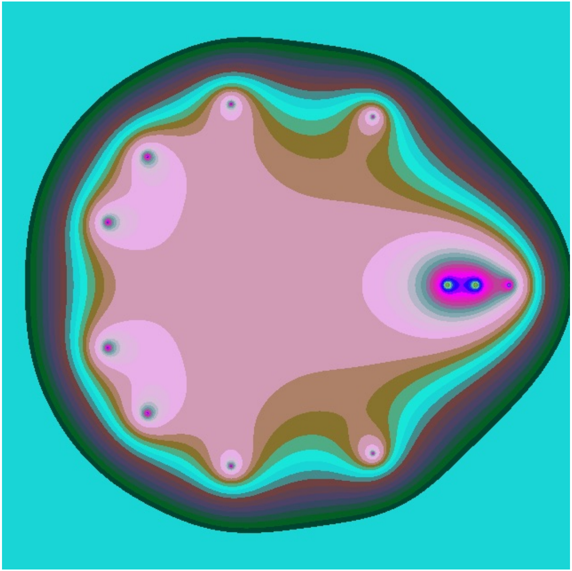
Surface automorphism $f_{a,b} : X_{a,b} \to X_{a,b}$ induces an isomorphism $f^*_{a,b} : H^2(X_{a,b}) \to H^2(X_{a,b})$.

THEOREM(Bedford-Kim, 2006).
If $(a, b) \in V_n$, then the characteristic polynomial of $f^*_{a,b}$ is

$$\chi_n(x) = x^{n+1}(x^3 - x - 1) + x^3 + x^2 - 1.$$

If $n \geq 7$, then the largest real root of χ_n is greater than 1.

Eigenvalues of a loxodromic automorphism



3. Invariant cubic curve

Γ_1-family

In our family of rational automorphisms

$$\varphi_{a,b} : (x, y) \mapsto (y, \frac{y + a}{x + b} + b),$$

we make a very special choice of parameters.

$$a = \frac{-1 - t + 2t^3 - t^5 - t^6}{2t^2(1 + t)^2}, \qquad b = \frac{1 - t^5}{2t^2(1 + t)}.$$

THEOREM(Bedford-Kim 2006).
$(a, b) \in V_n$ if and only if $\chi_n(t) = 0$.

Γ_1 family

In this case, $f_{a,b}$ has an invariant cubic curve C with a cuspidal point. The fixed points of $f_{a,b}$ are in the invariant cubic curve.

Eigenvalues of the jacobian matrix at cuspidal fixed point, say R, are t^2 and t^3.

Eigenvalues of the jacobian matrix at the other fixed point, say S, are t^{-1} and t^n.

The dynamics in the invariant curve is conjugate to $\zeta \mapsto t\zeta$, with correspondence $\zeta = 0 \leftrightarrow R$, $\zeta = \infty \leftrightarrow S$.

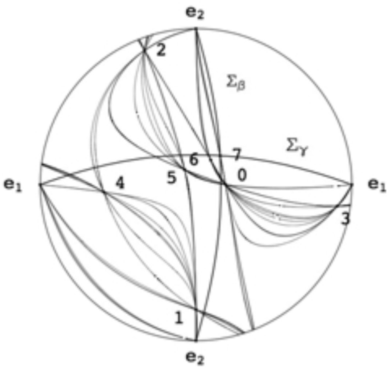
The polynomial h which defines the invariant cubic curve is derived from the following equation.

$$h \circ \varphi_{a,b} = t \det(D\varphi_{a,b})\ h.$$

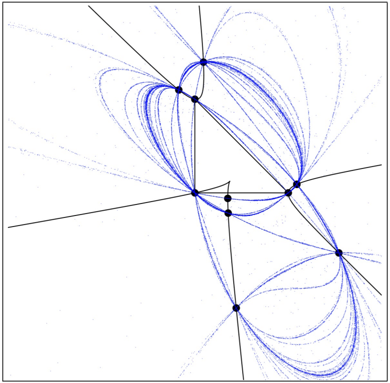
Meromorphic 2-form $\eta = \frac{dx \wedge dy}{h}$ satisfies $\varphi^*_{a,b}\eta = t^{-1}\eta$.

The determinant of a periodic point of period p is t^{-p}, except the fixed points.

