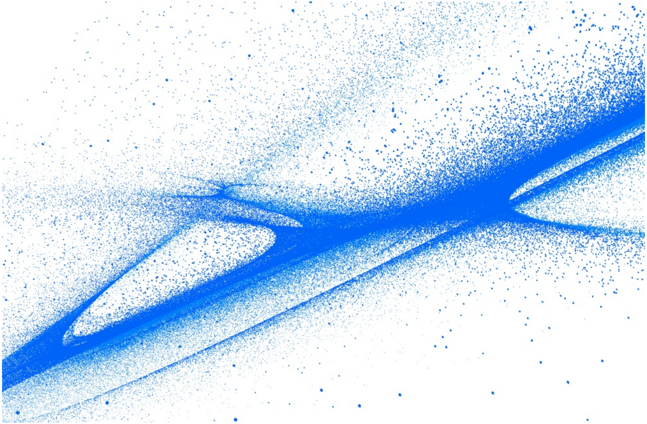
Strange Attractors in Surface Automorphisms



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Abstract

Computer assisted visualizations suggest the existence of strange attractors in dissipative complex dynamical system on complex surfaces.

Contents

1. Julia set

Fatou set

Let $f : X \to X$ be an automorphism of a compact complex manifold X.

A point $p \in X$ is a point of the **forward Fatou set** F_f^+ if there exists an open neighborhood U of p on which the sequence $\{f^n\}_{n\in\mathbb{N}}$ forms a normal family of holomorphic mappings from U to X.

Define the **backward Fatou set** F_f^- and the **Fatou set** F_f by

$$F_f^- = F_{f^{-1}}^+, \qquad F_f = F_f^+ \cap F_f^-.$$

Julia set

Define the **forward Julia set** J_f^+, the **backward Julia set** J_f^-, and the **Julia set** J_f as follows.

$$J_f^+ = X \setminus F_f^+, \quad J_f^- = X \setminus F_f^-, \text{ and } J_f = J_f^+ \cap J_f^-.$$

Let $J_f^\dagger$ denote the closure of the set of saddle periodic points.

Clearly, $J_f^\dagger \subset J_f$.

Loxodromic automorphism

Let f be an automorphism of a compact Kähler surface X.
Let $H^{1,1}(X,\mathbb{R}) = H^{1,1}(X,\mathbb{C}) \cap H^2(X,\mathbb{R})$.
Then, $f^* : H^{1,1}(X,\mathbb{R}) \to H^{1,1}(X,\mathbb{R})$ is an automorphism preserving the intersection pairing.
Define the **dynamical degree** λ_f by

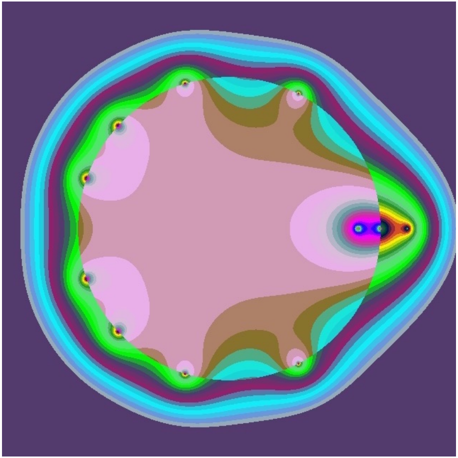
$$\lambda_f = \lim_{n\to\infty} ||f^{*n}||^{\frac{1}{n}}.$$

THEOREM. If $\lambda_f > 1$, then λ_f is an eigenvalue of f^* with multiplicity 1, and it is the unique eigenvalue with modulus > 1.

If $\lambda_f > 1$, then λ_f^{-1} is an eigenvalue, too. Other eigenvalues are of modulus 1.

f is said to be **loxodromic** if $\lambda_f > 1$.

Characteristic polynomial of a loxodromic automorphism



Invariant currents and invariant measures

Let f be a loxodromic automorphism of a compact Kähler surface X.

