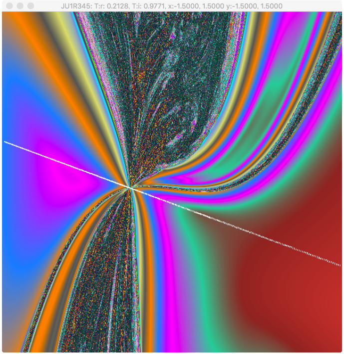
Invariant Curves in Complex Surface Automorphisms



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Abstract

Automorphisms of complex surfaces can have various invariant curves. In this note, we consider a family of rational surface automorphisms with an invariant caspidal cubic curve.

Such rational automorphism can have, at the same time, an invariant line, or an invariant quadratic curve, or a pair of lines intersecting at a point.

Dynamics in invariant curves are studied.

Contents

1. Rational surface

Surface automorphism of positive entropy

THEOREM (Cantat 1999).
Suppose $\mathcal{X}$ is a compact algebraic surface.
If $f \in \mathrm{Aut}(\mathcal{X})$ with $h_{top}(f) > 0$, then $\mathcal{X}$ is either

a torus $\mathbb{T}^2 = \mathbb{C}^2/\mathcal{L}$,
a $K3$ surface or an Enriques surface,
or a rational surface.

Blowups

THEOREM (Nagata 1960/1961).

Suppose f is an automorphism on a rational surface $\mathcal{X}$, and f_* has infinite order.

Then there is a holomorphic, birational map $\pi : \mathcal{X} \to \mathbb{P}^2$ where the map π is obtained by a finite blowup process.

Automorphisms of rational surfaces

THEOREM (Bedford-Kim 2006, McMullen 2007).
For each $n > 3$, there exist a, b which satisfy two polynomial equations $P_n(a, b) = 0, Q_n(a, b) = 0$ such that

$$f_{a,b} : (x, y) \mapsto \left(y, \frac{y + a}{x + b}\right)$$

induces an automorphism of a surface $\pi : \mathcal{X}_{a,b} \to \mathbb{P}^2$ where $\mathcal{X}_{a,b}$ is obtained by blowing up n points.

Rational surface automorphisms

Rational families studied by Bedford and Kim(2010,2012).

$$f(x,y) = (y, -x + cy + \sum_{\ell=1}^{k-1} \frac{a_\ell}{y^{2\ell}} + \frac{1}{y^{2k}}).$$

$$f(x,y) = (y, -\delta x + cy + y^{-1}).$$

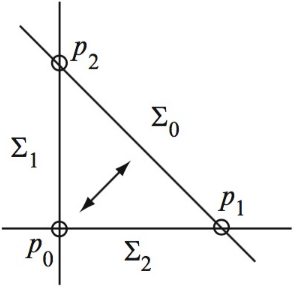
J. Diller (2011) gave a systematic method for constructing automorphisms with positive entropy on rational complex surfaces.

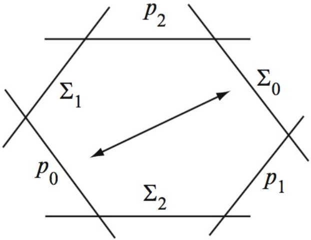
T. Uehara (2016) gave still more examples of rational surface automorphisms.

Cremona involution

Cremona involution J of $\mathbb{P}^2$ is defined by

$$J[x:y:z] = [x^{-1}:y^{-1}:z^{-1}] = [yz:zx:xy].$$





For linear transformations $L_1, L_2 \in PGL(\mathbb{P}^2)$,

$$f = L_1 \circ J \circ L_2$$

is a birational transformation.

Cremona transformations with invariant cubic curve

A birational transformation $f : \mathbb{P}^2 \to \mathbb{P}^2$ is called a Cremona transformation.

A quadratic transformation $f : \mathbb{P}^2 \to \mathbb{P}^2$ always acts by blowing up three (indeterminacy) points $I(f) = \{p_1^+, p_2^+, p_3^+\}$ in $\mathbb{P}^2$ and blowing down the (exceptional) lines joining them. The inverse map f^{-1} is also a quadratic transformation and $I(f^{-1}) = \{p_1^-, p_2^-, p_3^-\}$ consists of the images of the three exceptional lines.

$$p_i^- = f(\ell(p_j^+, p_k^+)) \quad \text{for} \quad \{i, j, k\} = \{1, 2, 3\}.$$

Here, $\ell(p, q)$ denotes the line passing through p and q.

Orbit data

Suppose that for natural numbers n_1, n_2, n_3, and a permutation $\sigma : \{1,2,3\} \to \{1,2,3\}$, f satisfies

$$f^{n_i - 1}(p_i^-) = p_{\sigma(i)}^+, \qquad i = 1, 2, 3.$$

$$\ell(p_j^+, p_k^+) \to p_i^- \to f(p_i^-) \to \cdots \to p_{\sigma(i)}^+ \to \ell(p_{\sigma(j)}^-, p_{\sigma(k)}^-).$$

By blowing up in $n_1 + n_2 + n_3$ points

$$p_1^-, f(p_1^-), \cdots, f^{n_1-1}(p_1^-) = p_{\sigma(1)}^+,$$

$$p_2^-, f(p_2^-), \cdots, f^{n_2-1}(p_2^-) = p_{\sigma(2)}^+,$$

$$p_3^-, f(p_3^-), \cdots, f^{n_3-1}(p_3^-) = p_{\sigma(3)}^+,$$

f lifts to a surface automorphism.

2. Surface automorphism

Quadratic Cremona transformation

THEOREM. (Diller 2011)

Let C be a cuspidal cubic curve, n_1, n_2, n_3 and $\sigma \in \Sigma_3$ be orbit data. If f is a quadratic transformation properly fixing C that tentatively realizes the orbit data, then the multiplier for $f|_{C_{\text{reg}}}$ is a root of the corresponding characteristic polynomial $P(\lambda)$. Conversely, there exists a tentative realization f for each root $\lambda = a$ of $P(\lambda)$ that is not a root of unity, and f is unique up to conjugacy of linear transformation preserving C.

Uehara's formula of birational transformation

Uehara(2016) obtained an explicit formula for Cremona transformations with an invariant cuspidal cubic curve.

$$\begin{cases} f_1([x:y:z]) = (d/3)\cdot\{(v_1^2-3v_2)x^2+v_1v_3z^2-3xy+2v_1yz-(v_1v_2-3v_3)zx\} \\ f_2([x:y:z]) = (d/3)^3\cdot\{v_1(v_1^3-9v_1v_2+27v_3)x^2-27y^2+v_1^3v_3z^2+9(2v_1^2-3v_2)xy \\ \qquad\qquad +(8v_1^3-27v_1v_2+27v_3)yz-v_1^2(v_1v_2-9v_3)zx\} \\ f_3([x:y:z]) = v_1x^2+v_3z^2-yz-v_2zx, \end{cases} \tag{13}$$

where $C=\{yz^2=x^3\}$ and $v_\ell = v_\ell(a)$ are given by

$$v_1 = a_1+a_2+a_3, \quad v_2 = a_1a_2+a_2a_3+a_3a_1, \quad v_3 = a_1a_2a_3.$$

Uehara's formula in non-homogeneous coordinates

For $d \in \mathbb{C}^\times$ and $a_1, a_2, a_3 \in \mathbb{C}$ with $a_1 + a_2 + a_3 \neq 0$,

$$X = d \cdot \left\{ x + \frac{\nu_1}{3} + \frac{\nu_1(y - x^3)}{\nu_1 x^2 - \nu_2 x + \nu_3 - y} \right\},$$

$$Y = d^3 \cdot \left\{ (x + \frac{\nu_1}{3})^3 + y - x^3 + \frac{\nu_1(y - x^3)}{\nu_1 x^2 - \nu_2 x + \nu_3 - y} (\nu_1(x + \frac{\nu_1}{3}) - \nu_2) \right\},$$

where $\nu_1 = a_1 + a_2 + a_3$, $\nu_2 = a_1 a_2 + a_2 a_3 + a_3 a_1$, and $\nu_3 = a_1 a_2 a_3$.

If $y = x^3$ then $Y = X^3$.

Orbit data

Parameters d, a_1, a_2, a_3 are determined from the orbit data (n_1, n_2, n_3) with permutation $\sigma \in \Sigma_3$.

Orbit data specifies the behavior of the indeterminate points and the exceptional lines.

The value d is chosen among the eigenvalues of the cohomology homomorphism $f^* : H^2(\mathcal{S}, \mathbb{Z}) \to H^2(\mathcal{S}, \mathbb{Z})$.

Parameters a_1, a_2, a_3 are computed from d.

Differential form $\eta = \frac{dx \wedge dy}{y - x^3}$ is an eigenform.

$$f^* \eta = d \cdot \eta.$$

This defines a meromorphic volume form $\eta \wedge \bar{\eta}$.

