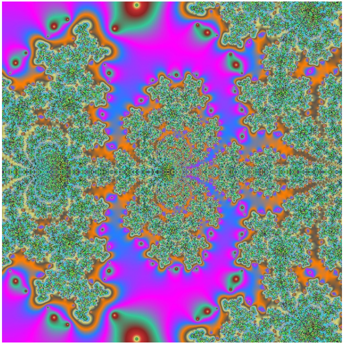
Rotation Attractors and Rotation Domains in Complex Surface Automorphisms



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Contents

1. Surface automorphism

Blow-up

The **blow-up** of a surface X at a point $p \in X$ is a surface $\tilde{X}$, together with a projection $\pi : \tilde{X} \to X$ such that the **exceptional fiber** $E = \pi^{-1}(p)$ is equivalent to $\mathbb{P}^1$, and $\pi : \tilde{X} \setminus E \to X \setminus \{p\}$ is biholomorphic.

$$\Gamma = \left\{((x,y),[\xi : \eta]) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid x\eta = y\xi\right\}$$

$$\pi((x,y),[\xi : \eta]) = (x,y).$$

Surface automorphisms with positive entropy

THEOREM (S. Cantat 1999, M. Nagata 1960/1961)
Let X be a connected compact complex surface. Assume that $\mathrm{Aut}(X)$ contains an automorphism with positive topological entropy. Then X is a Kähler surface, and

either X is obtained from the plane $\mathbb{P}^2(\mathbb{C})$ by a finite sequence of at least ten blowups,

or (the minimal model of) X is isomorphic to a torus, a K3 surface, or an Enriques surface.

(co)homology

Let $\pi : \mathcal{S} \to \mathbb{P}^2$ be a blow-up of $\mathbb{P}^2$ at n distinct points $p_1, \cdots, p_n$.

$$E_i = \pi^{-1}(p_i) \subset \mathcal{S}, \qquad \text{exceptional fiber}, i = 1, \cdots, n,$$

$$H \subset \mathcal{S}, \qquad \text{generic line.}$$

A basis of $H^2(\mathcal{S}; \mathbb{Z})$ is given by $[H]$ and $[E_i]$, $i = 1, \cdots, n$.

Intersection pairing and Minkowski lattice

Intersection pairing on $H^2(\mathcal{S};\mathbb{Z})$:

$$[H]\cdot[H]=1,\quad [E_i]\cdot[E_j]=-\delta_{ij},\quad [H]\cdot[E_i]=0.$$

Let $\mathbb{Z}^{1,n}$ denote the lattice $\mathbb{Z}^{n+1}$ equipped with the Minkowski inner product

$$x\cdot y=x_0y_0-x_1y_1-x_2y_2-\cdots-x_ny_n,$$

for basis $e_0,e_1,\cdots,e_n$.

$H^2(\mathcal{S};\mathbb{Z})$ is isomorphic to $\mathbb{Z}^{1,n}$.

Loxodromic automorphism

Let f be an automorphism of a compact Kähler surface $\mathcal{S}$.
Let $H^{1,1}(\mathcal{S};\mathbb{R}) = H^{1,1}(\mathcal{S};\mathbb{C}) \cap H^2(\mathcal{S};\mathbb{R})$.
Then $f^* : H^{1,1}(\mathcal{S};\mathbb{R}) \to H^{1,1}(\mathcal{S};\mathbb{R})$ is an isomorphism preserving the intersection pairing.
Define the **dynamical degree** λ_f by

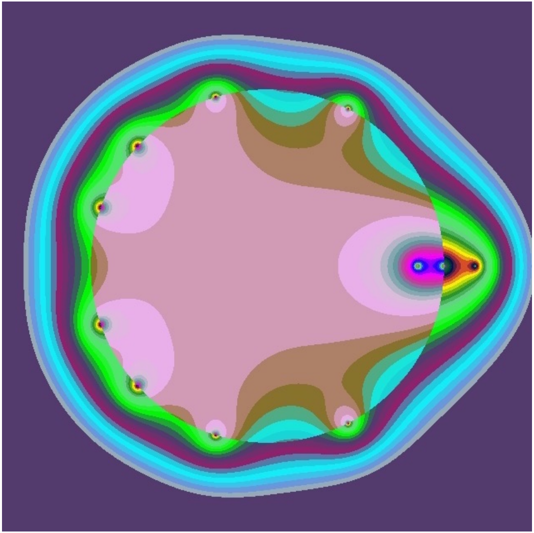
$$\lambda_f = \lim_{n\to\infty} ||f^{*n}||^{\frac{1}{n}}.$$

THEOREM. If $\lambda_f > 1$, then λ_f is an eigenvalue of f^* with multiplicity 1, and it is the unique eigenvalue with modulus > 1.