THEOREM(Cantat 2001, Dinh-Sibony 2005). There exist positive, closed currents T_f^+ and T_f^- with invariance property

$$f^* T_f^+ = \lambda_f T_f^+ \quad \text{and} \quad f^* T_f^- = \lambda_f^{-1} T_f^- .$$

We obtain an invariant measure $\mu_f = T_f^+ \wedge T_f^-$.

THEOREM(Bedford-Lyubich-Smilie 1993,Cantat 2003). Let $\Lambda(f,k)$ denote the set of saddle periodic points of f of period k. Then

$$\frac{1}{\lambda_f^k} \sum_{p \in \Lambda(f,k)} \delta_p$$

converges to μ_f as k goes to ∞.

Julia set J^*

We denote $\mathrm{supp}(\mu_f)$ by J_f^*.

If f is a loxodromic automorphism of a compact Kähler surface X, then

$$J_f^* \subset J_f^\dagger \subset J_f.$$

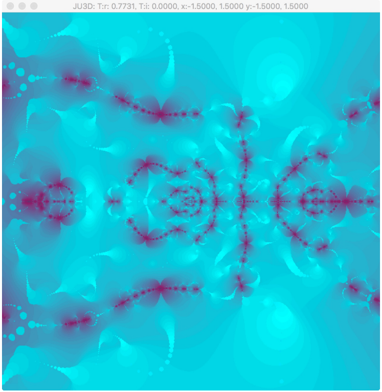
THEOREM(U., 2018). There exists a loxodromic automorphism f of a compact Kähler surface X such that

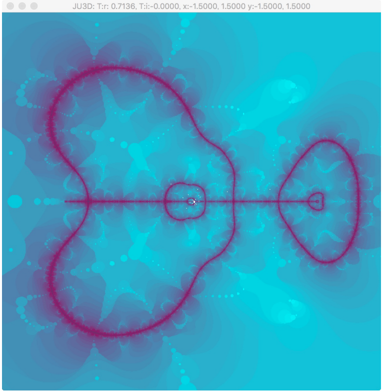
$$J_f^* \neq J_f^\dagger \neq J_f.$$

REMARK.
If f is a Hénon map, then $J_f^* = J_f^\dagger \subset J_f$.
If f is a hyperbolic Hénon map, then $J_f^* = J_f^\dagger = J_f$.