Characteristic polynomial

For surface automorphism $f : \mathcal{S} \to \mathcal{S}$ satisfying orbit data (n_1, n_2, n_3) with permutation $\sigma \in \Sigma_3$, the characteristic polynomial of the homomorphism $f^* : H^2(\mathcal{S}, \mathbb{Z}) \to H^2(\mathcal{S}, \mathbb{Z})$ is as follows. (see [BK1])

In the case of invariant cubic curve $y = x^3$, the indeterminacy points $p_i^+ = (a_i, a_i^3)$, $i = 1, 2, 3$, of f and the indeterminacy points $p_i^- = (b_i, b_i^3)$, $i = 1, 2, 3$, of f^{-1} are computed as follows.

σ is the identity

(case 1) $\sigma = id$.

$$\chi(d) = d^{n_1+n_2+n_3+1} - 2d^{n_1+n_2+n_3} + d^{n_1+n_2} + d^{n_2+n_3} + d^{n_3+n_1}$$
$$- d^{n_1+1} - d^{n_2+1} - d^{n_3+1} + 2d - 1.$$

$$a_i = -\frac{d^{n_i-1}(d-1)}{d^{n_i}-1} + \frac{1}{3} \qquad (i = 1, 2, 3).$$
$$b_i = -\frac{d-1}{d^{n_i}-1} + \frac{1}{3} \qquad (i = 1, 2, 3).$$

σ is a transposition

(case 2) σ is a transposition $(\sigma(1) = 2, \sigma(2) = 1, \sigma(3) = 3)$.

$$\chi(d) = d^{n_1+n_2+n_3+1} - 2d^{n_1+n_2+n_3} + d^{n_1+n_2} + (d-1)(d^{n_1+n_3} + d^{n_2+n_3})$$
$$-(d-1)(d^{n_1} + d^{n_2}) + d^{n_3+1} - 2d + 1.$$

$$a_i = -\frac{d^{n_j-1}(d^{n_i}+1)(d-1)}{d^{n_i+n_j}-1} + \frac{1}{3} \qquad ((i,j) = (1,2),(2,1)).$$
$$a_k = -\frac{d^{n_k-1}(d-1)}{d^{n_k}-1} + \frac{1}{3} \qquad (k=3).$$
$$b_i = -\frac{(d^{n_j}+1)(d-1)}{d^{n_i+n_j}-1} + \frac{1}{3} \qquad ((i,j) = (1,2),(2,1)).$$
$$b_k = -\frac{d-1}{d^{n_k}-1} + \frac{1}{3} \qquad (k=3).$$

σ is a cyclic permutation

(case 3) σ is a cyclic permutation $(\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1)$.

$$\chi(d) = d^{n_1+n_2+n_3+1} - 2d^{n_1+n_2+n_3} + (d-1)(d^{n_1+n_2} + d^{n_2+n_3} + d^{n_3+n_1})$$

$$+(d-1)(d^{n_1} + d^{n_2} + d^{n_3}) + 2d - 1.$$

$$a_i = -\frac{d^{n_k-1}(d^{n_j}(d^{n_i}+1)+1)(d-1)}{d^{n_i+n_j+n_k}-1} + \frac{1}{3}$$

$$((i,j,k) = (1,2,3), (2,3,1), (3,1,2)).$$

$$b_i = -\frac{(d^{n_k}(d^{n_j}+1)+1)(d-1)}{d^{n_i+n_j+n_k}-1} + \frac{1}{3}$$

$$((i,j,k) = (1,2,3), (2,3,1), (3,1,2)).$$

Orbit data to parameters

From orbit data (n_1, n_2, n_3),σ, parameters are determined by the followings. To simplify the computations, fixed point is fixed to $(\frac{1}{3}, \frac{1}{27})$.

$$a_1 + a_2 + a_3 = \frac{1}{d} - 1.$$

$$a_{\sigma(i)} - \frac{1}{3} = d^{n_i - 1}(b_i - \frac{1}{3}),$$
$$b_i - \frac{1}{3} = d \cdot (a_i - \frac{1}{3}) + d - 1,$$
$$\text{for} \quad i = 1, 2, 3.$$

These equations have a solution iff $\chi(d) = 0$ (assuming d is not a root of unity).

Eigen meromorphic form

J.Diller et al. [DJS] proved the existence of eigen meromorphic form.

THEOREM. (Diller-Jackson-Sommese 2007)
Let $f : \mathcal{S} \to \mathcal{S}$ be an algebraically stable birational map of a complex projective surface with $\lambda(f) > 1$. Let C be a connected f-invariant curve of genus one. By contracting curves in $\mathcal{S}$, one can arrange additionally that $-C$ is the divisor of a meromorphic two-form η satisfying $f^*\eta = c\eta$. The constant c is determined solely by the curve C and the induced automorphism $f : C \to C$.

Meromorphic form η

For our map, equality

$$Y - X^3 = \frac{1}{d}(y - x^3)\,\det Df_{(x,y)}.$$

can be verified by a direct computation.

PROPOSITION $\eta = \frac{dx\wedge dy}{y-x^3}$ is an eigen two-form for f^*.

$$f^*\eta = d\cdot\eta.$$

3. Rotating attractor

Attracting annulus(?)

Attracting annuli are observed numerically for dissipative cases with orbit data

$(3, 3, 4)$, cyclic permutation,
$(2, 3, 5)$, cyclic permutation,
$(3, 2, 5)$, cyclic permutation,
$(2, 3, 6)$, cyclic permutation.

In these cases, numerical observation tells us that the basin of attraction is open and dense in the surface.

Attracting annulus(?)

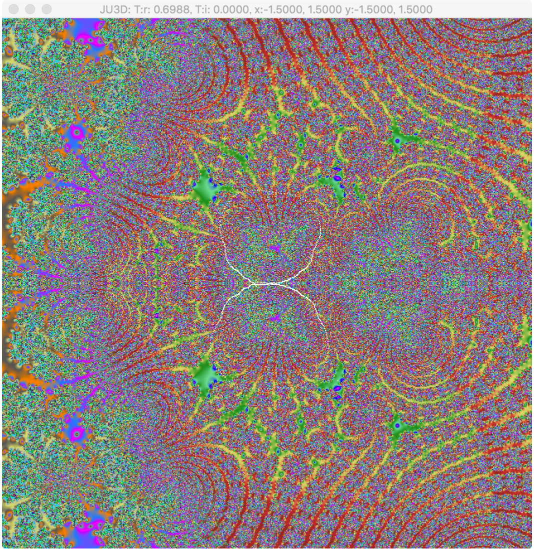
Existence of attracting annulus is a challenging problem.

Diagonal slice $\{x = y\}$, or horizontal slice $\{Y = 0\}$ is shown colored according to the norm of the derivative along each orbit.

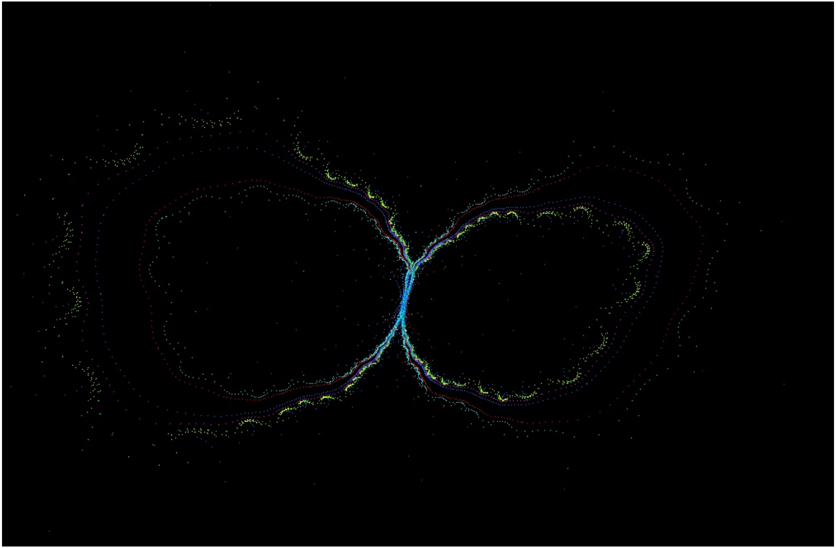
As, it seems, the Lypunov exponent $= 0$, norm of the derivative is estimated for some finite number of iterations, which suggests the transient behavior of the orbit before being attracted to the attractor.

Projection of an orbit to the slice is shown in the pictures.