Real slice

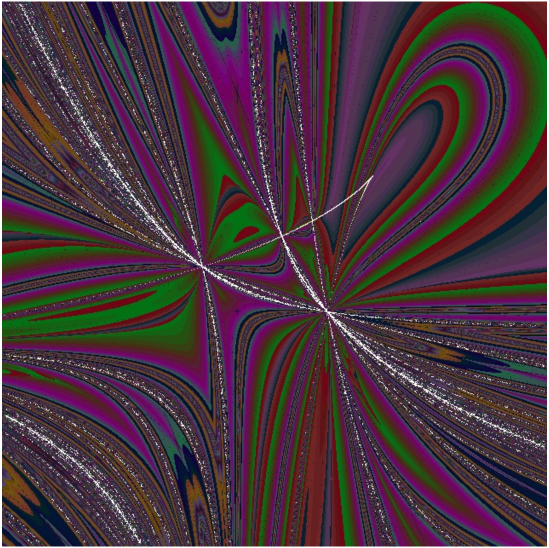


[Bedford-Kim 2006]



[McMullen 2005]

Real slice



4. Fixed points

Let λ_n denote the largest real root of χ_n.

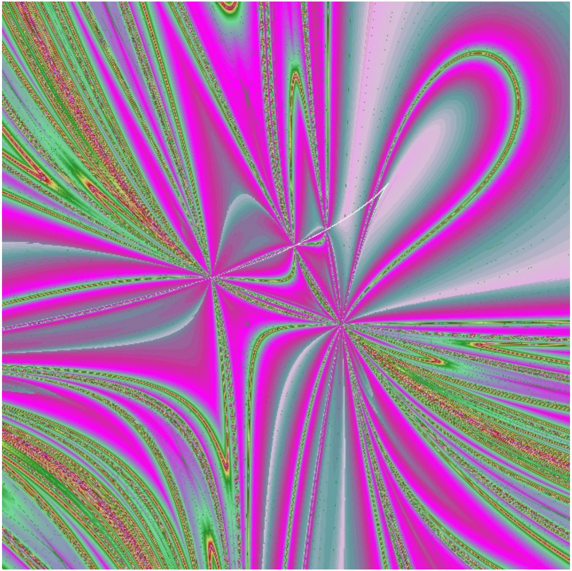
If $n \geq 7$, then $\lambda_n > 1$.

We take an $n \geq 7$, and we set $t = \lambda_n$ in the following. Parameters a and b are suppressed in the following.

$$\lambda_7 = \lambda_{\text{Lehmer}} \approx 1.17628081$$

Fixed point R is a repeller as the eigenvalues are t^2 and t^3.

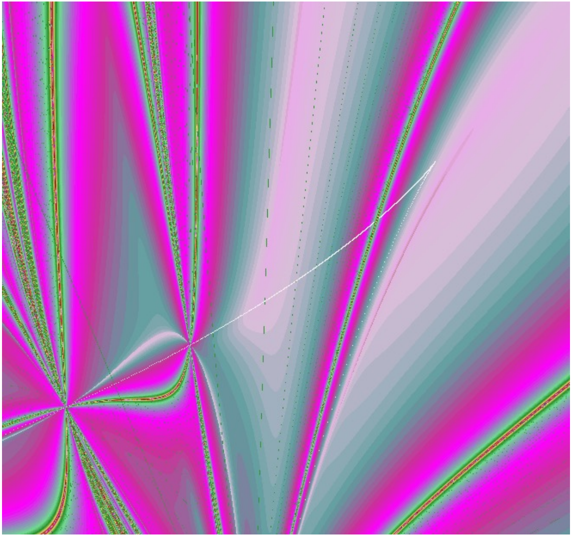
PROPOSITION. The fixed point R is linearizable. The unstable manifold $W^u(R)$ is isomorphic to $\mathbb{C}^2$. $W^u(R)$ is an invariant open set in X.

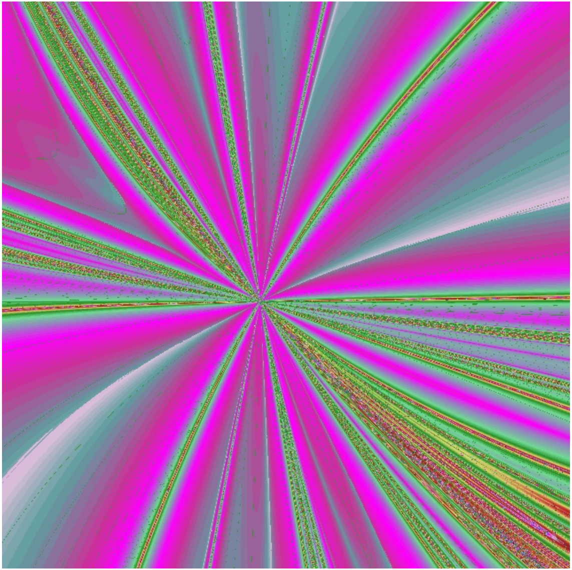


Fixed point S is a saddle as the eigenvalues are t^{-1} and t^n.