If $\lambda_f > 1$, then λ_f^{-1} is an eigenvalue, too. Other eigenvalues are of modulus 1.

f is said to be **loxodromic** if $\lambda_f > 1$.

$\chi_7(z)$



Invariant currents and invariant measures

Let f be a loxodromic automorphism of a compact Kähler surface $\mathcal{S}$.

THEOREM (Cantat 2001, Dinh-Sibony 2005). There exist positive, closed currents T_f^+ and T_f^- with invariance property

$$f^* T_f^+ = \lambda_f T_f^+ \quad \text{and} \quad f^* T_f^- = \lambda_f^{-1} T_f^-.$$

We obtain an invariant measure $\mu_f = T_f^+ \wedge T_f^-$.

THEOREM (Bedford-Lyubich-Smilie 1993, Cantat 2003). Let $\Lambda(f, k)$ denote the set of saddle periodic points of f of period k. Then

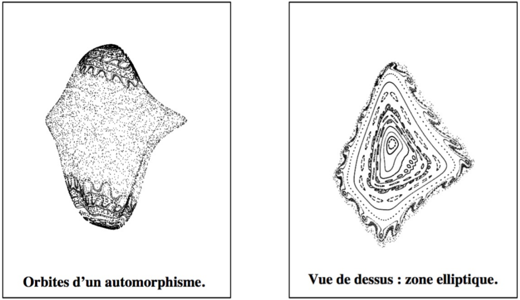
$$\mu_f = \lim_{k \to \infty} \frac{1}{\lambda_f^k} \sum_{p \in \Lambda(f,k)} \delta_p.$$

2. K3 surface

Automorphisms of K3 surface by Cantat

S. Cantat (2001) studied the dynamics of holomorphic diffeomorphisms of compact complex surfaces, especially in the case of projective K3 surfaces. He proved the existence of the invariant probability measure of maximal entropy when the topological entropy of the automorphism is strictly positive.

SERGE CANTAT

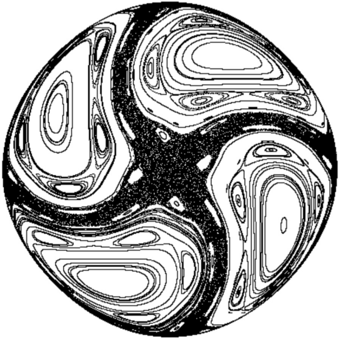


Orbites d’un automorphisme.

Vue de dessus : zone elliptique.

Automorphisms of K3 surface by McMullen

C. T. McMullen (2002) gave the first examples of $K3$ surface automorphisms $f : X \to X$ with Siegel disks (domains on which f acts by an irrational rotation). The set of such examples is countable, and the surface X must be non-projective to carry a Siegel disk.



3. Rational surface

Automorphisms of rational surfaces

THEOREM (Bedford-Kim 2006, McMullen 2007) For each $n > 3$, there exist a, b which satisfy two polynomial equations $P_n(a, b) = 0$, $Q_n(a, b) = 0$ such that

$$f_{a,b} : (x, y) \mapsto \left(y, \frac{y + a}{x + b}\right)$$

induces an automorphism of a surface $\pi : \mathcal{X}_{a,b} \to \mathbb{P}^2$ where $\mathcal{X}_{a,b}$ is obtained by blowing up n points.

Automorphisms of rational surfaces

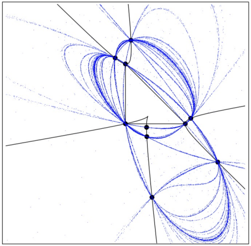
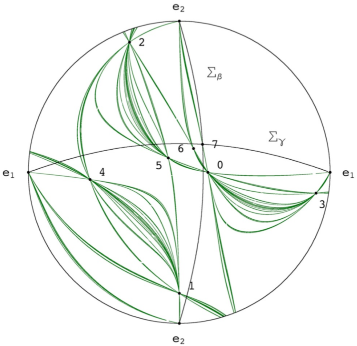


Figure 1. The expanding map F_{10} associated to Lehmer's number.



Rational surface automorphisms

Rational families studied by Bedford and Kim.

$$f(x,y) = (y, -\delta x + cy + y^{-1}).$$

$$f(x,y) = (y, -x + cy + \sum_{\ell=1}^{k-1} \frac{a_\ell}{y^{2\ell}} + \frac{1}{y^{2k}}).$$

Cremona transformations with invariant cubic curve

J. Diller (2011) gave a method for constructing automorphisms with positive entropy on rational complex surfaces.

A birational transformation $f : \mathbb{P}^2 \to \mathbb{P}^2$ is called a Cremona transformation.