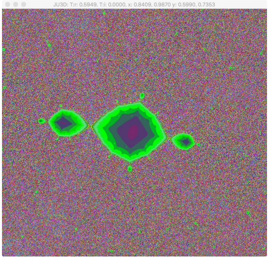
2. Attractor

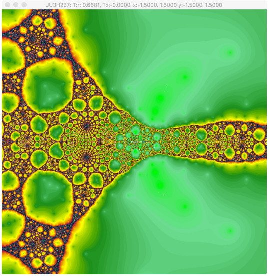
Attracting fixed point

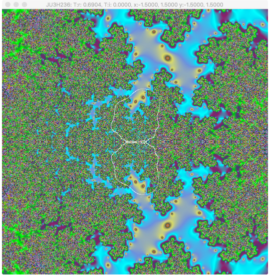




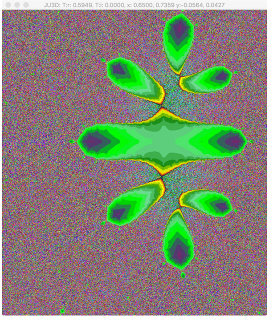
Pictures of F^+ and J^+

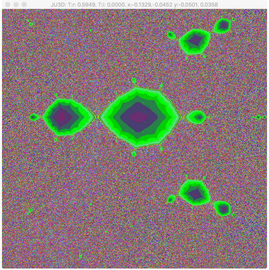


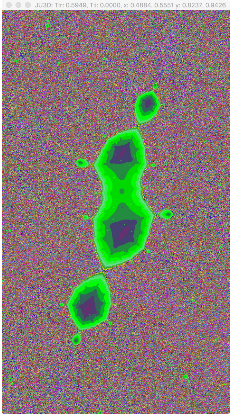


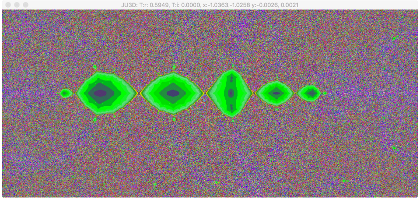


Pictures of attracting periodic point

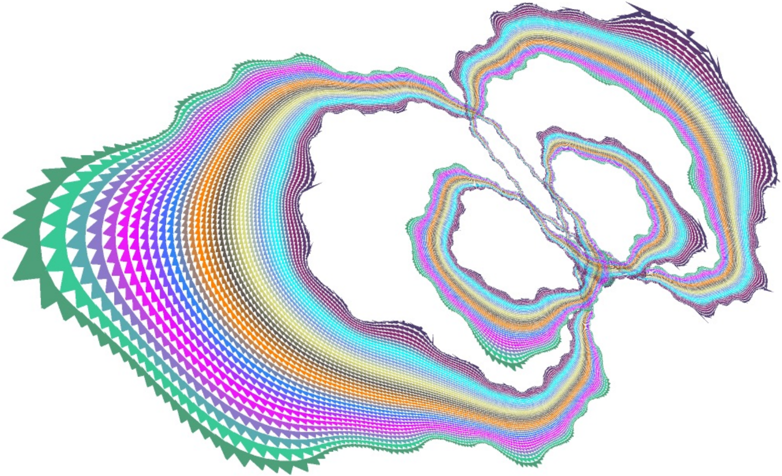




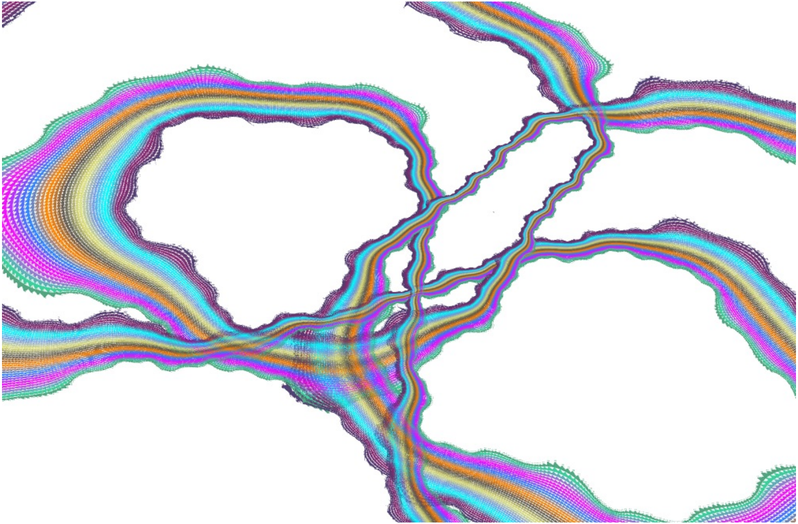




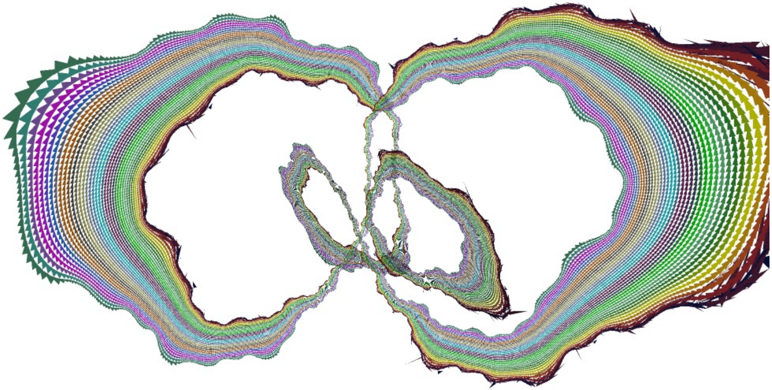
Attracting Herman ring (numerically observed)



attracting Herman ring



attracting Herman ring



Numerically observed attractor

We have observed (dissipative case):
Attracting fixed point,
Attracting periodic cycle,
Attracting Herman ring (numerical observation only),
Attracting Riemann sphere.