PROPOSITION. The stable manifold $W^s(S)$ is isomorphic to $\mathbb{C}$ and coincides with the strict transform of the cubic curve C in X with R omitted.

$$W^s(S) = C \setminus \{R\}$$





5. Proof of Main Theorem

Proof of the main Theorem.

By our choice of parameters, we have $t = \lambda_n = \lambda_f > 1$.

As the cubic curve C is invariant and the dynamics is conjugate to $\zeta \mapsto \lambda_f \zeta$, with correspondence $\zeta = 0 \leftrightarrow R$, $\zeta = \infty \leftrightarrow S$, we have

$$C \setminus \{S\} \subset W^u(R).$$

There exits a neighborhood V of S such that $V \setminus W^u(S) \subset W^u(R)$.

There is no homoclinic point of S, $W^u(S) \cap W^s(S) = \{S\}$.

There exists an open set U such that

$$W^u(S) \subset U \quad \text{and} \quad U \setminus W^u(S) \subset W^u(R).$$

Let $A := W^u(S) \cup W^u(R)$. A is an open invariant set.

Let $B := X \setminus A$. B is a closed invariant set.

f has many saddle periodic points.

THEOREM(Katok 1980).

Let f be a loxodromic automorphism of a compact Kähler surface X. The set of saddle periodic points of f is Zariski dense in X. The number $N(f,k)$ of saddle periodic points of f of period at most k grows like λ_f^k :

$$\limsup \frac{1}{k} \log(N(f,k)) \geq \log(\lambda_f).$$

Let Λ denote the set of saddle periodic points in X.

And let $\Lambda' = \Lambda \setminus \{S\}$.

As S is the only one periodic saddle point in A, we see that $\Lambda' \subset B$.

Since B is a closed invariant set, we have $\overline{\Lambda'} \subset B$.

THEOREM(Bedford-Lyubich-Smilie 1993,Cantat 2003). Let $\Lambda(f,k)$ denote the set of saddle periodic points of f of period k. Then

$$\frac{1}{\lambda_f^k} \sum_{p \in \Lambda(f,k)} \delta_p$$

converges to μ_f as k goes to ∞.

We see that

$$\mathrm{supp}(\mu_f) \subset \overline{\Lambda'} \subset B.$$

Since $W^u(S) \subset A$, we have $\text{supp}(\mu_f) \cap W^u(S) = \phi$.
So, $\quad S \notin \text{supp}(\mu_f)$.
With $S \in J_f^*$, we get

$$\text{supp}(\mu_f) \neq J_f^*.$$

The existence of a heteroclinic point :

$$W^u(S) \cap W^s(\overline{\Lambda'}) \neq \phi$$

is proved by looking at the construction of current T_f^- by the Ahlfors current of entire functions of unstable manifolds.

The normalized pushforwards $\lambda_f^{-n}[f^n(D^u)]$ of a disk $D^u \subset W^u(S)$ converge to a nonzero multiple of T_f^-. Then $T_f^+ \wedge \lambda_f^{-n}[f^n(D^u)]$ converge to a nonzero multiple of the measure μ_f. It follows that $T_f^+ \wedge \lambda_f^{-n}[f^n(D^u)]$ must be nonzero for some n.

This implies $W^u(S) \cap W^s(\overline{\Lambda'}) \neq \phi$.

And $W^u(S) \cap W^s(\overline{\Lambda'}) \subset J_f$.

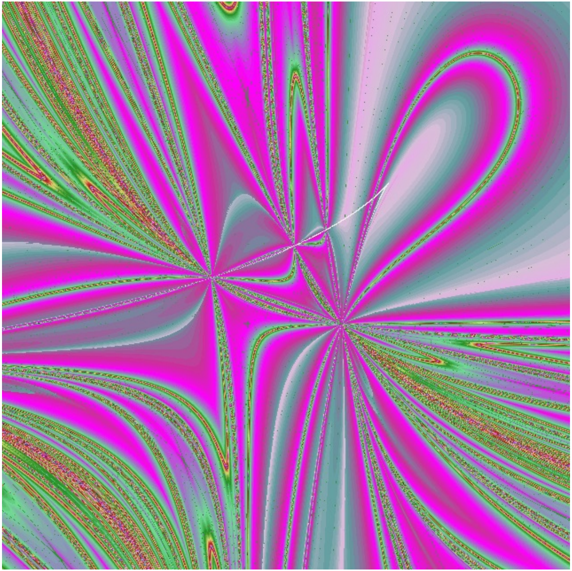
As $J_f^* \cap W^u(S) = \{S\}$, $J_f^* \cap W^u(S) \cap W^s(\overline{\Lambda'}) = \phi$.

