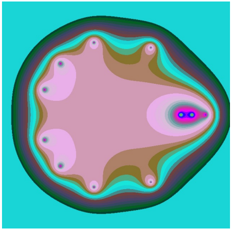
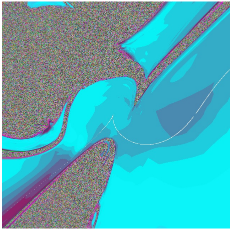
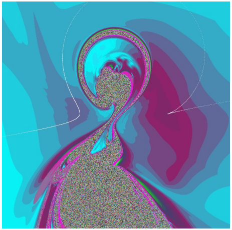
An elementary approach to invariant cubic curves of surface automorphisms







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Abstract

Rational surface is constructed by blowing-up the complex projective space $\mathbb{CP}^2$ in several points. Under certain conditions, birational map from $\mathbb{CP}^2$ to itself can be extended to the rational surface and defines an automorphism of the rational surface.

Such surface automorphisms are studied by E.Bedford, KH.Kim and C.McMullen. Some of them have invariant cubic curves.

In this note, we derive the explicit formula of invariant cubic curves by an elementary computation executable by hand.

Linear fractional recurrences

Birational map studied in [BK1],[BK2],

$$f_{a,b}(x,y) = (y, \frac{y+a}{x+b})$$

Birational map studied in [M],

$$f_{a,b}(x,y) = (a,b) + (y, \frac{y}{x})$$

Self-anti-conjugate map

$$f(x,y) = (y, \frac{y+\alpha}{x+i\beta} + i\beta),$$

Intrinsic parametrization of birational maps

These families of birational maps are equivalent under correspondence of parameters and change of coordinates.

When there exists an invariant cubic curve, the curve contains one or two of fixed points. And the cubic curve has a singularity there. So, we suppose the origin is a fixed point where the invariant cubic has a singularity, and choose the trace τ and the determinant δ of the Jacobian matrix at the fixed point, as our parameters of birational maps. Our family is as follows.

$$F(u, v) = (v, \frac{\tau v - \delta u}{\tau u + 1}).$$

Equation of the invariant cubic curve

Let $P(u,v)$ be a cubic polynomial which defines a cubic curve with a singularity at the origin.

The equation to be satisfied is as follows. (Formulated and solved in [BK2].)

$$(*) \qquad P \circ F(u,v) = tP(u,v)\det DF_{(u,v)}$$

for some $t \in \mathbb{C}$.

Rem: This equation is for meromorphic eigenform $\eta = \frac{du \wedge dv}{P(u,v)}$, with $tF^*\eta = \eta$.

Invariant cubic curves

Assume $\delta \neq 0$, $\tau \neq 0$, $t \neq 0, \pm 1$.

THEOREM A. The equation of invariant cubic polynomial has solutions (up to a constant multiple) in the following cases.

(Γ_1) $\tau = t^2 + t^3$, $\delta = t^5$,

$$P_1(u, v) = uv(tu + v) + \frac{1 + t + t^2}{1 + t}(tu - \frac{v}{t})^2.$$

(Γ_2) $\tau = -t - t^2$, $\delta = t^3$,

$$P_2(u, v) = (tu + v)((1 + t)uv + tu + v).$$

(Γ_3) $\tau = -t$, $\delta = t^2$,

$$P_3(u, v) = uv(tu + v).$$

In these three cases, invariant cubic curves were computed in [BK2]. But the formulas are complicated and not easy to understand.

(Γ_1) Irreducible cubic with a cusp.

(Γ_2) Line tangent to a quadric.

(Γ_3) Three lines passing through a point.

Uniformization

Theorem B. Uniformizing functions can be taken as follows.