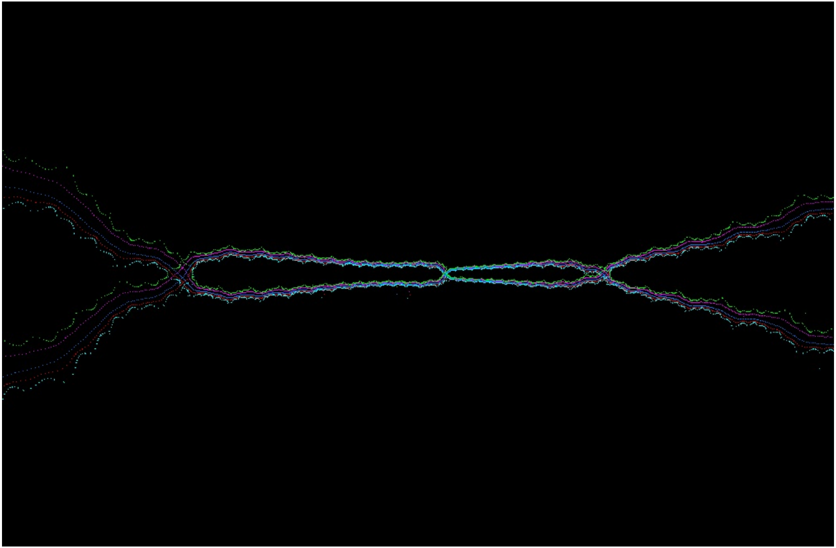
Orbit data (3,3,4), cyclic permutation, diagonal slice $\{y = x\}$



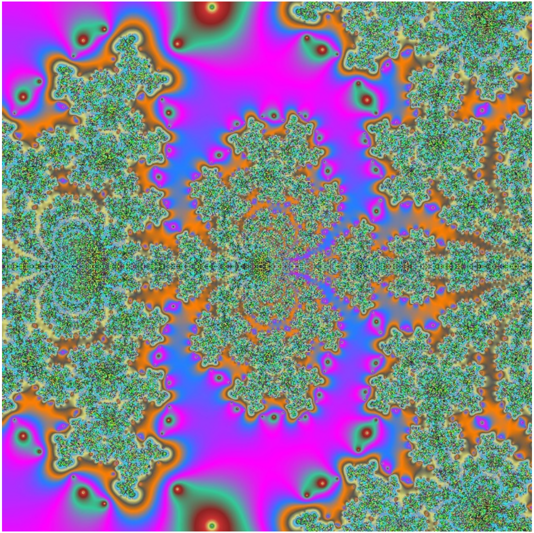
Attracting Hermann ring(?)



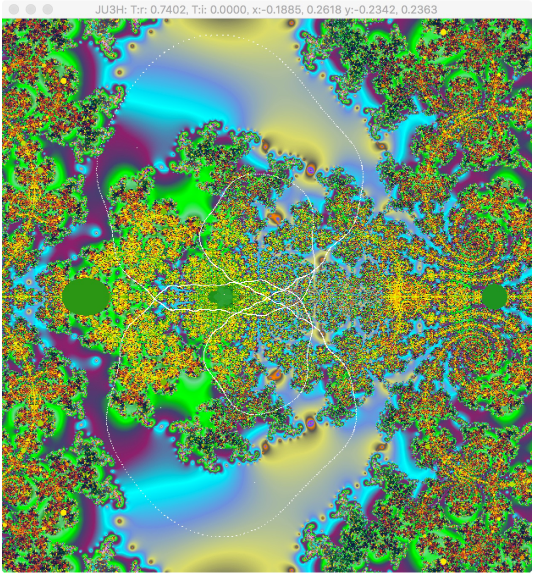
Attracting Hermann ring(?), enlarged



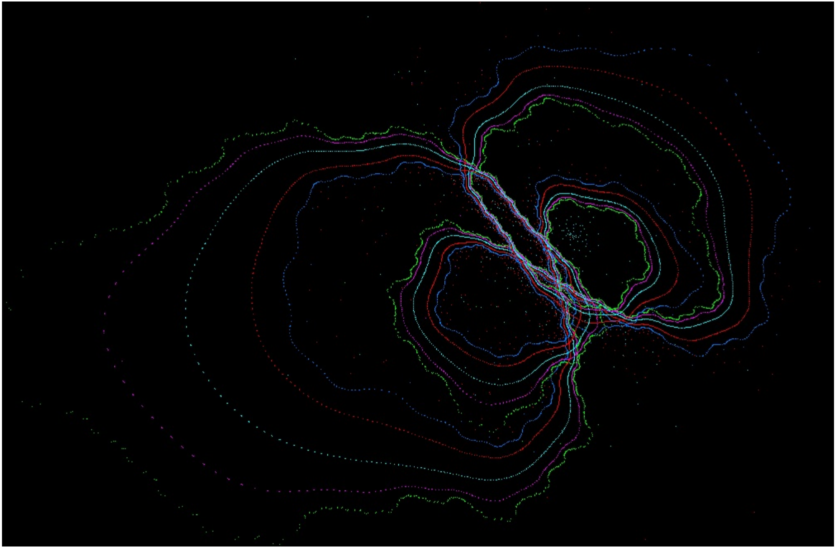
Orbit data (3,3,4), cyclic permutation, horizontal slice $\{y = 0\}$



Orbit data (3,2,5), cyclic permutation, horizontal slice $\{y = 0\}$



Attracting Hermann ring(?)



Attracting invariant line

In the dissipative case, $(0 < d < 1)$, the determinant with respect to the two-form η is equal to d.

If there is an invariant curve, disjoint from the cubic curve $\{y = x^3\}$, and the intrinsic dynamics is neutral, then this curve must be an attractor.

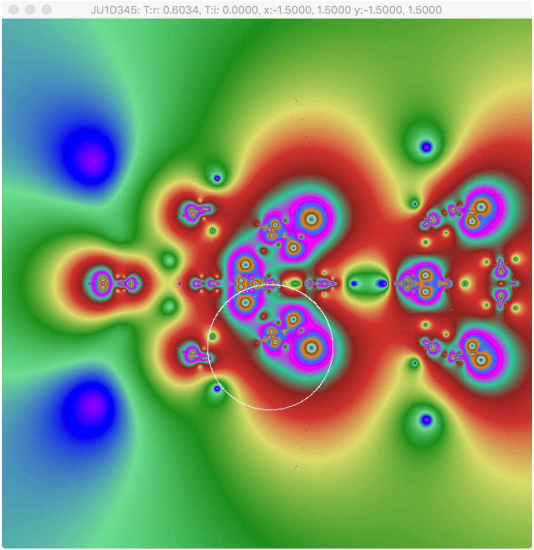
According to [DJS], invariant curve must be a tree of genus 0, if it is not contained in the cubic curve.

Invariant curve

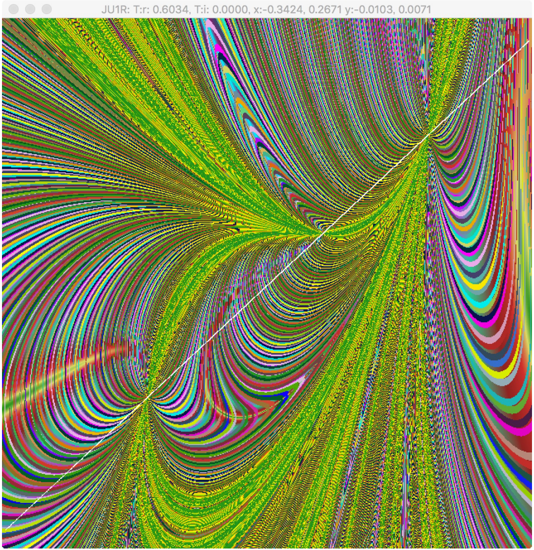
THEOREM. (Diller-Jackson-Sommese 2007)
Let $f : X \to X$ be an algebraically stable map with $\lambda(f) > 1$, and suppose that $V = f(V)$ is a connected curve with $g(V) = 1$. Then by contracting finitely many curves, one may further arrange the following.

(1) $V \sim -K_X$ is an anticanonical divisor.
(2) $I(f^n) \subset V$ for every $n \in \mathbb{Z}$.
(3) Any connected curve strictly contained in V has genus zero.
(4) If W is a connected f-invariant curve not completely contained in V, then W has genus zero, is disjoint from V, and is equal to a tree of smooth rational curves, each with self-intersection -2.

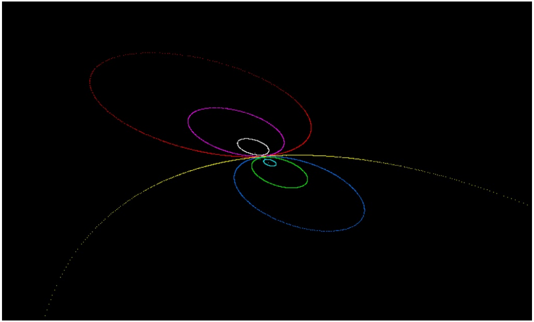
Orbit data (3,4,5), id, diagonal slice



Attracting invariant line with irrational(?) rotation, real slice



Attracting invariant line with irrational(?) rotation



Invariant line

THEOREM. In the case of orbit data $(3, n_2, n_3)$ with $\sigma(1) = 1$, the surface automorphism has an invariant line passing through three blowup points p_1^+, p_1^-, and $f(p_1^-)$.

REM. In this case, the self-intersection of the strict transform of this invariant line is -2.