We get

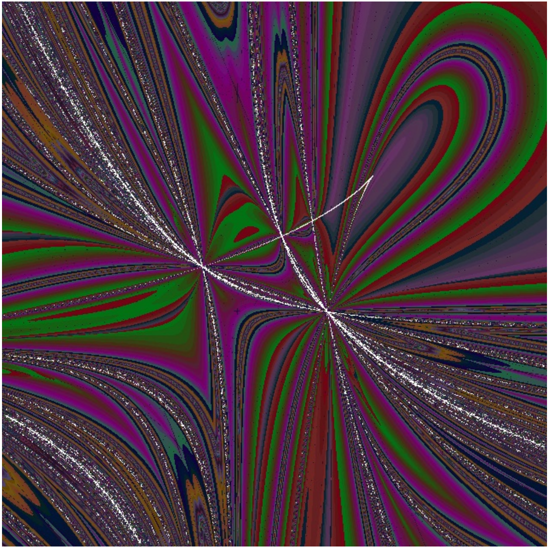
$$J_f^* \neq J_f.$$

6. Pictures

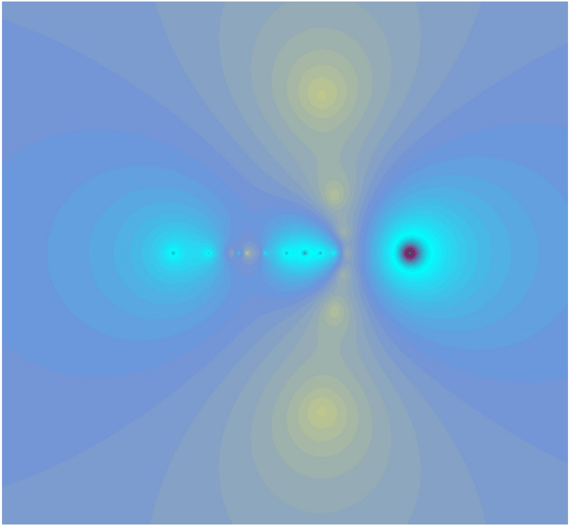
Real slice, colored by backward iteration