A quadratic transformation $f : \mathbb{P}^2 \to \mathbb{P}^2$ always acts by blowing up three (indeterminacy) points $I(f) = \{p_1^+, p_2^+, p_3^+\}$ in $\mathbb{P}^2$ and blowing down the (exceptional) lines joining them. The inverse map f^{-1} is also a quadratic transformation and $I(f^{-1}) = \{p_1^-, p_2^-, p_3^-\}$ consists of the images of the three exceptional lines.

T. Uehara (2016) gave still more examples of rational surface automorphisms.

Uehara's explicit formula of birational maps

Uehara(2016) obtained an explicit formula for Cremona transformations with an invariant cuspidal cubic curve.

For $d \in \mathbb{C}^\times$ and $a_1, a_2, a_3 \in \mathbb{C}$ with $a_1 + a_2 + a_3 \neq 0$,

$$X = d \cdot \left\{ x + \frac{\nu_1}{3} + \frac{\nu_1(y - x^3)}{\nu_1 x^2 - \nu_2 x + \nu_3 - y} \right\},$$

$$Y = d^3 \cdot \left\{ (x + \frac{\nu_1}{3})^3 + y - x^3 + \frac{\nu_1(y - x^3)}{\nu_1 x^2 - \nu_2 x + \nu_3 - y} (\nu_1(x + \frac{\nu_1}{3}) - \nu_2) \right\},$$

where $\nu_1 = a_1 + a_2 + a_3$, $\nu_2 = a_1 a_2 + a_2 a_3 + a_3 a_1$, and $\nu_3 = a_1 a_2 a_3$.

If $y = x^3$ then $Y = X^3$.

Orbit data

Parameters d, a_1, a_2, a_3 are determined from the orbit data (n_1, n_2, n_3) with permutation $\sigma \in \Sigma_3$.

Orbit data specifies the behavior of the indeterminate points and the exceptional lines.

The value d is chosen among the eigenvalues of the cohomology homomorphism $f^* : H^2(\mathcal{S}, \mathbb{Z}) \to H^2(\mathcal{S}, \mathbb{Z})$.

Parameters a_1, a_2, a_3 are computed from d.

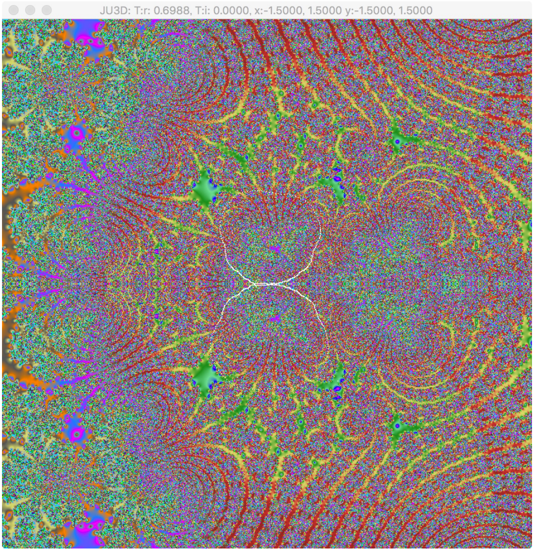
Differential form $\eta = \frac{dx \wedge dy}{y - x^3}$ is an eigenform.

$$f^* \eta = d \cdot \eta.$$

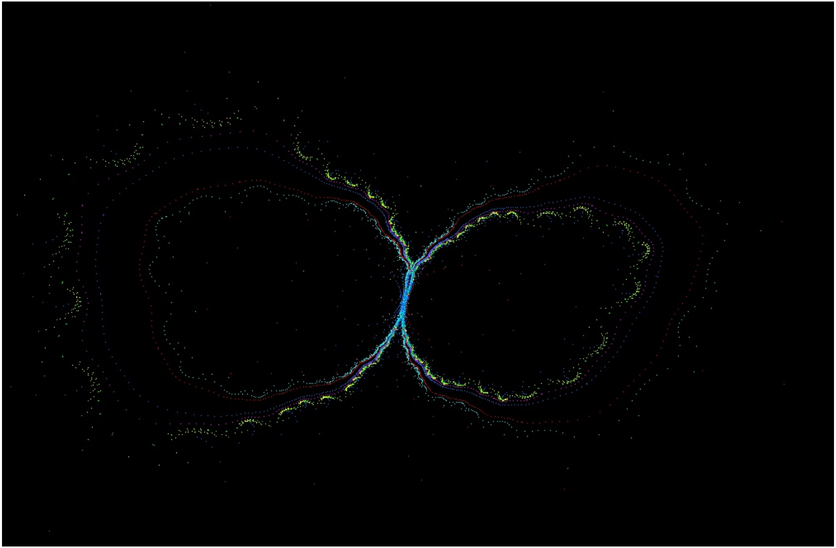
This defines a meromorphic volume form $\eta \wedge \bar{\eta}$.