PROOF. Let $p_1^+ = (a_1, a_1^3)$, $p_1^- = (b_1, b_1^3)$, and $f(p_1^-) = (c_1, c_1^3)$. Then,

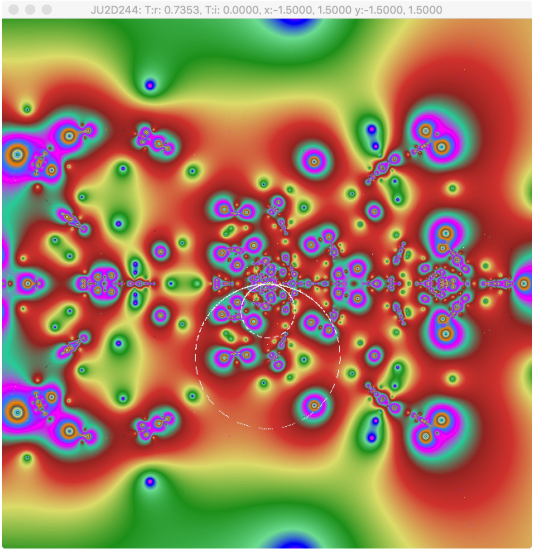
$$a_1 = -\frac{d^2(d-1)}{d^3-1} + \frac{1}{3}, \quad b_1 = -\frac{d-1}{d^3-1} + \frac{1}{3}, \quad c_1 = -\frac{d(d-1)}{d^3-1} + \frac{1}{3}.$$

Immediately we see that $a_1 + b_1 + c_1 = 0$. Hence three points $p_1^+, p_1^-, f(p_1^-)$ are on a line. Let L denote this line. As L passes through the indeterminate point p_1^+, its image $f(L)$ is a line. Since $f(L)$ passes through $p_1^+ = f^2(p_1^-)$ and $f(p_1^-)$, it coincides with L.

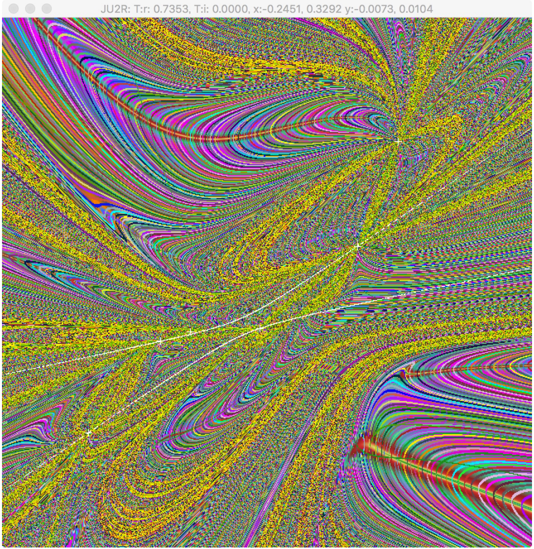
Attracting quadratic curve

There are cases where the attractor is an invariant quadratic curve, disjoint from the cubic curve.

Orbit data (2,4,4), transposition (1,2), diagonal slice



Attracting quadratic curve with irrational(?) rotation, real slice



Invariant quadratic curve

THEOREM. In the case of orbit data $(2, 4, n)$ with transposition (1,2), the surface automorphism has an invariant quadratic curve passing through six blowup points $p_1^+, p_1^-, p_2^+, p_2^-, f(p_2^-), f^2(p_2^-)$.

PROOF. Quadratic curve is mapped to a quadratic curve by Cremona transformation if the quadratic curve passes through exactly two indeterminate points. If there exists a quadratic curve passing through these 6 points, its image by f is a quadratic curve, since p_1^+ and p_2^+ are indeterminate points. Points $p_1^+ = f(p_1^-)$, $p_2^+ = f^3(p_2^-)$, $f(p_2^-)$, $f^2(p_2^-)$ are in the image quadratic curve, which must be the same quadratic curve, since 4 points determines the quadratic curve.

So, we only need to prove the existence of a quadratic curve passing through the 6 points.

Let

$$a_1 = -\frac{d(d^4+1)(d-1)}{d^6-1} + \frac{1}{3}, \quad a_2 = -\frac{d^3(d^2+1)(d-1)}{d^6-1} + \frac{1}{3},$$

$$b_1 = -\frac{(d^2+1)(d-1)}{d^6-1} + \frac{1}{3}, \quad b_2 = -\frac{(d^4+1)(d-1)}{d^6-1} + \frac{1}{3},$$

$$c_1 = -\frac{d(d^2+1)(d-1)}{d^6-1} + \frac{1}{3}, \quad c_2 = -\frac{d^2(d^2+1)(d-1)}{d^6-1} + \frac{1}{3}.$$

These are the x-coordinates of the blowup points.

$$p_1^+ = (a_1, a_1^3), \quad p_1^- = (b_1, b_1^3),$$
$$p_2^+ = (a_2, a_2^3), \quad p_2^- = (b_2, b_2^3),$$
$$f(p_2^-) = (c_1, c_1^3), \quad f^2(p_2^-) = (c_2, c_2^3).$$

Immediately, we see that

$$a_1 + a_2 + b_1 + b_2 + c_1 + c_2 = 0.$$

Consider polynomial of degree 6 :

$$P(z) = (z - a_1)(z - a_2)(z - b_1)(z - b_2)(z - c_1)(z - c_2)$$
$$= z^6 + A_4 z^4 + A_3 z^3 + A_2 z^2 + A_1 z + A_0.$$

Let $Q(x, y)$ be a quadratic polynomial defined by

$$Q(x, y) = y^2 + A_4 xy + A_3 y + A_2 x^2 + A_1 x + A_0.$$

The 6 points $p_1^+, p_1^-, p_2^+, p_2^-, f(p_2^-), f^2(p_2^-)$ satisfy $Q(x, y) = 0$. Hence the quadratic curve $Q(x, y) = 0$ passes through these 6 points.

We conclude that quadratic curve $\{Q(x, y) = 0\}$ is invariant under f.

REM. The strict transform of this quadratic curve has self-intersection -2.

4. Rotation domain

Rotation domain

Suppose Ω is a Fatou component of a volume preserving automorphism f with $f(\Omega) = \Omega$. Define the set of all limits of convergent subsequences $\mathcal{G}$ by

$$\mathcal{G} = \left\{ g = \lim_{n_j \to \infty} f^{n_j} : \Omega \to \overline{\Omega} \right\}.$$

If $g = \lim_{n_j \to \infty} f^{n_j}$ is such a limit, then g must preserve volume, and thus it is locally invertible. It follows that $g : \Omega \to \Omega$.

It is known that $\mathcal{G}$ is a compact Lie group, by a theorem of H. Cartan. The connected component $\mathcal{G}_0$ of the identity must be a (real) torus.

Rank of a rotation domain

In the volume preserving Hénon map case, known result is as follows.

THEOREM (Bedford-Smilie 1991).
$\mathcal{G}_0$ is isomorphic to $\mathbb{T}^\rho$ with $\rho = 1$ or 2.

Same result should hold for surface automorphism case.

Such a domain is called a **rotation domain**, and we refer to ρ as the **rank** of the rotation domain.

Reinhardt domain

Let $D \subset \mathbb{C}^2$ be a connected open set. We say that D is a **Reinhardt domain** if $(e^{i\theta}z, e^{i\phi}w) \in D$ for all $(z, w) \in D$ and all $\theta, \phi \in \mathbb{R}$.

If Ω is a rank 2 rotation domain, then the $\mathcal{G}$-action on Ω may be conjugated to the standard linear action on $\mathbb{C}^2$.

THEOREM. (Barrettt-Bedford-Dadok 1989) There are a Reinhardt domain $D \subset \mathbb{C}^2$, a linear map $L : (x, y) \mapsto (\alpha x, \beta y)$, $|\alpha| = |\beta| = 1$, and a biholomorphic map $\Phi : \Omega \to D$ such that $\Phi \circ f = L \circ \Phi$.

5. Exotic rotation domain

Exotic rotation domains are observed numerically by examining the slice comprising the fixed points of the involution related to reversibility.

Existence of exotic rotation domains is a challenging problem.

Reversible dynamics

We say that a map f is **reversible by an involution** τ if $\tau \circ f \circ \tau = f^{-1}$.

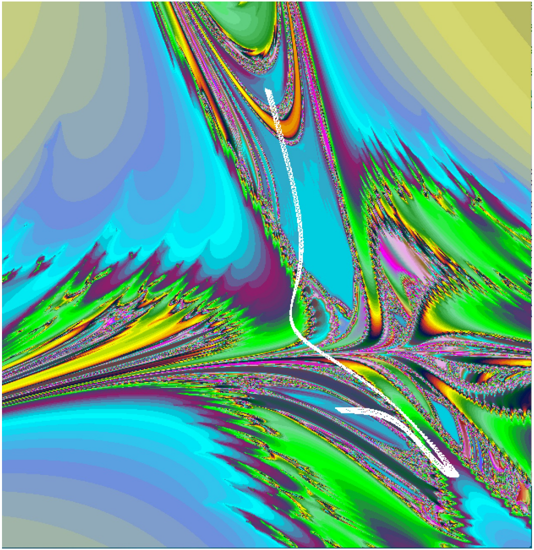
THEOREM. A Hénon map is reversible by the (anti-holomorphic) involution $\tau(x, y) = (\bar{y}, \bar{x})$ if and only if it has the form

$$f(x, y) = (y, \beta p(y) - \beta^2 x)$$

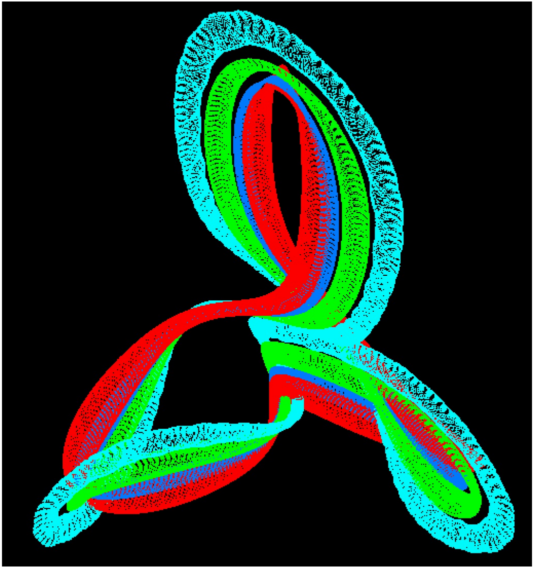
where $p(y)$ is a real polynomial and $|\beta| = 1$.