$$(\Gamma_1)\ \ \psi_C(\zeta) = \left(\frac{\xi_0\zeta^2}{(\zeta+1)(\zeta+t)}, \frac{\xi_0\zeta^2}{(\zeta+1)(\zeta+\frac{1}{t})} \right), \ \xi_0 = \frac{(1-t)(t^3-1)}{t^2}.$$

$$(\Gamma_2)\ \ \psi_L(\zeta) = \left(\frac{\zeta}{\zeta+1}\frac{1-t}{t}, \frac{\zeta}{\zeta+1}(t-1) \right),$$

$$\psi_Q(\zeta) = \left(\frac{t^{-1}\zeta}{t^{-1}\zeta+1}(t-1), \frac{t\zeta}{t\zeta+1}\frac{1-t}{t} \right),$$

$$(\Gamma_3)\ \ \psi_0(\zeta) = \left(\frac{\zeta}{\zeta+1}\frac{t^3-1}{t^2}, \frac{\zeta}{\zeta+1}\frac{1-t^3}{t} \right),$$

$$\psi_1(\zeta) = \left(\frac{t^{-1}\zeta}{t^{-1}\zeta+1}\frac{1-t^3}{t}, 0 \right), \quad \psi_2(\zeta) = \left(0, \frac{t\zeta}{t\zeta+1}\frac{t^3-1}{t^2} \right).$$

Schröder's equations

THEOREM C.

$$(\Gamma_1) \qquad F \circ \psi_P(\zeta) = \psi_P(t\zeta).$$

$$(\Gamma_2) \qquad F \circ \psi_L(\zeta) = \psi_Q(t\zeta),$$

$$F \circ \psi_Q(\zeta) = \psi_L(t\zeta).$$

$$(\Gamma_3) \qquad F \circ \psi_0(\zeta) = \psi_1(t\zeta),$$

$$F \circ \psi_1(\zeta) = \psi_2(t\zeta),$$

$$F \circ \psi_2(\zeta) = \psi_0(t\zeta).$$

Indeterminate points

The indeterminate point of F is $p = (\frac{-1}{\tau}, \frac{-\delta}{\tau^2})$, and the indeterminate point of F^{-1} is $q = (\frac{-\delta}{\tau^2}, \frac{-\delta}{\tau})$.
Let $\zeta_p = \frac{t}{t^3 - t - 1}, \ \ \zeta_q = \frac{t^2}{1 - t^2 - t^3}$.

THEOREM D.

$$(\Gamma_1) \qquad \psi_P(\zeta_p) = p, \quad \psi_P(\zeta_q) = q.$$

$$(\Gamma_2) \qquad \psi_Q(\zeta_p) = p, \quad \psi_Q(\zeta_q) = q.$$

$$(\Gamma_3) \qquad \psi_0(\zeta_p) = p, \quad \psi_0(\zeta_q) = q.$$

eigenvalues

THEOREM E. In these cases,

$$F^n(q) = p \quad \text{if and only if} \quad \zeta_p = t^n \zeta_q.$$

REM. $\chi_n(x) = x^{n+1}(x^3 - x - 1) + (x^3 + x^2 - 1)$ is the characteristic polynomial of $F^* : H^{1,1}(X) \to H^{1,1}(X)$. Here, when $F^n(q) = p$, X is a compact complex surface obtained by blowing up the projective plane $\mathbb{CP}^2$ at the $n+3$ points $e_1 = [0:1:0], e_2 = [0:0:1]$ and $F^j(q), 0 \le j \le n$.

$$\text{If } \zeta_p = t^n \zeta_q, \text{ then } \chi_n(t) = 0.$$

Coxeter element and Siegel ball

THEOREM. For all n sufficiently large, the standard Coxeter element $w \in W_n$ can be realized by a surface automorphism with a cycle of Siegel balls.

REM. McMullen proved the following.

THEOREM(McMullen, 2005). For all n sufficiently large with $n \neq 2, 4 \bmod 6$, the standard Coxeter element $w \in W_n$ can be realized by a surface automorphism with a Siegel disk.

Proof of Theorem A

Let

$$P(u, v) = C_0 u^3 + C_1 u^2 v + C_2 uv^2 + C_3 v^3 + D_0 u^2 + D_1 uv + D_2 v^2.$$

As $F(u, v) = (v, \frac{\tau v - \delta u}{\tau u + 1})$, we have $\det DF = \frac{\tau^2 v + \delta}{(\tau u + 1)^2}$. Then the equation $(*)$ gives rise to polynomial

$$\Phi_1(u, v) = (\tau u + 1)^3 \left(P \circ F(u, v) - tP(u, v) \det DF(u, v) \right),$$

which vanishes identically with respect to (u, v).

Expand Φ_1, and observe that it defines a system of linear equations.