4. Rotating attractor

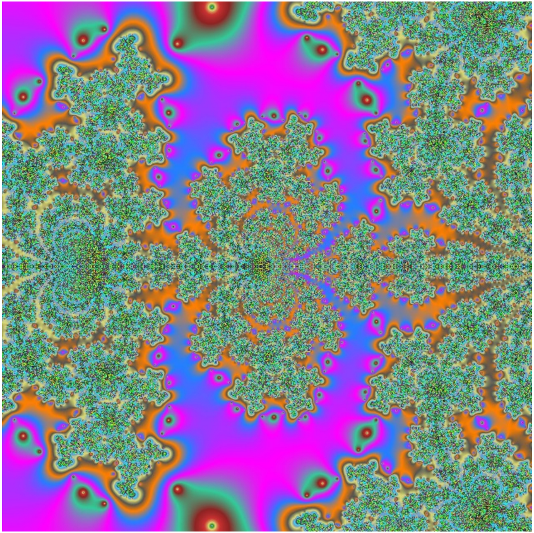
Orbit data (3,3,4), cyclic permutation



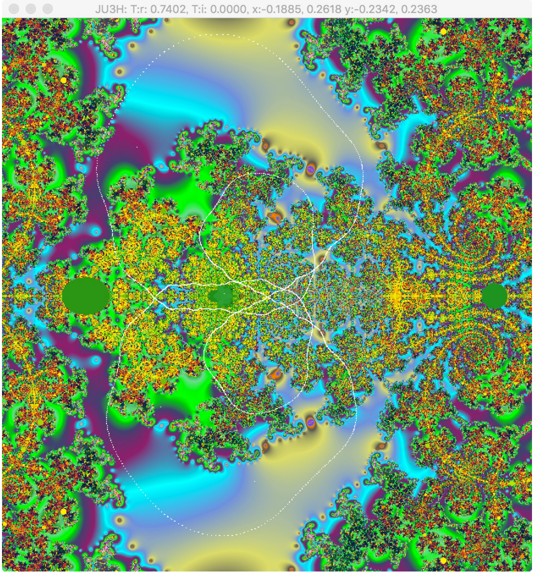
Attracting Hermann ring



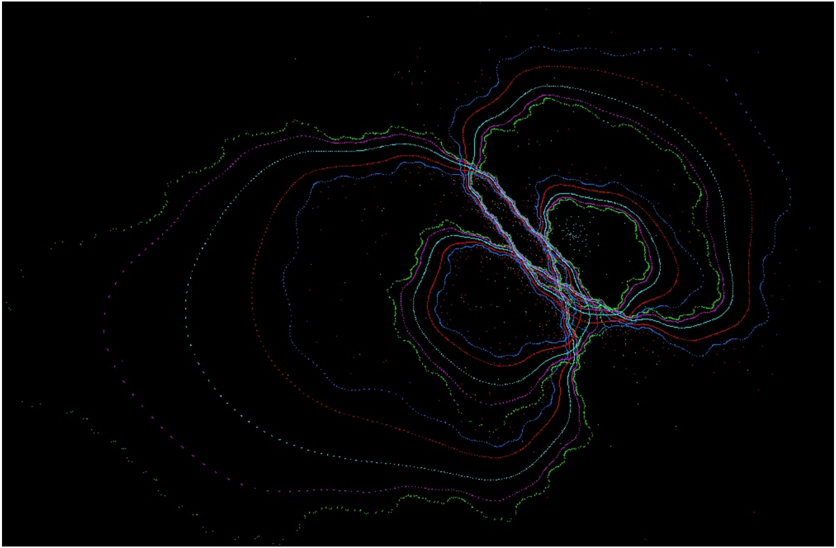
Orbit data (3,3,4), cyclic permutation



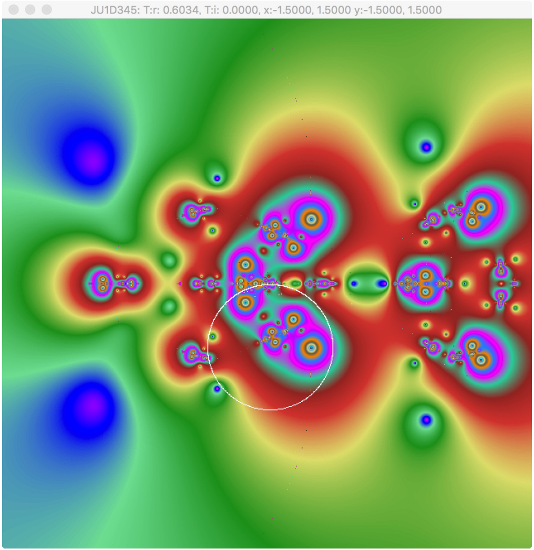
Orbit data (3,2,5), cyclic permutation



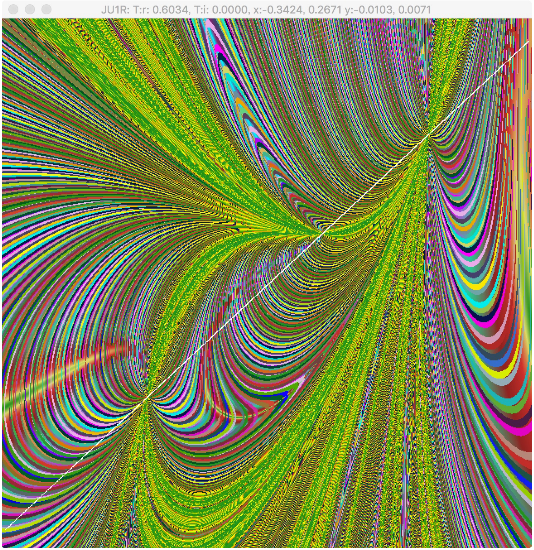
Attracting Hermann ring



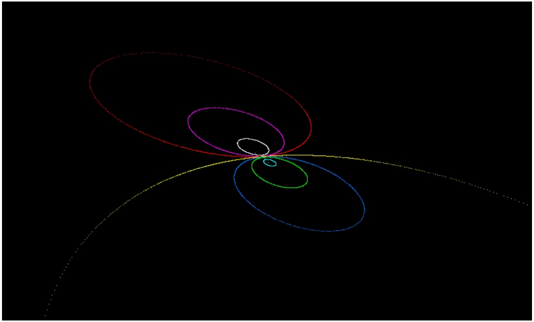
Orbit data (3,4,5), id



Attracting Riemann sphere with irrational rotation

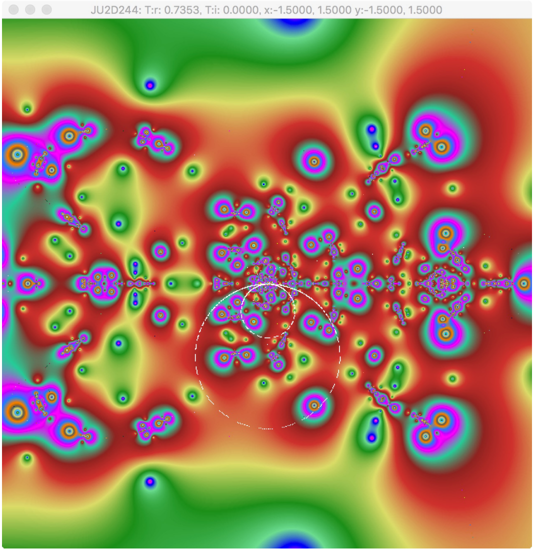


Attracting Riemann sphere with irrational rotation

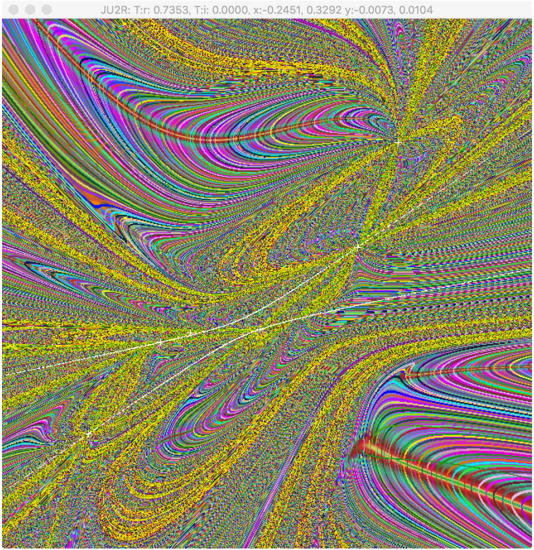


THEOREM. In the case of orbit data $(3, n_2, n_3)$ with $\sigma(1) = 1$, the surface automorphism has an invariant Riemann sphere passing through three blowup points p_1^+, p_1^-, and $f(p_1^-)$.

Orbit data (2,4,4), transposition (1,2)



Attracting quadratic curve with irrational rotation



THEOREM. In the case of orbit data $(2, 4, n)$ with transposition (1,2), the surface automorphism has an invariant quadratic curve passing through six blowup points $p_1^+, p_1^-, p_2^+, p_2^-, f(p_2^-), f^2(p_2^-)$.

5. Rotation domain

Rotation domain

Suppose Ω is a Fatou component of a volume preserving automorphism f with $f(\Omega) = \Omega$. Define the set of all limits of convergent subsequences $\mathcal{G}$ by

$$\mathcal{G} = \left\{ g = \lim_{n_j \to \infty} f^{n_j} : \Omega \to \overline{\Omega} \right\}.$$

If $g = \lim_{n_j \to \infty} f^{n_j}$ is such a limit, then g must preserve volume, and thus it is locally invertible. It follows that $g : \Omega \to \Omega$.