Conjugate diagonal $\Delta' = \{(x, \bar{x}) | x \in \mathbb{C}\}$ is the set of fixed points of involution τ.

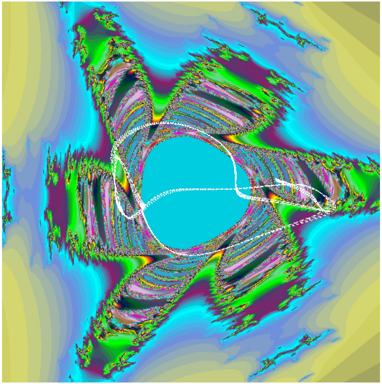
Conjugate diagonal slice for Hénon map

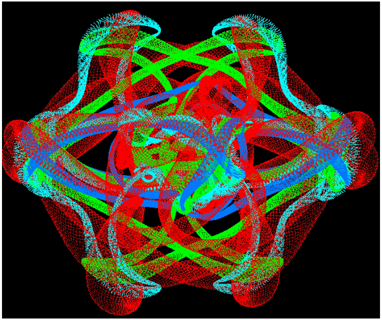


Tori(?) in an exotic rotation domain(?)



Conjugate diagonal slice for Hénon map





Conjugate reversible surface automorphisms

Let $T : \mathcal{S} \to \mathcal{S}$ be the involution of rational surface $\mathcal{S}$, defined by extending the complex conjugation $T(x, y) = (\bar{x}, \bar{y})$.

In the case of surface automorphism with invariant caspidal cubic curve, derived from a non-real eigenvalue d, some of them are reversible.

Theorem. For orbit data (n_1, n_2, n_3), with permutation $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$, the surface automorphism is reversible by T if $\sigma^{-1} = \sigma$, or $n_i = n_j$ for some $i \neq j$.

Reversibility, case 1

PROOF. There are three cases. In the followings, d is the determinant of the specified automorphism f_d. We assume d is a root of the characteristic polynomial for the orbit data, with is not a root of unity and $|d| = 1$.

(case 1) $\sigma = id$. In this case, as

$$a_i = -\frac{d^{n_i - 1}(d-1)}{d^{n_i} - 1} + \frac{1}{3}, \quad \text{and} \quad b_i = -\frac{d-1}{d^{n_i} - 1} + \frac{1}{3},$$

for $i = 1, 2, 3$, we have

$$\bar{a}_i = -\frac{d^{1-n_i}(d^{-1} - 1)}{d^{-n_i} - 1} + \frac{1}{3} = b_i.$$

Hence f_d is reversible by involution T, *i.e.*

$$f_d^{-1} = T \circ f_d \circ T.$$

Reversibility, case 2

(case 2) σ is a transposition. $(\sigma(1)=2, \sigma(2)=1, \sigma(3)=3)$
As in the preceding case, $\bar{a}_3 = b_3$. For $\{i,j\}=\{1,2\}$,

$$a_i = -\frac{d^{n_j-1}(d^{n_i}+1)(d-1)}{d^{n_i+n_j}-1} + \frac{1}{3}, \quad \text{and} \quad b_i = -\frac{(d^{n_j}+1)(d-1)}{d^{n_i+n_j}-1} + \frac{1}{3}.$$

We have, for $i=1,2$,

$$\bar{a}_i = -\frac{d^{1-n_j}(d^{-n_i}+1)(d^{-1}-1)}{d^{-n_i-n_j}-1} + \frac{1}{3} = b_j.$$

Hence f_d is reversible by involution T, *i.e.*

$$f_d^{-1} = T \circ f_d \circ T.$$

Reversibility, case 3

(case 3) σ is a cyclic permutation. $(\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1)$
For $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$,

$$a_i = -\frac{d^{n_k - 1}(d^{n_i + n_j} + d^{n_j} + 1)(d - 1)}{d^{n_i + n_j + n_k} - 1} + \frac{1}{3},$$

$$b_i = -\frac{(d^{n_i + n_j} + d^{n_j} + 1)(d - 1)}{d^{n_i + n_j + n_k} - 1} + \frac{1}{3}.$$

And

$$\begin{aligned} \bar{a}_i &= -\frac{d^{1 - n_k}(d^{-n_i - n_j} + d^{-n_j} + 1)(d^{-1} - 1)}{d^{-n_i - n_j - n_k} - 1} + \frac{1}{3} \\ &= -\frac{(d^{n_i + n_j} + d^{n_i} + 1)(d - 1)}{d^{n_i + n_j + n_k} - 1} + \frac{1}{3}. \end{aligned}$$

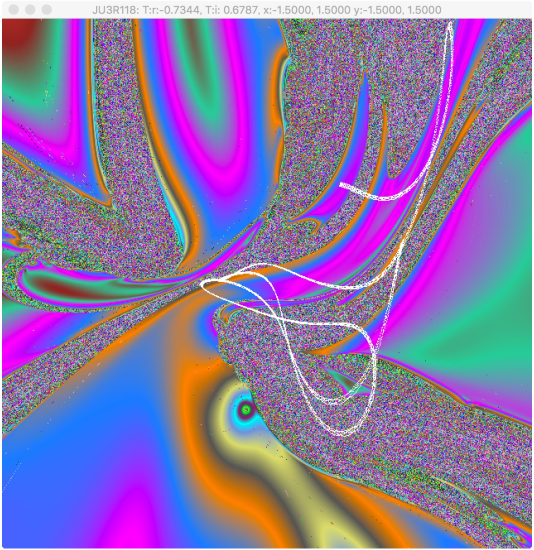
If $n_i = n_j \neq n_k$, then

$$b_i = \bar{a}_i, \quad b_j = \bar{a}_k, \quad b_k = \bar{a}_j,$$

which imply the reversibility

$$f_d^{-1} = T \circ f_d \circ T.$$

Orbit data (1,1,8), cyclic, $\mathbb{A} \times \mathbb{D}(?)$, t_3, rank 2(?).

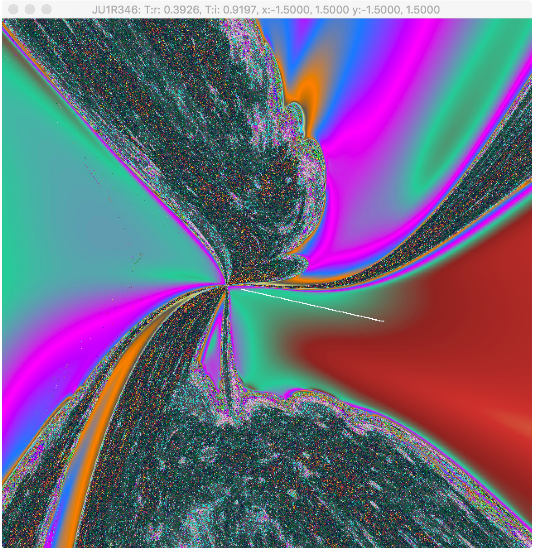


6. Invariant lines

In the volume preserving case, there exist surface automorphisms with invariant line or invariant quadratic curve.

The dynamics in the invariant line (or in the invariant quadratic curve) is conjugate to a Möbius transformation of a Riemann sphere.

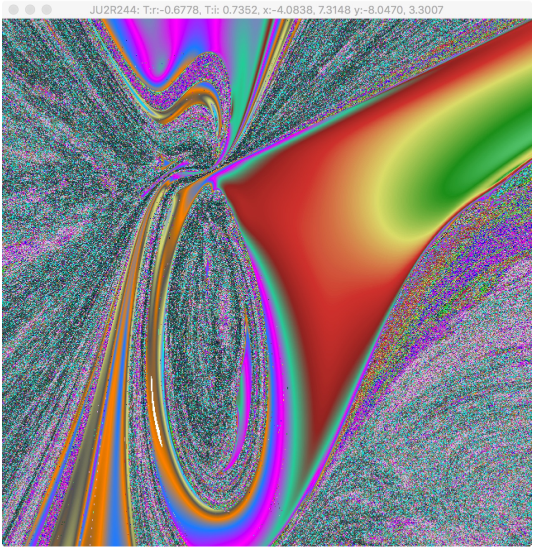
Orbit data (3,4,6), id, $\mathbb{P} \times \mathbb{D}(?)$, t_1, rank 2(?).