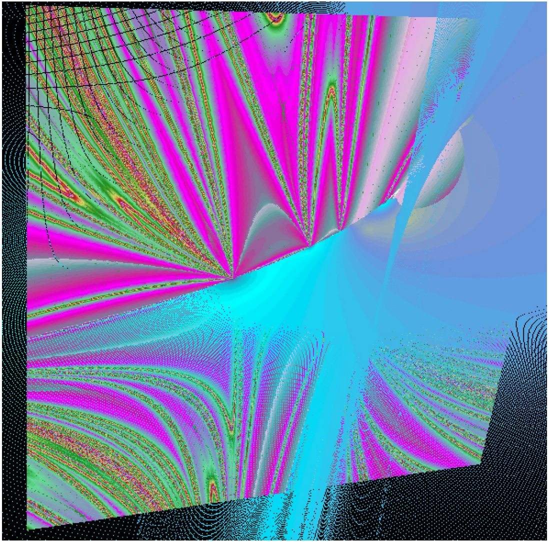
Real slice with Julia set



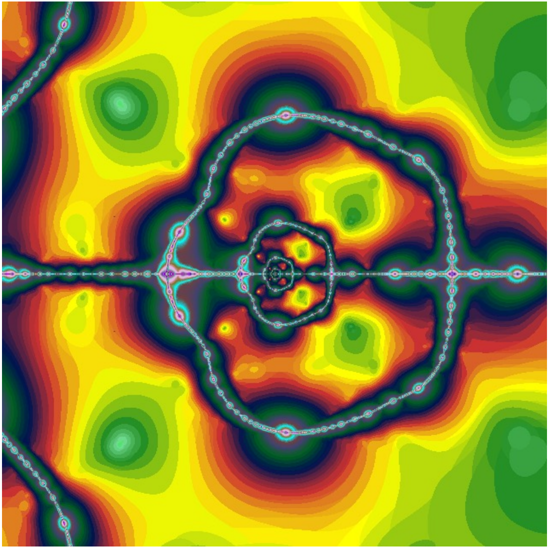
$W^s(S)$, colored by backward iteration



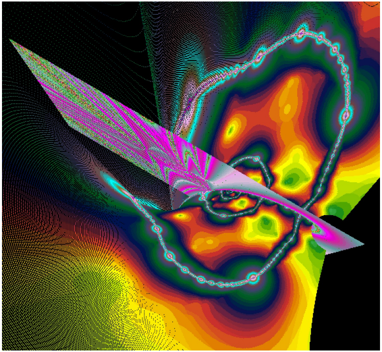
Real slice and $W^s(S)$

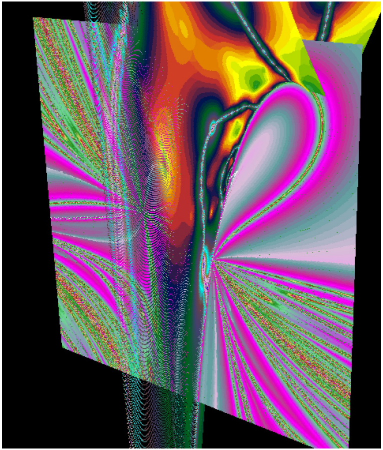


$W^u(S)$, colored by forward iteration

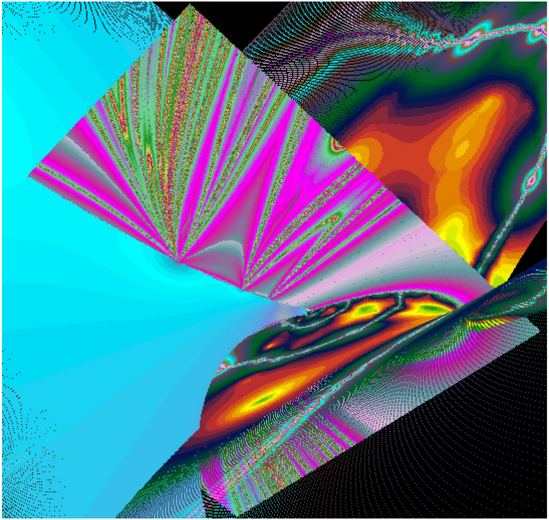


$W^u(S)$, and real plane

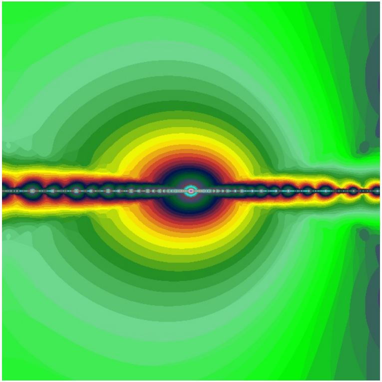


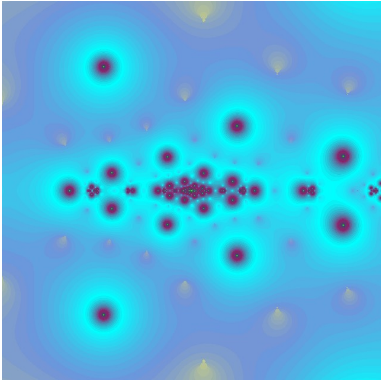


$W^u(S)$, $W^s(S)$ and real plane

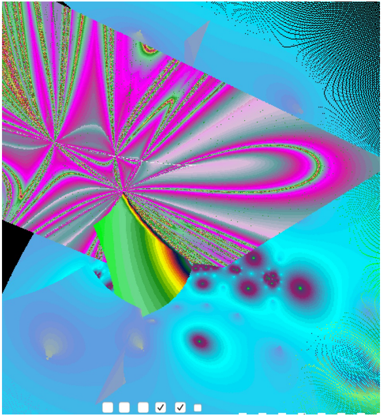


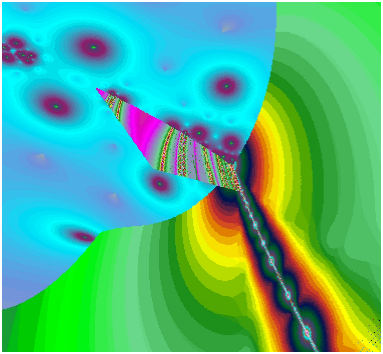
$W^u(P_3)$ and $W^s(P_3)$, $P_3 \in \Lambda$



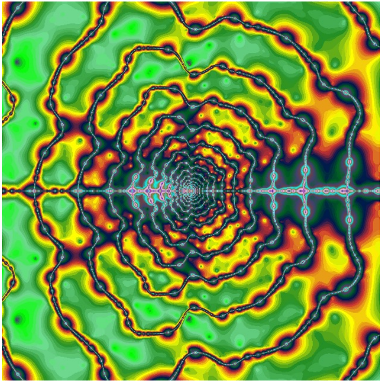
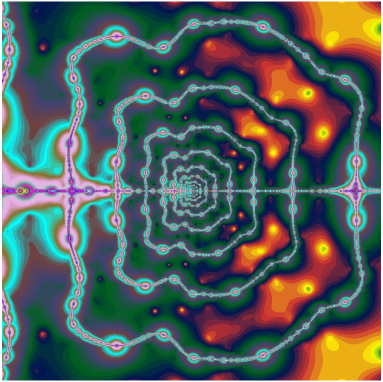