It is known that $\mathcal{G}$ is a compact Lie group, by a theorem of H. Cartan. The connected component $\mathcal{G}_0$ of the identity must be a (real) torus.

Rank of a rotation domain

In the volume preserving Hénon map case, known result is as follows.

THEOREM (Bedford-Smilie 1991).
$\mathcal{G}_0$ is isomorphic to $\mathbb{T}^\rho$ with $\rho = 1$ or 2.

Same result should hold for surface automorphism case.

Such a domain is called a **rotation domain**, and we refer to ρ as the **ramk** of the rotation domain.

Reinhardt domain

Let $D \subset \mathbb{C}^2$ be a connected open set. We say that D is a **Reinhardt domain** if $(e^{i\theta}z, e^{i\phi}w) \in D$ for all $(z, w) \in D$ and all $\theta, \phi \in \mathbb{R}$.

If Ω is a rank 2 rotation domain, then the $\mathcal{G}$-action on Ω may be conjugated to the standard linear action on $\mathbb{C}^2$.

THEOREM. (Barrettt-Bedford-Dadok 1989) There are a Reinhardt domain $D \subset \mathbb{C}^2$, a linear map $L : (x, y) \mapsto (\alpha x, \beta y)$, $|\alpha| = |\beta| = 1$, and a biholomorphic map $\Phi : \Omega \to D$ such that $\Phi \circ f = L \circ \Phi$.

Reversible dynamics

We say that a map f is **reversible by an involution** τ if $\tau \circ f \circ \tau = f^{-1}$.

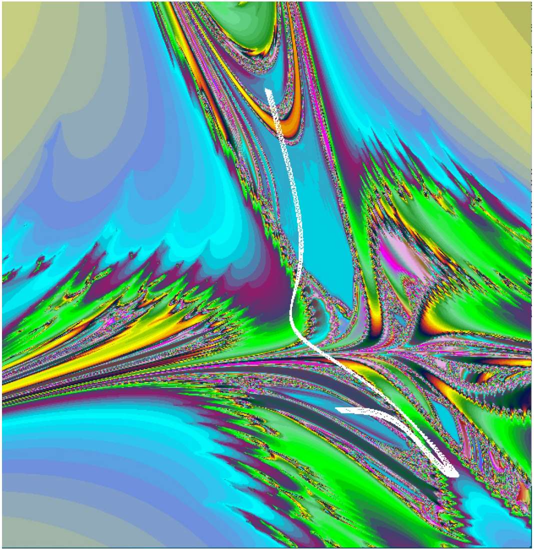
THEOREM. A Hénon map is reversible by the (anti-holomorphic) involution $\tau(x, y) = (\bar{y}, \bar{x})$ if and only if it has the form

$$f(x, y) = (y, \beta p(y) - \beta^2 x)$$

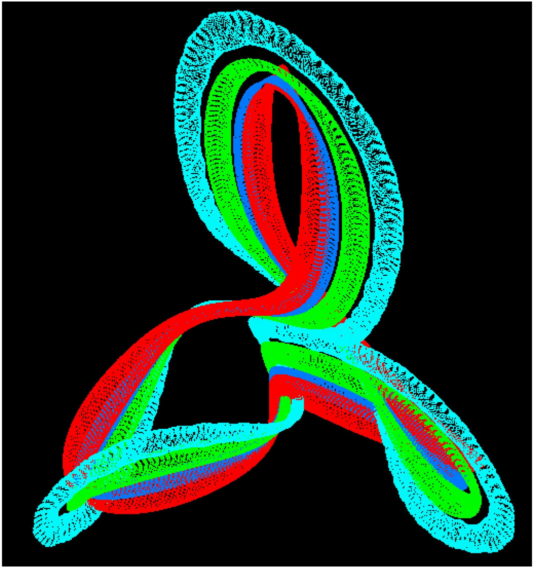
where $p(y)$ is a real polynomial and $|\beta| = 1$.

Conjugate diagonal $\Delta' = \{(x, \bar{x}) | x \in \mathbb{C}\}$ is the set of fixed points of involution τ.