THEOREM. In the case of orbit data $(3, n_2, n_3)$ with $\sigma(1) = 1$, the surface automorphism has an invariant line passing through three blowup points p_1^+, p_1^-, and $f(p_1^-)$. The strict transform of this line is a curve of genus 0 with self-intersection -2.

PROOF. The proof is same as the theorem for dissipative case.

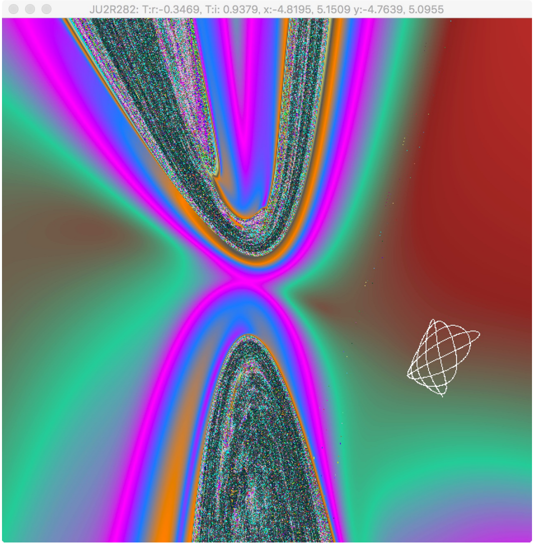
Orbit data (2,4,4), transposition (1,2), $Q \times \mathbb{D}(?)$, t_3, rank 2(?).



THEOREM. In the case of orbit data $(2, 4, n)$ with transposition (1,2), the surface automorphism has an invariant quadratic curve passing through six blowup points $p_1^+, p_1^-, p_2^+, p_2^-, f(p_2^-), f^2(p_2^-)$. The strict transform of this quadratic curve is a curve of genus 0 with self-intersection -2.

PROOF. The proof is same as the theorem for dissipative case.

Orbit data (2,8,2), transposition (2,3), $(C \cup \mathbb{P}) \times \mathbb{D}(?)$, t_3, rank 1.



Invariant line of self-intersection -1

THEOREM. In the case of orbit data $(2, n_2, n_3)$ with $\sigma(1) = 1$, the surface automorphism has an invariant line passing through two blowup points p_1^+, p_1^-, and the fixed point $p_0 = (\frac{1}{3}, \frac{1}{27})$. The strict transform of this line is a curve of genus 0 with self-intersection -1.

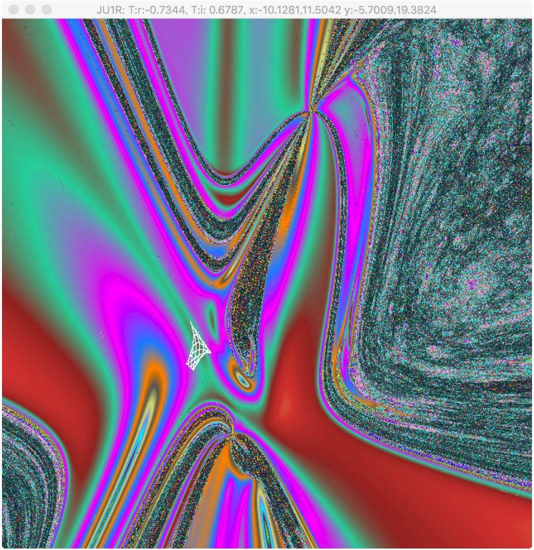
PROOF. In this case,

$$a_1 = -\frac{d(d-1)}{d^2-1} + \frac{1}{3}, \qquad b_1 = -\frac{d-1}{d^2-1} + \frac{1}{3}.$$

So, $a_1 + b_1 + \frac{1}{3} = 0$ holds. Hence, p_1^+, p_1^-, and p_0 are on a line, say L. As $p_1^+ \in L$, $f(L)$ is a line. And $f(L)$ passes through $p_1^+ = f(p_1^-)$ and $p_0 = f(p_0)$. Therefore, $f(L) = L$.

REM. In this case, the fixed point p_0 of surface automorphism f is a fixed point of Möbius transformation $f|_L : L \to L$. The eigenvalue at p_0 of $f|_L$ is $d^{3-n_1-n_2-n_3}$. The eigenvalues of the other fixed point of f contained in L are $d^{n_1+n_2+n_3-3}$ and $d^{4-n_1-n_2-n_3}$.

Orbit data (2,3,7), id, $(C \cup \mathbb{P}_1 \cup \mathbb{P}_2) \times \mathbb{D}(?)$, t_3, rank 1.

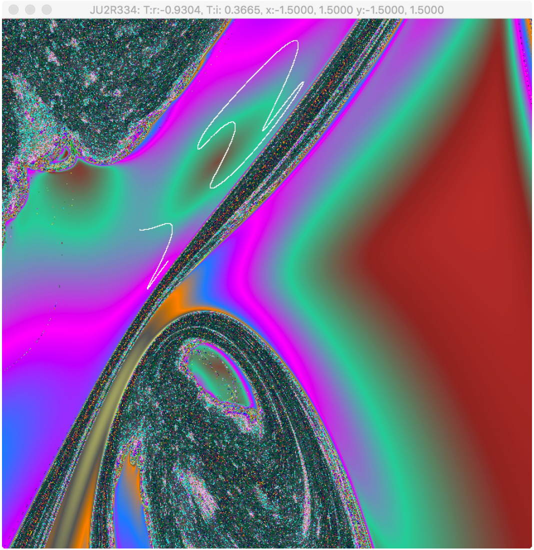


Special case

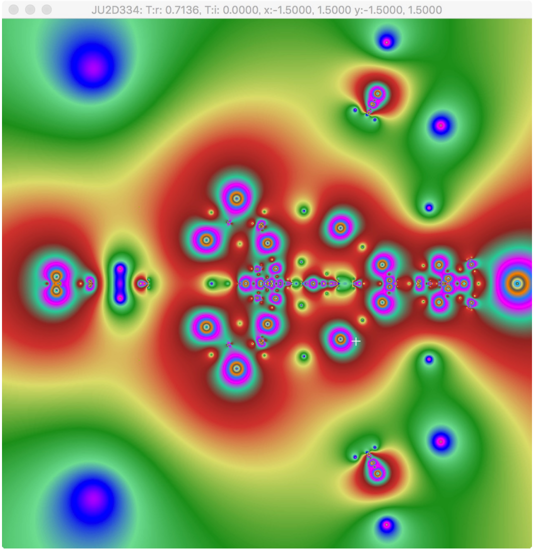
THEOREM. In the case of orbit data $(3,3,n)$, $\sigma = id.$ or $\sigma = (1,2)$, with $n \geq 4$, the surface automorphism f has an invariant line of period-three periodic points.

PROOF. Similarly as in the case of $(3, n_2, n_3)$, $\sigma(1) = 1$, f has an invariant line, say L, passing through points p_1^+, p_1^-, and $f(p_1^-)$. In this case we have $p_2^+ = p_1^+$ and $p_2^- = p_1^-$. The image $f(p_1^+)$ is the line passing through p_2^- and p_3^-. The point in the strict transform of L must be mapped to a point in the same line. So p_1^+ is mapped to p_2^-. This shows that the Möbius transformation $f|_L$ has a periodic point of period 3. Consequently, all the points of L, except for two fixed points, are periodic points of period 3.

Orbit data (3,3,4), id, $\mathbb{P} \times \mathbb{D}(?)$, t_3, rank 1.



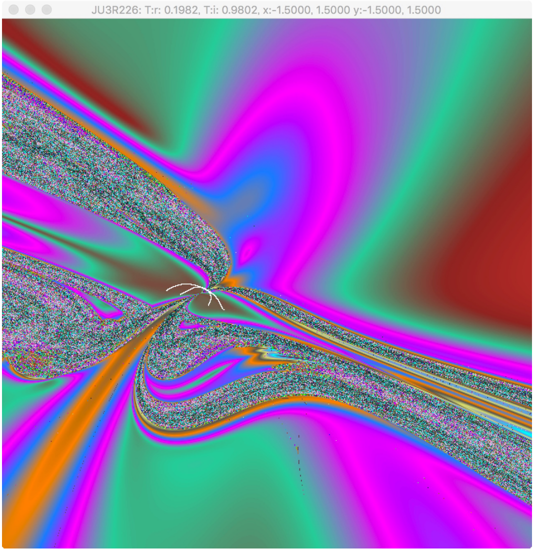
Orbit data (3,3,4), id, $\mathbb{P}$, t_r, attractor, diagonal slice



Special case

THEOREM. In the case of orbit data $(2, 2, n)$, cyclic permutation, with $n \geq 6$, the surface automorphism f has an invariant quadratic curve consisting of two lines intersecting at a point. The two lines are mapped to each other and the fixed point is linearizable.

Orbit data (2,2,6), cyclic, $(\mathbb{P} \cup \mathbb{P}) \times \mathbb{D}(?)$, t_2, rank 1.