Though we have not found parabolic basin and Siegel disk as attractor, in these cases,

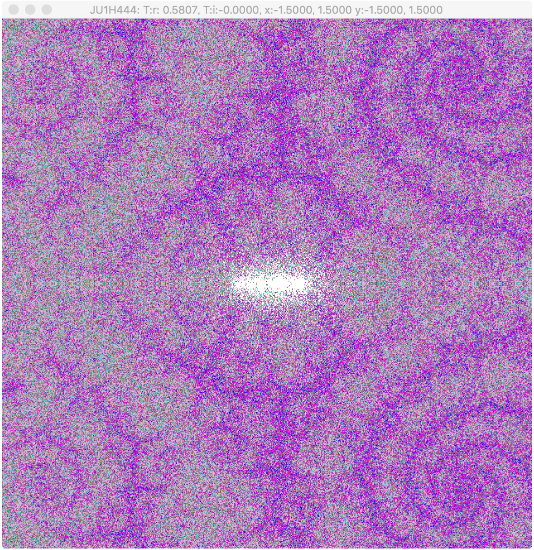
$$F^+ \neq \emptyset.$$

In this note, we report examples of surface automorphisms suggesting

$$F^+ = \emptyset, \quad \text{and} \quad J^+ = X,$$

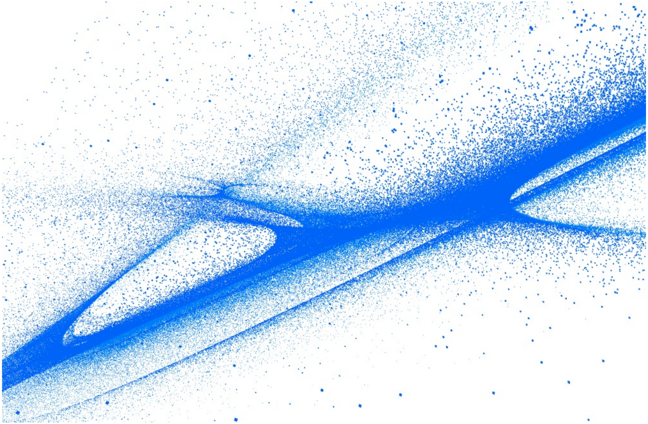
which are found numerically. (No mathematical proof.)

Chaotic slice



Plot of an orbit

This picture represents the forward orbit of a randomly chosen initial point by an automorphism $f : X \to X$ of complex surface.



Our conjecture

Our conjecture is:

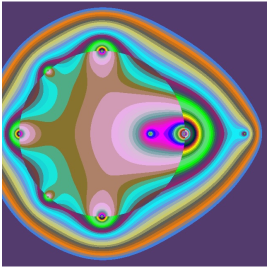
There exists a surface automorphism for which $\mathrm{supp}(\mu_f)$ is an attractor.

In the following sections, we construct a surface automorphism which seems to have a strange attractor.

As general construction of surface automorphisms is complicated, we explain only concrete construction.

3. Surface automorphism

Quadratic birational transformation



Let $\lambda = 0.580691832...$ be the smallest positive real root of equation

$$z^4 - z^3 - z^2 - z + 1 = 0.$$

Let $a = 0.240694602...$ be given by

$$a = \frac{\lambda^3(\lambda - 1)}{1 - \lambda^4} + \frac{1}{3}.$$

Explicit formula for invariant cuspidal cubic curve case

Define quadratic birational map $f : \mathbb{C}^2 \dashrightarrow \mathbb{C}^2$, $(x, y) \mapsto (X, Y)$ by

$$X = \lambda \left(x + a + \frac{3a(y - x^3)}{3ax^2 - 3a^2x + a^3 - y} \right),$$