It may be hard to expand the formula ...

$$\Phi_1(u,v) = C_0 v^3(\tau u+1)^3 + C_1 v^2(\tau v - \delta u)(\tau u+1)^2$$
$$+ C_2 v(\tau v-\delta u)^2(\tau u+1) + C_3(\tau v-\delta u)^3 + D_0 v^2(\tau u+1)^3$$
$$+D_1 v(\tau v-\delta u)(\tau u+1)^2 + D_2(\tau v-\delta u)^2(\tau u+1)$$
$$-t(\tau^2 v+\delta)(\tau u+1)(C_0u^3+C_1u^2v+C_2uv^2+C_3v^3+D_0u^2+D_1uv+D_2v^2).$$

Only one term contains u^4. Hence $-t\delta\tau C_0 = 0$. As we assumed $t\delta\tau \neq 0$, necessarily $C_0 = 0$.

Similarly from the term containing v^4, we have $C_3 = 0$.

Then from $\Phi_1(u,0) = \delta u^2(\delta D_2 - tD_0)(\tau u+1)$, we get $D_2 = \frac{t}{\delta} D_0$.

From these we get ...

$$
\begin{aligned}
\Phi_2(u,v) &= (\Phi_1(u,v)-\Phi_1(u,0))/(v(\tau u+1)) \\
&= C_1\{v(\tau v-\delta u)(\tau u+1) - tu^2(\tau^2 v+\delta)\} \\
&\quad + C_2\{(\tau v-\delta u)^2 - tuv(\tau^2 v+\delta)\} \\
&\quad + D_1\{(\tau v-\delta u)(\tau u+1) - tu(\tau^2 v+\delta)\} \\
&\quad + D_0\{v(\tau u+1)^2 - t\tau^2 u^2 + \frac{t}{\delta}(\tau^2 v - 2\tau\delta u - tv(\tau^2 v+\delta))\}.
\end{aligned}
$$

And from $\quad \Phi_2(u,0) = \{\delta(\delta C_2 - tC_1) - \tau(\delta D_1 + t\tau D_0)\}u^2$

$$
- \{\delta(1+t)D_1 + 2t\tau D_0\}u,
$$

we have $\quad D_1 = \dfrac{-2t\tau}{(1+t)\delta} D_0 \quad$ and $\quad \delta C_2 - tC_1 = \dfrac{t(t-1)}{(1+t)\delta} D_0.$

Assume these conditions to get ....

$$\Phi_3(u,v) = (\Phi_2(u,v) - \Phi_2(u,0))/v$$

$$= C_1\{(\tau v - \delta u)(\tau u + 1) - t\tau^2 u^2\} + C_2\{\tau^2 v - 2\tau\delta u - tu(\tau^2 v + \delta)\}$$

$$+D_1\{\tau(\tau u + 1) - t\tau^2 u\} + D_0\{(\tau u + 1)^2 - t^2 + \frac{t\tau^2}{\delta}(1 - tv)\}$$

And from $\Phi_3(u,0) \equiv 0$, we get

$$D_1 = \frac{\delta(t^2 - 1) - t\tau^2}{\tau}\tau\delta D_0,$$

$$C_1 + (2\tau + t)C_2 = \frac{\tau}{\delta}(\tau(1-t)D_1 + 2D_0),$$

$$(\delta + t\tau)C_1 = \tau D_0.$$

Finally, from $\quad \Phi_4(u,v) = (\Phi_3(u,v) - \Phi_3(u,0))/v$

$$= \tau(\tau u + 1)C_1 + \tau^2(1 - tu)C_2 - \frac{t^2\tau^2}{\delta} D_0 \equiv 0,$$

we get $\quad C_1 = tC_2, \quad$ and $\quad C_1 + \tau C_2 = \dfrac{t^2\tau}{\delta} D_0.$

Clearly, if all the conditions above are satisfied, they define an invariant cubic curve.

Summing up the above conditions under $C_0 = C_3 = 0$, $D_2 = \frac{t}{\delta} D_0$, and $t\tau\delta \neq 0$, $t \neq \pm 1$,we obtained the following system of seven equations.