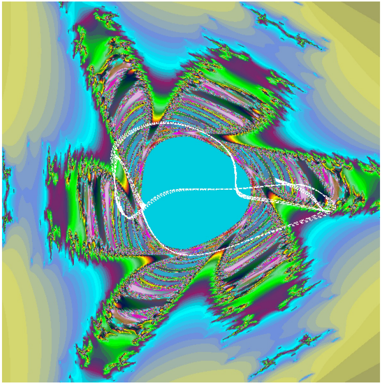
Conjugate diagonal slice for Hénon map

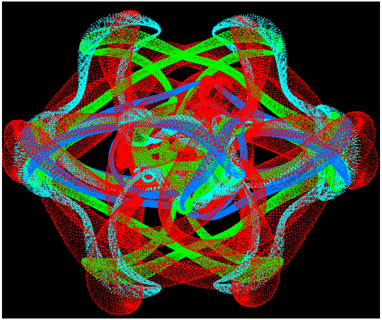


Tori in an exotic rotation domain



Conjugate diagonal slice for Hénon map





Conjugate reversible automorphisms

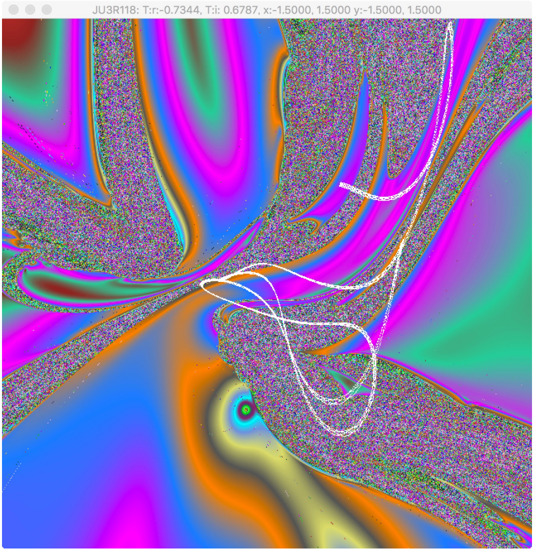
Let $T : \mathcal{S} \to \mathcal{S}$ be the involution of rational surface $\mathcal{S}$, defined by extending the complex conjugation $T(x, y) = (\bar{x}, \bar{y})$.

In the case of surface automorphism with invariant caspidal cubic curve, some of them are reversible.

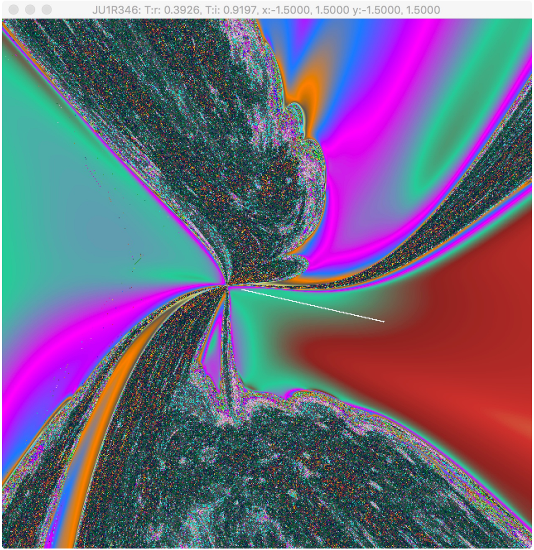
THEOREM. For orbit data (n_1, n_2, n_3), with permutation $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$, the surface automorphism is reversible by T if $\sigma^{-1} = \sigma$, or $n_i = n_j$ for some $i \neq j$.

6. Exotic rotation domain

Orbit data (1,1,8), cyclic permutation, $\mathbb{A} \times \mathbb{D}$, t_3, rank 2.

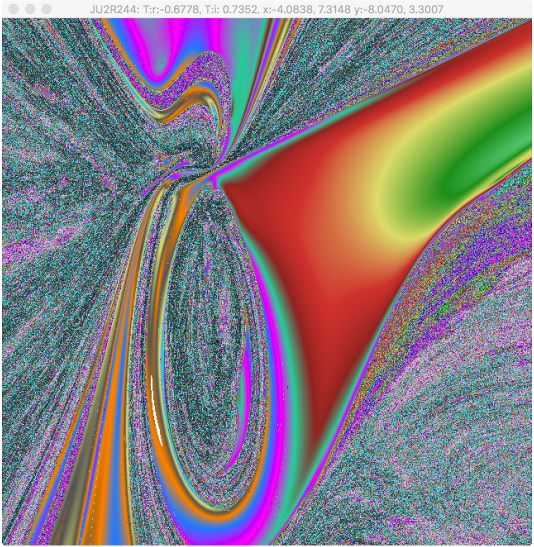


Orbit data (3,4,6), id, $\mathbb{P} \times \mathbb{D}$, t_1, rank 2.