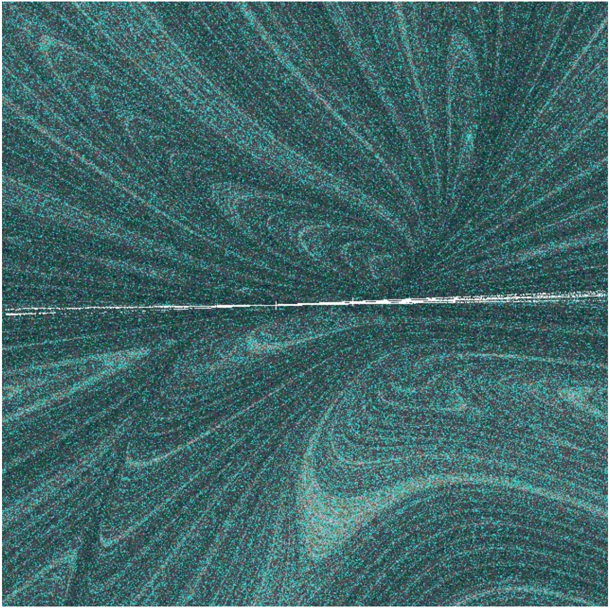
$$Y = \lambda^3 \left((x + a)^3 + (y - x^3)(1 + \frac{9a^2x}{3ax^2 - 3a^2x + a^3 - y}) \right).$$

This map preserves cubic curve $\{y = x^3\}$.
Point (a, a^3) is a point of indeterminacy (degenerated).
f is symmetric with respect to the complex conjugacy.

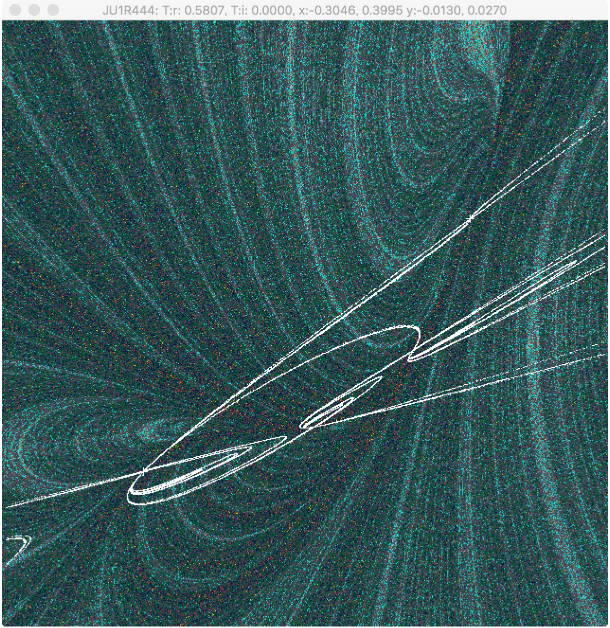
$$f(\bar{x}, \bar{y}) = (\bar{X}, \bar{Y}).$$

The real subspace is invariant under f.

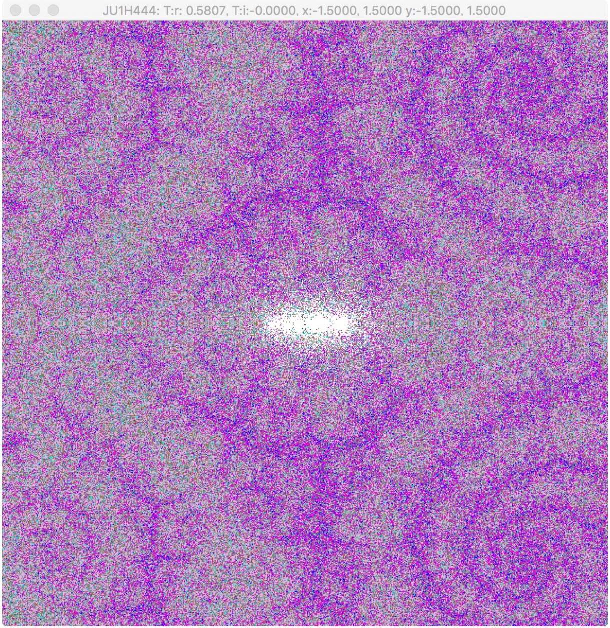
Real slice (forward orbit)