Equations

$$\text{(eq.1)} \qquad D_1 = \frac{-2t\tau}{(1+t)\delta} D_0,$$

$$\text{(eq.2)} \qquad D_1 = \frac{\delta(t^2-1) - t\tau^2}{\tau\delta} D_0,$$

$$\text{(eq.3)} \qquad C_1 = tC_2,$$

$$\text{(eq.4)} \qquad (\delta + t\tau)C_1 = \tau D_0,$$

$$\text{(eq.5)} \qquad C_1 + \tau C_2 = \frac{t^2\tau}{\delta} D_0,$$

$$\text{(eq.6)} \qquad \delta C_2 - tC_1 = \frac{t(t-1)\tau^2}{(1+t)\delta} D_0,$$

$$\text{(eq.7)} \qquad C_1 + (2\tau + t)C_2 = \frac{\tau}{\delta}((1-t)\tau D_1 + 2D_0).$$

Equations (eq.1),(eq.3),(eq.4) determine the solution for given D_0. Other equations should be compatible with these equations.

To have (eq.1) and (eq.2) hold, we have two cases.

$$\text{(case I)} \qquad D_0 \neq 0, \quad \text{and} \quad \frac{-2t\tau}{(1+t)\delta} = \frac{\delta(t^2-1) - t\tau^2}{\tau\delta}.$$

$$\text{(case II)} \qquad D_1 = D_0 = 0.$$

In case I, we have

$$\delta = \frac{t\tau^2}{(1+t)^2} \quad \text{and} \quad D_1 = \frac{-2(1+t)}{\tau} D_0.$$

PROPOSITION. In case I, $(*)$ has a nontrivial solution if and only if

$$(**) \qquad \tau^2 + (t - t^3)\tau - t^3(1+t)^2 = 0.$$

Eliminate δ, C_1 and D_1 from (eq.4) $\cdots$ (eq.7) to get

$$\text{(eq.4')} \qquad \frac{t^3}{t(1+t)^2}(\tau + (1+t)^2)C_2 = D_0,$$

$$\text{(eq.5')} \qquad \frac{1}{t(1+t)^2}(\tau^2 + t\tau)C_2 = D_0,$$

$$\text{(eq.6')} \qquad \frac{t\tau^2 - t^2(1+t)^2}{(1+t)^2(t^2-1)}C_2 = D_0,$$

$$\text{(eq.7')} \qquad \frac{1}{t(1+t)^2}(\tau^2 + t\tau)C_2 = D_0.$$

Evidently, (eq.5') and (eq.7') are equivalent.

Equation (eq.6') can be rewritten as

$$\frac{1}{t(1+t)^2}\left\{\tau^2 + t\tau + \frac{1}{t^2-1}\{\tau^2 + t\tau - t^3(\tau + (1+t)^2)\}\right\} C_2 = D_0.$$

If (eq.4') and (eq.5') has a nontrivial solution,

$$(**) \qquad \tau^2 + (t - t^3)\tau - t^3(1+t)^2 = 0$$

must be satisfied. Conversely, if $(**)$ is satisfied, (eq.4'), (eq.5'), (eq.6'), and (eq.7') are all satisfied.

In case I, there are two subcases.
Equation $(**)$ can be factorized as

$$(\tau - t^2 - t^3)(\tau + t + t^2) = 0.$$

We obtained two cases

$$\text{(case } \Gamma_1\text{)} \qquad \tau = t^2 + t^3, \quad \delta = t^5.$$

$$\text{(case } \Gamma_2\text{)} \qquad \tau = -t - t^2, \quad \delta = t^3.$$

Now, we go back to (eq.1)$\cdots$(eq.7) and consider the case II, *i.e.*, $D_1 = D_0 = 0$. We look for a non-trivial solution of

$$\text{(eq.3'')} \qquad C_1 = tC_2,$$

$$\text{(eq.4'')} \qquad (\delta + t\tau)C_1 = 0,$$

$$\text{(eq.5'')} \qquad C_1 + \tau C_2 = 0,$$

$$\text{(eq.6'')} \qquad \delta C_2 - tC_1 = 0,$$

$$\text{(eq.7'')} \qquad C_1 + (2\tau + t)C_2 = 0.$$

If $\delta + t\tau \neq 0$, then these equations has no non-trivial solutions. Hence if there is a non-trivial solution, we must have

$$\text{(case } \Gamma_3) \qquad \tau = -t, \quad \text{and} \quad \delta = t^2.$$

In cases (case Γ_1),(case Γ_2),(case Γ_3), solutions of $(*)$ is obtained as in THEOREM A.