$W^u(P_3)$, $W^s(P_3)$ and real plane

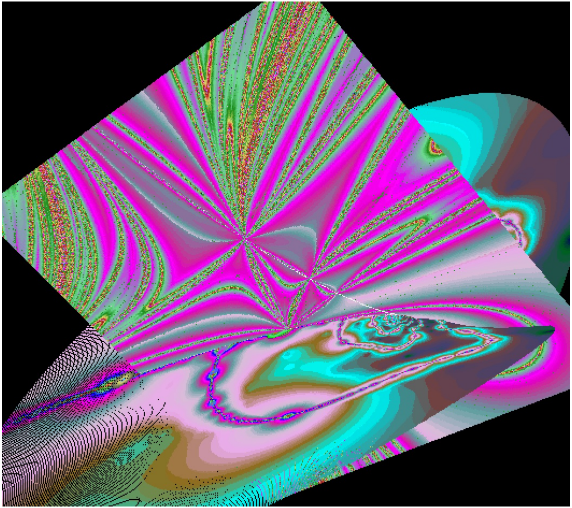




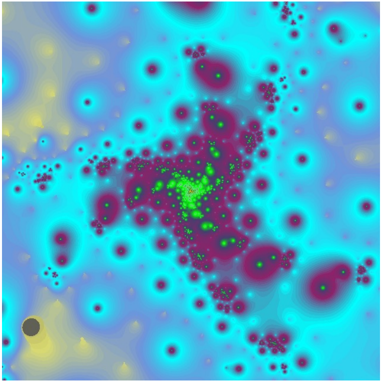
$W^u(R)$

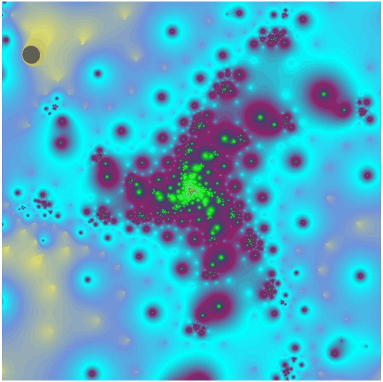
 

$W^u(R)$ and real plane

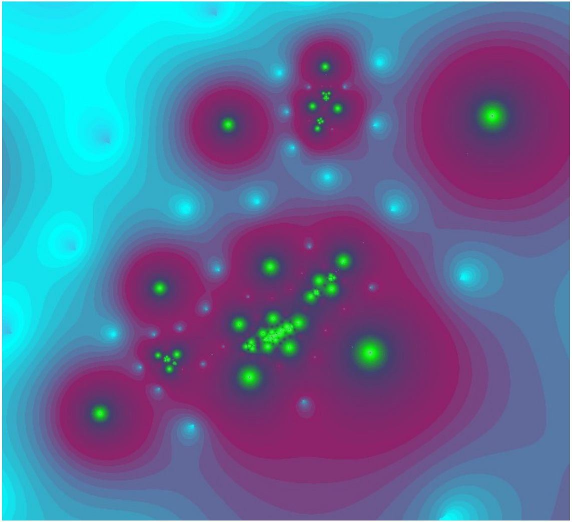


$W^s(A_2)$, A_2 is the 2-cycle attractor

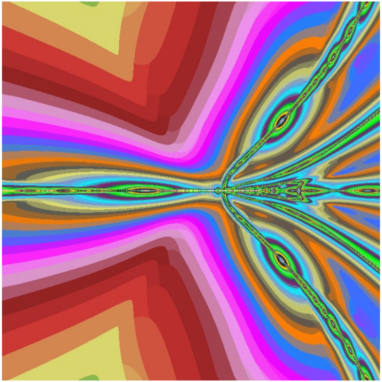


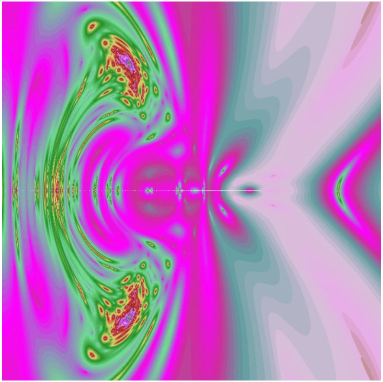


$W^s(A_2)$

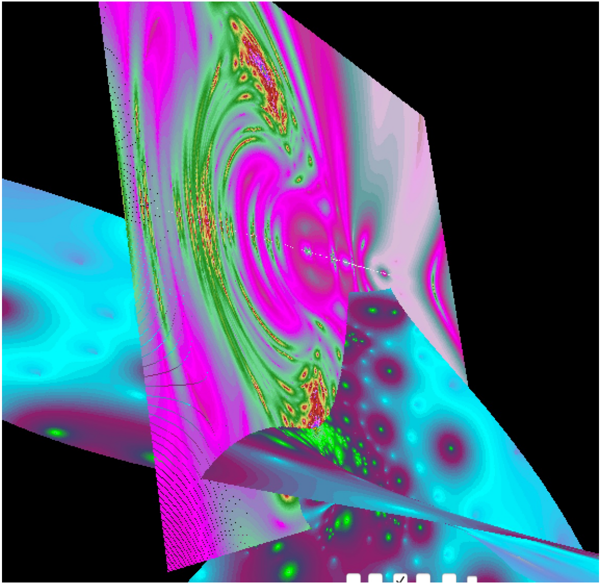


Conjugate diagonal slice, forward and backward iteration





$W^s(A_2)$ and conjugate diagonal plane



Γ_2-family

Same results hold for Γ_2-family and Γ_3-family of surface automorphisms. Γ_2-family is given by

$$a = \frac{1 + t + 2t^2 + t^3 + t^4}{2t(1+t)^2}, \qquad b = \frac{t^3 - 1}{2t(1+t)}.$$

THEOREM(Bedford-Kim 2006).
Assume $n \geq 8$ is even. Then $(a, b) \in V_n$ if and only if $\chi_n(t) = 0$.

In this case, $f_{a,b}$ has an invariant cubic curve consisting of a line and a quadric tangent to each other in a point. A fixed point of $f_{a,b}$ and a cycle of period 2 are in the invariant cubic curve. Eigenvalues of the jacobian matrix at the fixed point, are $-t$ and $-t^2$. Eigenvalues of the jacobian matrix of the cycle of period 2, are t^{-2} and t^{n+2}. The dynamics in the invariant curve is conjugate to $\zeta \mapsto t\zeta$, with correspondence $\zeta = 0 \leftrightarrow$ fixed point, $\zeta = \infty \leftrightarrow$ 2-cycle.

Γ_3-family

Γ_3-family is given by

$$a = \frac{(1+t)^2}{2t}, \qquad b = \frac{t^2-1}{2t}.$$

THEOREM(Bedford-Kim 2006).
Assume $n \geq 9$ is a multiple of 3. Then $(a,b) \in V_n$ if and only if $\chi_n(t) = 0$.