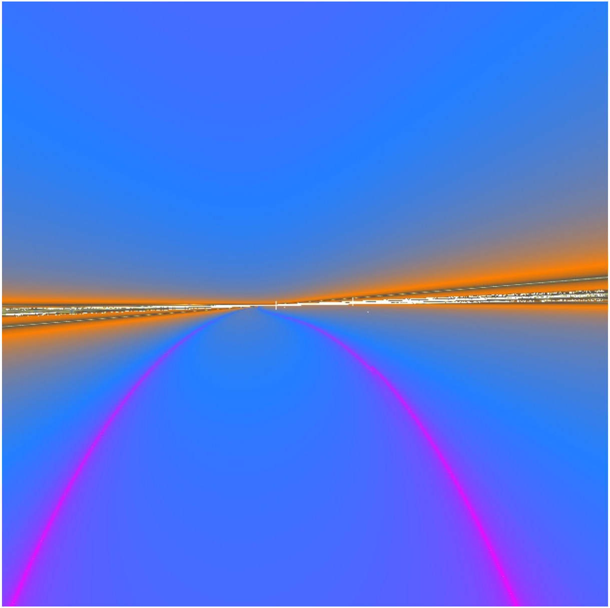
Real slice (forward orbit)



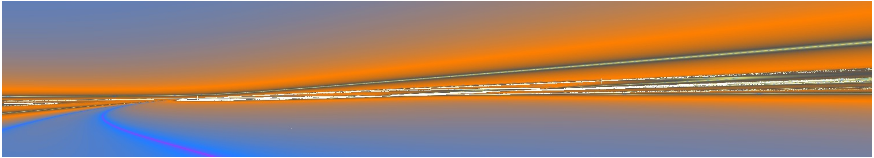
Horizontal slice(forward orbit)



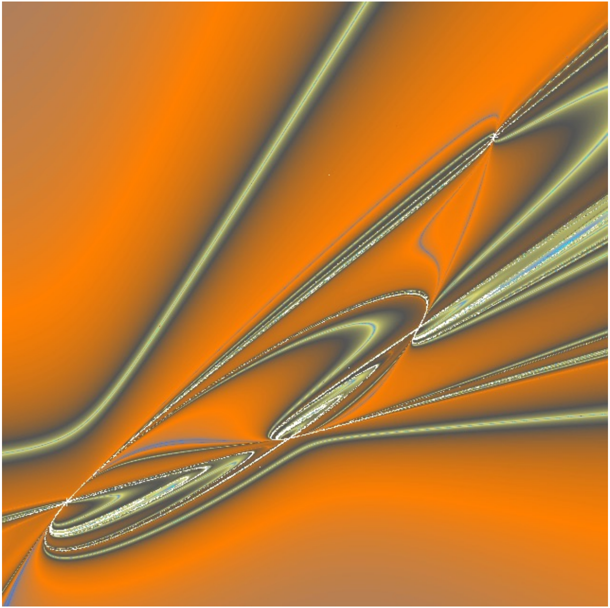
Real slice(backward orbit)



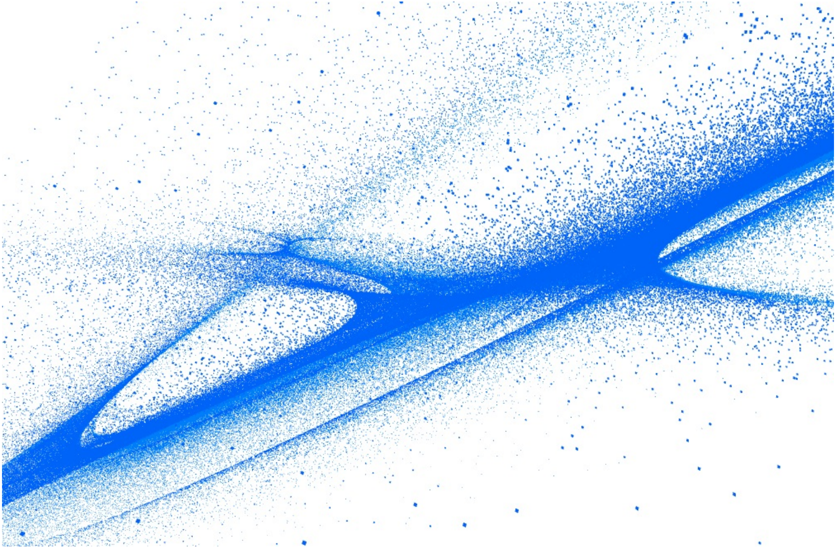
Real slice(backward orbit)