(Γ_1) $\tau = t^2 + t^3$, $\delta = t^5$,

$$P_1(u,v) = uv(tu+v) + \frac{1+t+t^2}{1+t}(tu - \frac{v}{t})^2.$$

(Γ_2) $\tau = -t - t^2$, $\delta = t^3$,

$$P_2(u,v) = (tu+v)((1+t)uv + tu + v).$$

(Γ_3) $\tau = -t$, $\delta = t^2$,

$$P_3(u,v) = uv(tu+v).$$

Theorem B

THEOREM B. Uniformizing functions can be taken as follows.

$$(\Gamma_1)\ \ \psi_C(\zeta) = \left(\frac{\xi_0 \zeta^2}{(\zeta+1)(\zeta+t)}, \frac{\xi_0\zeta^2}{(\zeta+1)(\zeta+\frac{1}{t})} \right),\ \xi_0 = \frac{(1-t)(t^3-1)}{t^2}.$$

$$(\Gamma_2)\ \ \psi_L(\zeta) = \left(\frac{\zeta}{\zeta+1}\frac{1-t}{t}, \frac{\zeta}{\zeta+1}(t-1) \right),$$

$$\psi_Q(\zeta) = \left(\frac{t^{-1}\zeta}{t^{-1}\zeta+1}(t-1), \frac{t\zeta}{t\zeta+1}\frac{1-t}{t} \right),$$

$$(\Gamma_3)\ \ \psi_0(\zeta) = \left(\frac{\zeta}{\zeta+1}\frac{t^3-1}{t^2}, \frac{\zeta}{\zeta+1}\frac{1-t^3}{t} \right),$$

$$\psi_1(\zeta) = \left(\frac{t^{-1}\zeta}{t^{-1}\zeta+1}\frac{1-t^3}{t}, 0 \right),\quad \psi_2(\zeta) = \left(0, \frac{t\zeta}{t\zeta+1}\frac{t^3-1}{t^2} \right).$$

Proof of Theorem B

The invariant cubic curve $\{P(u, v) = 0\}$ has a uniformization coordinate $\psi(\zeta)$.
Theorem B gives the explicit formulas for each cases. It was computed in [BK2]. Here, we try to execute the computation by hand.

In (u, v)-coordinates, the other fixed point is (ξ_0, ξ_0), with

$$\xi_0 = \frac{\tau - \delta - 1}{\tau}.$$

In case (Γ_1), $\quad \xi_0 = -(t-1)^2(1+t+t^2)/t^2.$

case (Γ_1)

In this case, $\tau = t^2 + t^3$, $\delta = t^5$, and

$$P_1(u,v) = uv(tu+v) + \frac{1+t+t^2}{1+t}(tu - \frac{v}{t})^2.$$

The cubic curve $C = \{P_1(u,v) = 0\}$ has a caspidal singular point at the origin. The cubic curve contains the other fixed point (ξ_0, ξ_0).
We look for a uniformizing rational function $\psi_C : \mathbb{C} \to C$, satisfying

$$\psi_C(0) = (0,0), \quad \psi_C(\infty) = (\xi_0, \xi_0), \quad P_1(\psi_C(\zeta)) \equiv 0,$$

and

$$F \circ \psi_C(\zeta) = \psi_C(t\zeta).$$

We set $\psi_C(\zeta) = (u(\zeta), v(\zeta))$, and

$$u(\zeta) = \frac{a_1\zeta + b_1\zeta^2 + \xi_0\zeta^3}{1 + a_0\zeta + b_0\zeta^2 + \zeta^3},$$

$$v(\zeta) = \frac{a_2\zeta + b_2\zeta^2 + \xi_0\zeta^3}{1 + a_0\zeta + b_0\zeta^2 + \zeta^3},$$

where constants a_i, b_i depend on t.