In this case, $f_{a,b}$ has an invariant cubic curve consisting of three lines intersecting in a point. A fixed point of $f_{a,b}$ and a cycle of period 3 are in the invariant cubic curve. Eigenvalues of the jacobian matrix at the fixed point, are ωt and $\omega^2 t$. Eigenvalues of the jacobian matrix of the cycle of period 3, are t^{-3} and t^{n+3}. The dynamics in the invariant curve is conjugate to $\zeta \mapsto t\zeta$, with correspondence $\zeta = 0 \leftrightarrow$ fixed point, $\zeta = \infty \leftrightarrow$ 3-cycle.

Appendix

Numerical observations

$$F^+ = W^s(A_2) \approx \mathbb{C}^2 \sqcup \mathbb{C}^2,$$

$$F^- = W^u(R) \approx \mathbb{C}^2,$$

$$F = W^s(A_2) \cap W^u(R),$$

$$J^+ = \overline{\{R\} \cup W^s(S) \cup W^s(\Lambda)},$$

$$J^- = \overline{A_2 \cup W^u(\Lambda) \cup W^u(S)},$$

$$J = \overline{\Lambda \cup (W^u(S) \cap W^s(\Lambda))}.$$

Entropy

THEOREM(Diller-Kim 2017). Let $f : X \to X$ be an automophism as above. And let $f_{\mathbb{R}} : X_{\mathbb{R}} \to X_{\mathbb{R}}$ be its restriction to the real sub surface. Then $h(f) = h(f_{\mathbb{R}})$.

THEOREM(Bedford-Lyubich-Smilie 1993, Cantat 2001) Let X be a smooth projective surface over the real numbers $\mathbb{R}$. Let f be an automorphism of X defined over $\mathbb{R}$. The entropy of $f : X(\mathbb{R}) \to X(\mathbb{R})$ is equal to the entropy of $f : X(\mathbb{C}) \to X(\mathbb{C})$ if and only if all saddle periodic points of f which are not contained in rational periodic curves are contaioned in $X(\mathbb{R})$.

COROLLARY. There is no saddle periodic point of f in $X \setminus X_{\mathbb{R}}$.