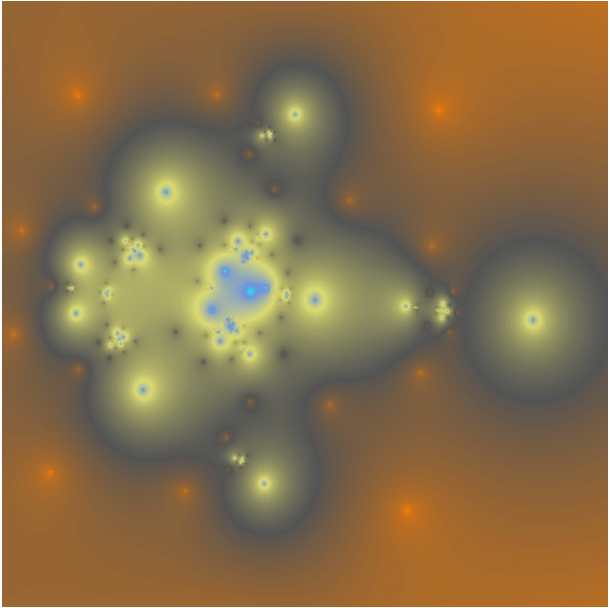
Real slice(backward orbit)



Plot of an orbit



Vertical slice(backward orbit)



Volume

The cubic curve $C = \{y = x^3\}$ is invariant under f.
Let

$$\eta = \frac{1}{y - x^3} dx \wedge dy.$$

η defines a meromorphic (2,0)-form on the surface X, with a simple pole along C and no other poles or zeros.

The form η determines a natural volume measure

$$\mathrm{vol}(U) = \int_U \eta \wedge \bar{\eta},$$

locally finite on $X \setminus C$, but of infinite total mass.

Determinant

$f^*\eta$ is also a meromorphic (2,0)-form on X with a simple pole along C and no other poles or zeros.

It is propotional to η :

$$f^*\eta = \delta(f) \cdot \eta.$$

$\delta(f)$ is called the **determinant** of f.

THEOREM. $\delta(f) = \lambda$.

Here λ is the multiplier $D(f|_C)$.

When $0 < \lambda < 1$, f is dissipative.

$$\int_{f(U)} \eta \wedge \bar{\eta} = \int_U f^*\eta \wedge f^*\bar{\eta} = \lambda\bar{\lambda} \int_U \eta \wedge \bar{\eta}.$$

Fixed points in C

The point $(0, \infty)$ and $(\frac{1}{3}, \frac{1}{27})$ are fixed points in C, with eigenvalues

$$\begin{array}{ll} \lambda^{-2} \text{ and } \lambda^{-3} & \text{at } (0, \infty), \\ \lambda \text{ and } \lambda^{-9} & \text{at } (\frac{1}{3}, \frac{1}{27}) \ . \end{array}$$

$(0, \infty)$ is a source (repeller), and $P_0 = (\frac{1}{3}, \frac{1}{27})$ is a saddle.

The stable manifold $W^s(P_0)$ of this saddle coincides with the invariant cubic curve C.

This saddle point does not have homoclinic points.