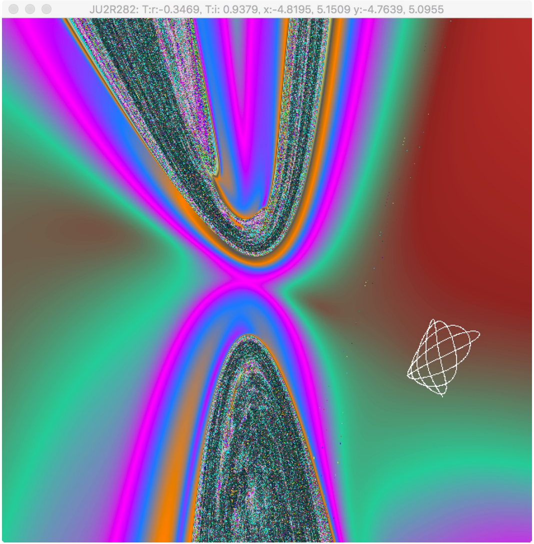
THEOREM. In the case of orbit data $(3, n_2, n_3)$ with $\sigma(1) = 1$, the surface automorphism has an invariant Riemann sphere passing through three blowup points p_1^+, p_1^-, and $f(p_1^-)$.

Orbit data (2,4,4), transposition (1,2), $Q \times \mathbb{D}$, t_3, rank 2.

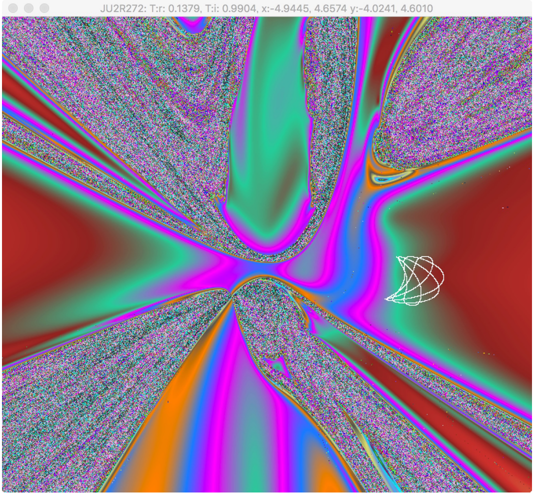


THEOREM. In the case of orbit data $(2, 4, n)$ with transposition (1,2), the surface automorphism has an invariant quadratic curve passing through six blowup points $p_1^+, p_1^-, p_2^+, p_2^-, f(p_2^-), f^2(p_2^-)$.

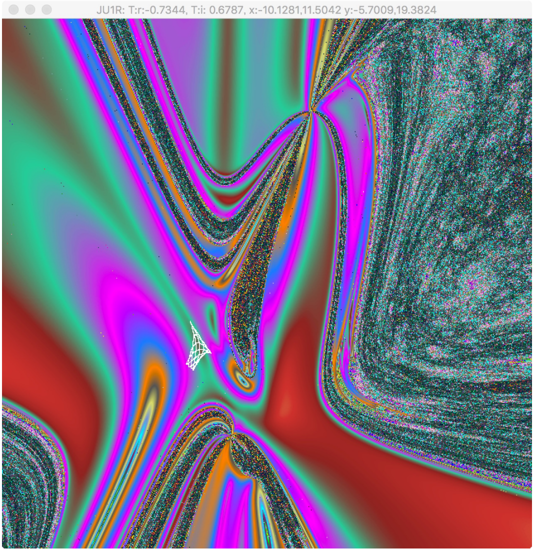
Orbit data (2,8,2), transposition (1,2), $(C \cup \mathbb{P}) \times \mathbb{D}$, t_3, rank 1.



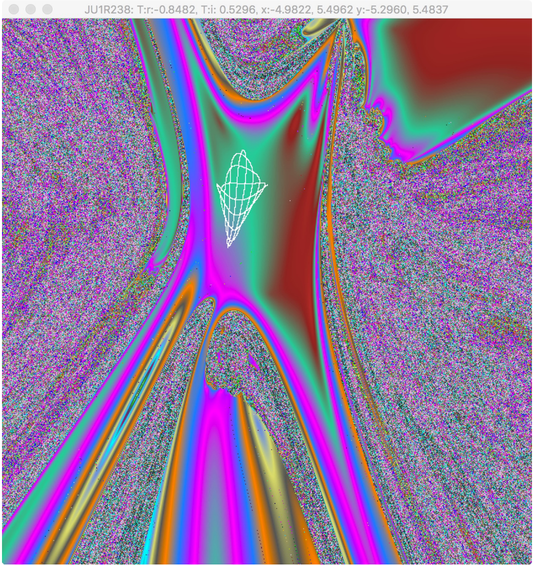
Orbit data (2,7,3), transposition (1,2), $\mathbb{D} \times \mathbb{D}$, t_2, rank 1.



Orbit data (2,3,7), id, $(C \cup \mathbb{P}_1 \cup \mathbb{P}_2) \times \mathbb{D}$, t_3, rank 1.



Orbit data (2,3,8), id, $(C \cup \mathbb{P}_1 \cup \mathbb{P}_2) \times \mathbb{D}$, t_4, rank 1.



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