Cycles of period ≤ 30

Fixed points : 1 repeller(R), 1 saddle(S)
Period 2 : 1 attracting cycle(A_2)
Period 3 : 1 saddle cycle(P_3)
Period 5 : 1 saddle cycle
Period 7 : 1 saddle cycle
Period 11 : 1 saddle cycle
Period 13 : 1 saddle cycle
Period $16 \leq k \leq 22$: 1 saddle cycle
Period $23 \leq k \leq 26$: 2 saddle cycles
Period 27, 28 : 3 saddle cycles
Period 29, 30 : 4 saddle cycles

Ahlfors current

Let $\xi : \mathbb{C} \to X$ be an entire curve.

$$A(r) = \int_{t=0}^{r} \int_{\theta=0}^{2\pi} ||\xi'(te^{i\theta})||_{\kappa}^{2} \, t \, d\theta \, dt,$$

$$N(r) = \frac{\int_0^r [\xi(\mathbb{D}_t)] \frac{dt}{t}}{\int_0^r A(t) \frac{dt}{t}}.$$

THEOREM. Let X be a compact Kähler surface with Kähler form κ. Let $\xi : \mathbb{C} \to X$ be an entire curve. There exist sequences of radii (r_n) going to ∞ such that $(N(r_n))$ converge toward a closed positive current T. If $\xi(\mathbb{C})$ is not contained in a compact curve, then $[T]$ intersects all classes of curves positively, and $\langle [T], [T] \rangle \geq 0$.

Fixed-point formulas

LEFSCHETZ'S FORMULA

$$\sum_{f(p)=p} \mathrm{sign}(\det(Df_p - Id)) = \sum_{k=0}^{\dim_{\mathbb{R}} X} (-1)^k \mathrm{trace}(f^*_{|H^k(X,\mathbb{R})}).$$

ATIYAH-BOTT FORMULA

For $r = 0, \cdots, \dim_{\mathbb{C}} X$,

$$\sum_{f(p)=p} \frac{\mathrm{trace}(\wedge^r Df_p)}{\det(Id - Df_p)} = \sum_{s=0}^{\dim_{\mathbb{C}} X} (-1)^s \mathrm{trace}(f^*_{|H^{r,s}(X,\mathbb{C})}).$$

Trace

The characteristic polynomial of $f^*_{|H^2}$ is :

$$\chi_n(x) = x^{n+1}(x^3 - x - 1) + x^3 + x^2 - 1 = \sum_{j=0}^{n+4} a_j x^{n+4-j}.$$

Let $\lambda_1, \cdots, \lambda_{n+4}$ be the roots of χ_n.

$$\text{trace}(f^*_{|H^2}) = \sum_{j=1}^{n+4} \lambda_j = a_1 = 0.$$

$$\text{trace}(f^{*2}_{|H^2}) = \sum_{j=1}^{n+4} \lambda_j^2 = a_1^2 - 2a_2 = 2.$$

$$\text{trace}(f^{*3}_{|H^2}) = \sum_{j=1}^{n+4} \lambda_j^3 = a_1^3 - 3a_1 a_2 - 3a_3 = 3.$$

References

References

[BK1] E. Bedford and K. Kim : Periodicities in Linear Fractional Recurrences : Degree Growth of Birational Surface Maps, Michigan Math., 647-670, J. 54 (2006).

[BK2] E. Bedford and K. Kim : Dynamics of Rational Surface Automorphisms: Linear Fractional Recurrences, J. Geom. Anal., 553-583, 19(2009).

[BLS] E.Bedford, M.Lyubich, J.Smilie : Polynomial diffeomorphismsms of $\mathbb{C}^2$. IV. The measure of maximal entropy and laminar currents. Invent. Math. 112(1), 77-125, 1993.

[C1] S. Cantat : Dynamique des automorphismes des surface $K3$, Acta. Math. 187(1), 1-57, 2001.

[C2] S. Cantat : Dynamics of automorphisms of compact complex surfaces, In *Frontiers in complex dynamics*, volume 51 of *Princeton Math. Ser.*, pp 463-514, Princrton Univ. Press, Princeton, NJ, 2014.

[DK] J.Diller and K.Kim : Entropy of rational surface automorphisms, arXiv:1710.07665v1, [math.DS] 2017.
[DS] T.-C. Dinh and Nessim Sibony : Green currents for holomorphic automorphisms of compact Kähler manifolds. J. Amer.Math.Soc., 18(2):291-312, 2005.
[K] A.Katok : Lyapunov exponents, entropy and periodic orbits for diffeomorphisms. Publ. Math. Inst. Hautes Études Sci., (51):137-173, 1980.
[M] C. T. McMullen : Dynamics on blowups of the projective plane. Publ. Math. Inst. Hautes Études Sci., (105):49-89, 2007.