Then compute

$$\Psi_1(\zeta) = \frac{(1 + a_0\zeta + b_0\zeta^2 + \zeta^3)^3}{\zeta^2} P_1(u(\zeta), v(\zeta)).$$

May appear cumbersome ...

$$\Psi_1(\zeta) = \zeta(a_1 + b_1\zeta + \xi_0\zeta^2)(a_2 + b_2\zeta + \xi_0\zeta^2)$$
$$\times(ta_1 + a_2 + (tb_1 + b_2)\zeta + (t+1)\xi_0\zeta^2)$$
$$+\frac{1+t+t^2}{1+t}(1 + a_0\zeta + b_0\zeta^2 + \zeta^3)$$
$$\times(ta_1 - \frac{a_2}{t} + (tb_1 - \frac{b_1}{t})\zeta + (t - \frac{1}{t})\xi_0\zeta^2)^2.$$

As $\Psi_1(\zeta) \equiv 0$, from $\Psi_1(0) = 0$, we have $a_2 = t^2 a_1$. From $\Psi_1'(0) = 0$, we have $a_1 a_2(ta_1 + a_2) = 0$. Thus, we have

$$a_1 = a_2 = 0,$$

provided $t \neq 0$, $t \neq -1$.

Next, let

$$\Psi_2(\zeta) = \frac{1}{\zeta^2}\Psi_1(\zeta)$$

$$= \zeta^2(b_1 + \xi_0\zeta)(b_2 + \xi_0\zeta)(tb_1 + b_2 + (t+1)\xi_0\zeta)$$

$$+\frac{1+t+t^2}{1+t}(1 + a_0\zeta + b_0\zeta^2 + \zeta^3)(tb_1 - \frac{b_2}{t} + (t - \frac{1}{t})\xi_0\zeta)^2.$$

From $\Psi_2(0) = 0$, we have $b_2 = t^2 b_1$.
Then from the terms of lowest degree, we have

$$b_1 b_2(tb_1 + b_2) + \frac{1+t+t^2}{1+t}(t - \frac{1}{t})^2\xi_0^2 = 0.$$

Hence we get

$$t^3 b_1^3 = \xi_0^3.$$

We choose $b_1 = \frac{1}{t}\xi_0$. Then $b_2 = t\xi_0$. (Other cases give similar formulas with slight change of coordinates.)
Then

$$\Psi_3(\zeta) = \frac{1}{\zeta^2}\Psi_2(\zeta)$$

$$= \xi_0^3(1+t)\left((\frac{1}{t}+\zeta)(t+\zeta)(1+\zeta) - (1 + a_0\zeta + b_0\zeta^2 + \zeta^3)\right).$$

Hence we get

$$a_0 = b_0 = 1 + t + \frac{1}{t}.$$

And for $\psi_C(\zeta) = (u(\zeta), v(\zeta))$,

$$u(\zeta) = \frac{\xi_0\zeta^2}{(\zeta+1)(\zeta+t)}, \qquad v(\zeta) = \frac{\xi_0\zeta^2}{(\zeta+1)(\zeta+\frac{1}{t})}.$$

Note that around the other fixed point (ξ_0, ξ_0), the invariant curve is regular and can be uniformized by $\chi = \zeta^{-1}$ as

$$u - \xi_0 = \frac{-\xi_0 \chi(\chi + 1 + \frac{1}{t})}{(\chi + 1)(\chi + \frac{1}{t})}, \qquad v - \xi_0 = \frac{-\xi_0 \chi(\chi + 1 + \frac{1}{t})}{(\chi + 1)(\chi + t)}.$$

case (Γ_2)

In case (Γ_2), $\tau = -t - t^2$, $\delta = t^3$, and

$$P_2(u, v) = (tu + v)((1 + t)uv + tu + v).$$

The cubic consists of a line $L = \{tu + v = 0\}$, and a quadric $Q = \{(1 + t)uv + tu + v = 0\}$. They are tangent at the origin.