Proof

PROOF. As stated in the reversibility case 3, we have

$$b_1 = \bar{a}_3, \quad b_2 = \bar{a}_2, \quad b_3 = \bar{a}_1.$$

Moreover, in this case, we have

$$a_3 = b_1.$$

For, as

$$a_3 = -\frac{(d-1)d}{d^{n+4}-1}(d^{n+2}+d^2+1)+\frac{1}{3},$$
$$b_1 = -\frac{d-1}{d^{n+4}-1}(d^{n+2}+d^n+1)+\frac{1}{3},$$

we have

$$a_3 - b_1 = -\frac{d-1}{d^{n+4}-1}(d^{n+3}-d^{n+2}-d^n+d^3+d-1).$$

Proof ($a_3 = b_1$)

On the other hand, in this case, d is a zero of characteristic polynomial

$$\chi(d) = (d^2 - d + 1)((d^3 - d^2 - 1)d^n + d^3 + d - 1),$$

which is not a root of unity, we conclude

$$a_3 = b_1.$$

Next, we show that three points, $p_3^+ = p_1^- = (a_3, a_3^3)$, $p_2^- = (b_2, b_2^3)$, and $p_2^+ = (a_2, a_2^3)$, are on a line.
Instead of showing $a_3 + b_2 + a_2 = 0$, we show

$$2(a_2 + b_2) + (a_3 + b_1) = 0.$$

Proof (L)

As

$$a_2 = -\frac{(d-1)d}{d^{n+4}-1}(d^{n+2}+d^n+1)+\frac{1}{3},$$
$$b_2 = -\frac{d-1}{d^{n+4}-1}(d^{n+2}+d^2+1)+\frac{1}{3},$$

we have

$$2(a_2+b_2)+(a_3+b_1) = -\frac{d+1}{d^{n+4}-1}((d^3-d^2-1)d^n+(d^3+d-1)).$$

Again, this value vanishes as d is not a root of unity and satisfies $\chi(d)=0$. Hence, three points p_3^+, p_2^+, and p_2^- are on a line.

Proof ($\tilde{L} \cap E$)

Let L denote the line passing through p_3^+, p_2^+, and p_2^-. The image of L is a blowdown point p_1^-.

Let E denote the exceptional fiber obtained by blowing up at p_1^-. E is also the exceptional fiber blown up at p_3^+. Points p_3^+ and p_1^- should be considered as a point on the invariant cubic curve C. So, we blow up at $\tilde{C} \cap E$ to have the surface automorphism, where $\tilde{C}$ denotes the strict transform of C. Hence, E is mapped to a line passing through p_1^- and p_2^-, which is L.

$\tilde{L}$ and E are mapped to each other. The intersection point $\tilde{L} \cap E$ is a fixed point.

Since the two-form $\eta = \frac{dx \wedge dy}{y - x^3}$ induces a two-form on the surface. The determinant of Df at the fixed point $\tilde{L} \cap E$ is d.

As $\tilde{L} \to E$, and $E \to \tilde{L}$, trace of Df at the fixed point is 0.
The eigenvalues at the fixed point are $\pm\sqrt{-d}$.

Proof (linearization)

Now, let us prove the linearizability of the fixed point.

Let $\lambda = \sqrt{-d}$ and $\mu = -\lambda$ be the eigenvalues of the fixed point. If there exist positive numbers c and M, such that for all integers $m \geq 0$, $n \geq 0$ with $m + n \geq 2$,

$$|\lambda^m \mu^n - \lambda| > \frac{c}{|m+n|^M}, \quad |\lambda^m \mu^n - \mu| > \frac{c}{|m+n|^M}$$

holds.

Clearly, λ is an algebraic number. It is not a root of unity. So, $\lambda^k - 1 = 0$ holds if and only if $k = 0$. And by Fel'dman's result using the Gel'fond-Baker method, applied to this case, we have

$$|k_0 2\pi i + k_1 \log \lambda| > \exp(-M(\delta + \log k_1)),$$

where δ is the degree of the algebraic number λ, M is a constant depending only on λ and δ.

Proof (Diophantine condition)

From this we can find a positive constant c such that for all $k \neq 0$,

$$|\pm\lambda^k - 1| > \frac{c}{|k|^M}$$

holds.

Now,

$$|\lambda^m \mu^n - \lambda| = |\lambda^{m-1}\mu^n - 1| = |(-1)^n \lambda^{m+n-1} - 1| > \frac{c}{|m+n-1|^M},$$

$$|\lambda^m \mu^n - \mu| = |\lambda^m \mu^{n-1} - 1| = |(-1)^m \lambda^{m+n-1} - 1| > \frac{c}{|m+n-1|^M}.$$

These indicate that the Diophantine condition at the fixed point is satisfied and the surface automorphism is linearlizable in a neighborhood of the fixed point.

7. Dynamics in L

Invariant line

In the case of orbit data $(3, m, n)$ with $\sigma(1) = 1$, there is an invariant line L.

The invariant line L passes through three points

$$p_1^+ = (a_1, a_1^3), \quad p_1^- = (b_1, b_1^3), \quad f(p_1^-) = (c_1, c_1^3),$$

with $a_1 + b_1 + c_1 = 0$.

The invariant line L is given by equation

$$y = (a_1^2 + a_1 b_1 + b_1^2)x + a_1 b_1 c_1.$$

Let z be the coordinate of L defined by

$$x = z, \qquad y = (a_1^2 + a_1 b_1 + b_1^2)z + a_1 b_1 c_1.$$

Dynamics in L

As, along L,

$$y - x^3 = -(x - a_1)(x - b_1)(x - c_1),$$

$$\nu_1 x^2 - \nu_2 x + \nu_3 - y = (x - a_1)(\nu_1 x + b_1 c_1 - a_2 a_3),$$

the dynamics $z \mapsto Z$ is given by

$$Z = d\left(z + \frac{\nu_1}{3} - \frac{\nu_1(z - b_1)(z - c_1)}{\nu_1 z + b_1 c_1 - a_2 a_3}\right).$$

This is in fact a Möbius transformation

$$Z = \frac{d(\frac{\nu_1^2}{3} - \nu_1 a_1 - a_2 a_3 + b_1 c_1)z - \frac{d}{3}\nu_1(a_2 a_3 + 2b_1 c_1)}{\nu_1 x + b_1 c_1 - a_2 a_3}.$$

Fixed point in L

The fixed points of this Möbius transformation are given by quadratic equation

$$z^2 + \tau z + \delta = 0,$$

where

$$\tau = d(a_1 - \frac{\nu_1}{3} + b_1 c_1 - a_2 a_3), \quad \delta = \frac{d}{3}(a_2 a_3 + 2 b_1 c_1).$$

(We used $\nu_1 = \frac{1-d}{d}$.)

Fixed points

PROPOSITION. $\tau,\ \delta \in \mathbb{R}$.

PROOF. If d is real, then all constants are real.

If d is not real, then the inverse map is the automorphism for $\bar{d}$. Hence all corresponding constants are complex conjugates. This means that the equation for the fixed points in L is given by

$$z^2 + \bar{\tau} z + \bar{\delta} = 0.$$

But the invariant line L and the fixed points are same. It follows that $\tau,\ \delta \in \mathbb{R}$.

Multipliers

Let $\Delta = \tau^2 - 4\delta$.
If $\Delta > 0$, then fixed points are on the real axis of L.
If $\Delta < 0$, then fixed points are not real and complex conjugate to each other.

Let λ denote the multiplier at a fixed point. (The multiplier at the other fixed point is λ^{-1}.)

PROPOSITION.
If $d \in \mathbb{R}$ and $\Delta > 0$, then $\lambda \in \mathbb{R}$.
If $d \in \mathbb{R}$ and $\Delta < 0$, then $|\lambda| = 1$.
If $|d| = 1$ and $\Delta > 0$, then $|\lambda| = 1$.
If $|d| = 1$ and $\Delta < 0$, then $\lambda \in \mathbb{R}$.

Proof (multiplier)

PROOF. Let the Möbius transformation be written as

$$Z = \frac{Az + B}{Cz + D}.$$

(case 1) If $d \in \mathbb{R}$, then we can suppose $A, B, C, D \in \mathbb{R}$. The quadratic equation of the fixed points is given by

$$Cz^2 + (D - A)z - B = 0.$$

And we have $\tau = \frac{D-A}{C}$, $\delta = -\frac{B}{C}$, and $\Delta = \frac{(D-A)^2+4BC}{C^2}$.