Periodic points of period 2 are $R_0 = (\frac{1-t}{t}, t - 1)$, and $R_1 = (t - 1, \frac{1-t}{t})$.

$$R_0 \in L, \quad \text{and} \quad R_1 \in Q.$$

Uniformizing rational function $\psi_L : \mathbb{C} \to L$ satisfies $\psi_L(0) = (0, 0)$, and $\psi_L(\infty) = R_0$. We can take

$$\psi_L(\zeta) = \left(\frac{\zeta}{\zeta + 1} \frac{1 - t}{t}, \quad \frac{\zeta}{\zeta + 1}(t - 1) \right).$$

Uniformizing rational function $\psi_Q : \mathbb{C} \to Q$ should satisfy $\psi_L(0) = (0,0)$, $\psi_Q(\infty) = R_1$, and

$$F \circ \psi_L(\zeta) = \psi_Q(t\zeta), \qquad F \circ \psi_Q(\zeta) = \psi_L(t\zeta).$$

Hence we have

$$\psi_Q(\zeta) = \left(\frac{t^{-1}\zeta}{t^{-1}\zeta + 1}(t-1), \quad \frac{t\zeta}{t\zeta + 1} \frac{1-t}{t} \right).$$

case (Γ_3)

In case (Γ_3), $\tau = -t$, $\delta = t^2$, and

$$P_3(u, v) = uv(tu + v).$$

The cubic consists of three lines $L_0 = \{tu + v = 0\}$, $L_1 = \{v = 0\}$, and $L_2 = \{u = 0\}$. They intersect at the origin.

Periodic point of period 3 are $S_0 = (\frac{t^3-1}{t^2},\ \frac{1-t^3}{t})$, $S_1 = (\frac{1-t^3}{t},\ 0)$, and $S_2 = (0,\ \frac{t^3-1}{t^2})$.

$$S_0 \in L_0, \quad S_1 \in L_1, \quad S_2 \in L_2,$$

$$S_1 = F(S_0), \quad S_2 = F(S_1), \quad S_0 = F(S_2).$$

Uniformizing rational functions $\psi_i : \mathbb{C} \to L_i, \quad i = 0, 1, 2$ should satisfy

$$\psi_i(0) = (0,0), \quad \psi_i(\infty) = S_i, \qquad i = 0,1,2,$$
$$F \circ \psi_i(\zeta) = \psi_{i+1}(t\zeta), \qquad (i \bmod 3).$$

Hence we have

$$\psi_0(\zeta) = \left(\frac{\zeta}{\zeta+1} \frac{t^3-1}{t^2}, \frac{\zeta}{\zeta+1} \frac{1-t^3}{t} \right),$$

$$\psi_1(\zeta) = \left(\frac{t^{-1}\zeta}{t^{-1}\zeta+1} \frac{1-t^3}{t}, 0 \right), \quad \psi_2(\zeta) = \left(0, \frac{t\zeta}{t\zeta+1} \frac{t^3-1}{t^2} \right).$$

Proof of Theorems C, D, E

Proof of Theorems C, D, E are straightforward.

Theorem C.

THEOREM C.

$$(\Gamma_1) \qquad F \circ \psi_P(\zeta) = \psi_P(t\zeta).$$

$$(\Gamma_2) \qquad F \circ \psi_L(\zeta) = \psi_Q(t\zeta),$$

$$F \circ \psi_Q(\zeta) = \psi_L(t\zeta).$$

$$(\Gamma_3) \qquad F \circ \psi_0(\zeta) = \psi_1(t\zeta),$$

$$F \circ \psi_1(\zeta) = \psi_2(t\zeta),$$

$$F \circ \psi_2(\zeta) = \psi_0(t\zeta).$$

Theorem D.

The indeterminate point of F is $p = (\frac{-1}{\tau}, \frac{-\delta}{\tau^2})$, and the indeterminate point of F^{-1} is $q = (\frac{-\delta}{\tau^2}, \frac{-\delta}{\tau})$.
Let $\zeta_p = \frac{t}{t^3-t-1}$, $\zeta_q = \frac{t^2}{1-t^2-t^3}$.

THEOREM D.

$$(\Gamma_1) \qquad \psi_P(\zeta_p) = p, \quad \psi_P(\zeta_q) = q.$$

$$(\Gamma_2) \qquad \psi_Q(\zeta_p) = p, \quad \psi_Q(\zeta_q) = q.$$

$$(\Gamma_3) \qquad \psi_0(\zeta_p) = p, \quad \psi_0(\zeta_q) = q.$$

Theorem E.

THEOREM E. In these cases,

$$F^n(q) = p \quad \text{if and only if} \quad \zeta_p = t^n \zeta_q.$$

REM. $\chi_n(x) = x^{n+1}(x^3 - x - 1) + (x^3 + x^2 - 1)$ is the characteristic polynomial of $F^* : H^{1,1}(X) \to H^{1,1}(X)$. Here, when $F^n(q) = p$, X is a compact complex surface obtained by blowing up the projective plane $\mathbb{CP}^2$ at the $n+3$ points $e_1 = [0:1:0], e_2 = [0:0:1]$ and $F^j(q), 0 \le j \le n$.

$$\text{If } \zeta_p = t^n \zeta_q, \text{ then } \chi_n(t) = 0.$$

from δ, τ to α, β

Let us derive the parameter correspondence between our parameters δ, τ for family

$$F(u, v) = (v, \frac{\tau v - \delta u}{\tau u + 1}),$$

and parameters α, β of our family of self-anti-conjugate maps

$$f(x, y) = (y, \frac{y + \alpha}{x + i\beta} + i\beta),$$

Fixed point (x_s, x_s) of f is given by

$$x_s^2 - x_s = \alpha - \beta^2.$$

And the Jacobian matrix of f at the fixed point is

$$Df = \begin{pmatrix} 0 & 1 \\ -\frac{x_s+\alpha}{(x_s+i\beta)^2} & \frac{1}{x_s+i\beta} \end{pmatrix}.$$

So, we have $\tau = \frac{1}{x_s+i\beta}$, and $\delta = \frac{x_s+\alpha}{(x_s+i\beta)^2}$. From these equations, we get

$$\alpha = \frac{2\delta - \tau - \tau\delta}{2\tau^2}, \qquad i\beta = \frac{1-\delta}{2\tau},$$

$$a = \alpha + i\beta = \frac{\delta(1-\tau)}{\tau^2}, \qquad b = 2i\beta = \frac{1-\delta}{\tau}.$$

Finally, we get the following.

$$(\Gamma_1) \qquad \tau = t^2 + t^3, \quad \delta = t^5,$$

$$\alpha = \frac{-1 - t + 2t^3 - t^5 - t^6}{2t^2(1+t)^2}, \qquad i\beta = \frac{1 - t^5}{2t^2(1+t)},$$

$$(\Gamma_2) \qquad \tau = -t - t^2, \qquad \delta = t^3,$$

$$\alpha = \frac{1 + t + 2t^2 + t^3 + t^4}{2t(1+t)^2}, \qquad i\beta = \frac{t^3 - 1}{2t(1+t)},$$

$$(\Gamma_3) \qquad \tau = -t, \qquad \delta = t^2,$$

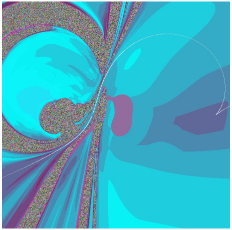
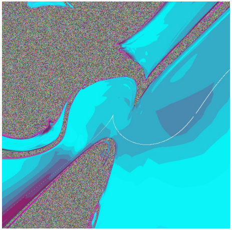
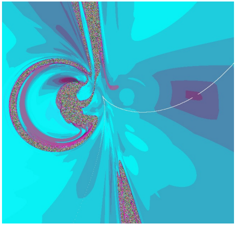
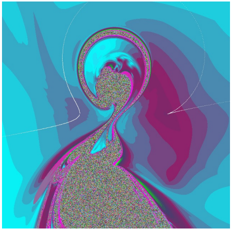
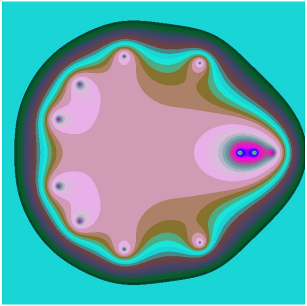
$$\alpha = \frac{(1+t)^2}{2t}, \qquad i\beta = \frac{t^2 - 1}{2t},$$

In [BK2] functions $\varphi_j(t)$ are as follows.

$$(\Gamma_1) \quad \varphi_1(t) = \left(\frac{t - t^3 - t^4}{(1+t)^2},\ \frac{1 - t^5}{t^2(1+t)} \right).$$

$$(\Gamma_2) \quad \varphi_2(t) = \left(\frac{t(1 + t + t^2)}{(1+t)^2},\ \frac{t^3 - 1}{t(1+t)} \right).$$

$$(\Gamma_3) \quad \varphi_3(t) = \left(1 + t,\ t - \frac{1}{t} \right).$$

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