Let

$$\Lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The discriminant of the characteristic polynomial of this matrix is given by

$$(A + D)^2 - 4(AD - BC) = C^2\Delta.$$

Let t_1, t_2 be the eigenvalues of Λ. Then the multiplier at a fixed point of the Möbius transformation is given by $\lambda = \frac{t_1}{t_2}$.
Hence,

$$\Delta > 0 \Rightarrow C^2\Delta > 0 \Rightarrow t_1, t_2 \in \mathbb{R} \Rightarrow \lambda \in \mathbb{R}.$$

And

$$\Delta < 0 \Rightarrow C^2\Delta < 0 \Rightarrow t_2 = \bar{t_1} \Rightarrow |\lambda| = 1.$$

(case 2) If $|d| = 1$, then by the reversibility of the automorphism, the inverse Möbius transformation is given by

$$z = \frac{\bar{A}Z + \bar{B}}{\bar{C}Z + \bar{D}}.$$

Necessarily, we have

$$\bar{A}B + \bar{B}D = 0, \quad \bar{C}A + \bar{D}C = 0, \quad A\bar{A} + \bar{B}C = B\bar{C} + D\bar{D}.$$

From the third equation,

$$i\mathbb{R} \ni \bar{B}C - B\bar{C} = D\bar{D} - A\bar{A} \in \mathbb{R}.$$

So, we have

$$B\bar{C} = \bar{B}C, \quad A\bar{A} = D\bar{D}.$$

We can assume $A + D \neq 0$, since the Möbius transformation becomes a real map if $A + D = 0$. Let

$$A' = \frac{A}{A+D}, \quad B' = \frac{B}{A+D}, \quad C' = \frac{C}{A+D}, \quad D' = \frac{D}{A+D}.$$

Then, by $|A'| = |D'|$, $A' + D' = 1$, we have $D' = \bar{A}'$. And from

$$Z = \frac{A'z + B'}{C'z + \bar{A}'}, \quad z = \frac{\bar{A}'Z + \bar{B}'}{\bar{C}'Z + A'},$$

We get $B', C' \in i\mathbb{R}$.

Now, we set $B' = i\beta, \;\; C' = i\gamma, \; A' = r + i\alpha \;\; (\alpha, \beta, \gamma \in \mathbb{R})$.
The Möbius transformation is rewritten as

$$Z = \frac{(r + i\alpha)z + i\beta}{i\gamma z + r - i\alpha}.$$

The equation of fixed points is given by

$$\gamma z^2 - 2\alpha z - \beta = 0.$$

And we have

$$\tau = -\frac{2\alpha}{\gamma}, \quad \delta = -\frac{\beta}{\gamma}.$$

So,

$$Z = \frac{(\frac{r}{\gamma} - \frac{\tau}{2}i)z - i\delta}{iz + \frac{r}{\gamma} + \frac{\tau}{2}i}.$$

Let

$$\Lambda = \begin{pmatrix} \frac{r}{\gamma} - \frac{\tau}{2}i & -\delta i \\ i & \frac{r}{\gamma} + \frac{\tau}{2}i \end{pmatrix}.$$

The discriminant of the characteristic polynomial is

$$(\frac{2r}{\gamma})^2 - 4((\frac{r}{\gamma})^2 + \frac{\tau^2}{4} - \delta) = -(\tau^2 - 4\delta) = -\Delta.$$

Hence,

$$\Delta > 0 \Rightarrow -\Delta < 0 \Rightarrow t_2 = \bar{t_1} \Rightarrow |\lambda| = 1.$$

And

$$\Delta < 0 \Rightarrow -\Delta > 0 \Rightarrow t_1, t_2 \in \mathbb{R} \Rightarrow \lambda \in \mathbb{R}.$$

8. $f \circ f = g$

f, g

Surface automorphisms for orbit data $(1, 1, 8)$, *cyclic permutation*, and orbit data $(2, 4, 4)$, *cyclic permutation* are related.

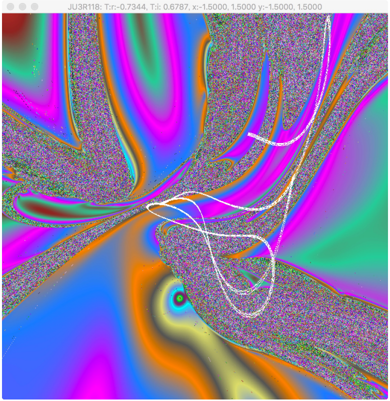
Let d, which is not a root of unity, denote an eigenvalue for orbit data $(1, 1, 8)$, *cyclic permutation*.

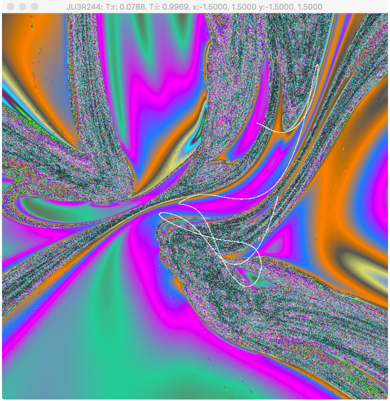
Let $f : \mathcal{S} \to \mathcal{S}$ be the surface automorphism in our family for this orbit data and determinant d.

Let $g : \mathcal{S}' \to \mathcal{S}'$ be the surface automorphism for orbit data $(2, 4, 4)$, *cyclic permutation*, with determinant $t = d^2$,

THEOREM. $\mathcal{S} = \mathcal{S}'$ and $g = f \circ f$.

Orbit data (1,1,8), cyclic, t_3, (2,4,4), cyclic, t_2.





$\psi(d^2) = -\chi(d)\chi(-d)$

Let $\chi(z)$ denote the characteristic polynomial of $f^* : H^2(\mathcal{S}) \to H^2(\mathcal{S})$.

Let $\psi(z)$ denote the characteristic polynomial of $g^* : H^2(\mathcal{S}') \to H^2(\mathcal{S}')$.

PROPOSITION. $\psi(d^2) = -\chi(d)\chi(-d)$.

PROOF. By direct computations.

$$\chi(d) = d^{11} - d^9 - d^8 + d^3 + d^2 - 1.$$

$$\psi(t) = (t-1)(t^{10} - t^9 - t^7 + t^6 - t^5 + t^4 - t^3 - t + 1).$$

$$-\chi(d)\chi(-d) = (d^{11} - d^9 + d^3)^2 - (-d^8 + d^2 - 1)^2$$

$$= (d^2 - 1)(d^{20} - d^{18} - d^{14} + d^{12} - d^{10} + d^8 - d^6 - d^2 + 1).$$

$f^* \circ f^* : H^2(\mathcal{S}) \to H^2(\mathcal{S})$

PROPOSITION. $f^* \circ f^* \simeq g^*$.

PROOF. The pullback cohomology homomorphism $f^* : H^2(\mathcal{S}) \to H^2(\mathcal{S})$ is represented as follows.

$$f^* \begin{cases} H \mapsto 2H - e_{3,1} - e_{2,1} - e_{1,8}, \\ e_{3,1} \mapsto H - e_{3,1} - e_{2,1}, \\ e_{2,1} \mapsto H - e_{2,1} - e_{1,8}, \\ e_{1,8} \mapsto e_{1,7} \mapsto e_{1,6} \mapsto \cdots \mapsto e_{1,1} \mapsto H - e_{3,1} - e_{1,8}. \end{cases}$$

$$f^* \circ f^* \begin{cases} H \mapsto 2H - e_{3,1} - e_{1,7} - e_{1,8}, \\ e_{3,1} \mapsto e_{2,1} \mapsto H - e_{3,1} - e_{1,7}, \\ e_{1,7} \mapsto e_{1,5} \mapsto e_{1,3} \mapsto e_{1,1} \mapsto H - e_{1,8} - e_{1,7}, \\ e_{1,8} \mapsto e_{1,6} \mapsto e_{1,4} \mapsto e_{1,2} \mapsto H - e_{3,1} - e_{1,8}. \end{cases}$$

This shows that $f \circ f$ is a quadratic Cremona transformation with indeterminate points at the base of $e_{3,1}, e_{1,7}, e_{1,8}$.

$f^* \circ f^* \simeq g^*$

$$g^* \begin{cases} H \mapsto 2H - E_{2,2} - E_{3,4} - E_{1,4}, \\ E_{2,2} \mapsto E_{2,1} \mapsto H - E_{2,2} - E_{3,4}, \\ E_{3,4} \mapsto E_{3,3} \mapsto E_{3,2} \mapsto E_{3,1} \mapsto H - E_{3,4} - E_{1,4}, \\ E_{1,4} \mapsto E_{1,3} \mapsto E_{1,2} \mapsto E_{1,1} \mapsto H - E_{1,4} - E_{2,2}. \end{cases}$$

The indeterminate points of g are the base points of $E_{2,2}$,$E_{3,4}$,$E_{1,4}$.

Comparing these formulas, we see that $f^* \circ f^* \simeq g^*$.

Quadratic Cremona transformation, preserving cuspidal cubic curve $\{y = x^3\}$ and fixing $(\frac{1}{3}, \frac{1}{27})$, is uniquely determined by orbit data $(n_1, n_2, n_3), \sigma$, and the determinant d, which is an eigenvalue, not a root of unity, of the cohomology homomorphism.

$\mathcal{S} = \mathcal{S}'$

As $f^* \circ f^* \simeq g^*$, $f \circ f$ and g has the same orbit data $(2, 4, 4)$, *cyclic permutation*.

The determinant of $f \circ f$ is d^2, and the determinant of g is $t = d^2$.

By the uniqueness of quadratic Cremona transformation in our family, we conclude

$$f \circ f = g, \quad \text{and} \quad \mathcal{S} = \mathcal{S}'.$$

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