The unstable manifold $W^u(P_0)$ of this saddle does not intersect with $\mathrm{supp}(\mu_f)$,

but its closure contains $\mathrm{supp}(\mu_f)$.

expected picture

Let Q be a saddle periodic point in $X \setminus C$.

$$\mathrm{supp}(\mu_f) = \overline{W^u(Q)} = \overline{W^u(P_0)} \setminus W^u(P_0).$$

4. Invariant cuspidal cubic curve

Cubic curve

Let C denote the cubic curve $\{y = x^3\}$ in $\mathbb{C}^2 \subset \mathbb{P}^2$.

This curve has a parametrization

$$p : \mathbb{C} \to C, \quad p(t) = (t, t^3).$$

We want to find birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, which maps C onto itself.

$$f(C) = C.$$

f has indeterminate points $I(f)$. The equality should be understood "modulo exceptional points".

$$f(C) = \overline{f(C \setminus I(f))}.$$

f induces an automorphism of the cubic curve C, which can be described by an affine map $t \mapsto \lambda(t+\mu)$ for some constants $\lambda \in \mathbb{C}^\times, \mu \in \mathbb{C}$.

Proposition. For $\lambda \in \mathbb{C}^\times$ and $a_1, a_2, a_3 \in \mathbb{C}$ with $a_1 + a_2 + a_3 \neq 0$, there exists a quadratic birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, such that

$$f(C) = C, \quad I(f) = \{p(a_1), p(a_2), p(a_3)\},$$

inducing $t \mapsto \lambda(t + \frac{\nu_1}{3})$, with $\nu_1 = a_1 + a_2 + a_3$.

In affine coordinates, we set $f(x,y) = (\frac{f_1(x,y)}{f_3(x,y)}, \frac{f_2(x,y)}{f_3(x,y)})$

PROOF. Let $\nu_2 = a_1a_2 + a_2a_3 + a_3a_1$ and $\nu_3 = a_1a_2a_3$.
The indeterminate points, $p(a_i) = (a_i, a_i^3), i = 1, 2, 3$, are common zeros of the system of equations

$$\begin{cases} y - x^3 & = & 0 \\ x^3 - \nu_1 x^2 + \nu_2 x - \nu_3 & = & 0 \end{cases}.$$

As quadratic polynomial $f_3(x,y)$ must vanish in these indeterminacy points, we can choose

$$f_3(x,y) = \nu_1 x^2 - \nu_2 x + \nu_3 - y.$$

Since $f(p(t)) = p(\lambda(t + \frac{\nu_1}{3}))$ for $t \in \mathbb{C}$, $f : (x, y) \mapsto (X, Y)$ can be written as

$$X = \lambda \left(x + \frac{\nu_1}{3} + \frac{(y - x^3)U(x, y)}{f_3(x, y)} \right),$$

$$Y = \lambda^3 \left((x + \frac{\nu_1}{3})^3 + \frac{(y - x^3)V(x, y)}{f_3(x, y)} \right),$$

where polynomials $U(x, y), V(x, y)$ are chosen so that f becomes a quadratic rational map.

To determine polynomials $U(x,y)$ and $V(x,y)$ we require that

$$f_1(x,y) = \lambda\left((x+\frac{\nu_1}{3})f_3(x,y) + (y-x^3)U(x,y)\right),$$

$$f_2(x,y) = \lambda^3\left((x+\frac{\nu_1}{3})^3 f_3(x,y) + (y-x^3)V(x,y)\right)$$

are quadratic polynomials. We get

$$U(x,y) = \nu_1,$$

$$V(x,y) = \nu_1 x^2 + (\nu_1^2 - \nu_2)x - y + \frac{\nu_1^3}{3} - \nu_1\nu_2 + \nu_3.$$

This gives the explicit formula for the quadratic birational map f.

Explicit formula for invariant cuspidal cubic curve case

PROPOSITION. The quadratic birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ in the previous proposition is given by

$$X = \lambda \left(x + \frac{\nu_1}{3} + \frac{\nu_1 (y - x^3)}{\nu_1 x^2 - \nu_2 x + \nu_3 - y} \right),$$

$$Y = \lambda^3 \left((x + \frac{\nu_1}{3})^3 + (y - x^3)(1 + \frac{\nu_1^2 x + \frac{\nu_1^3}{3} - \nu_1 \nu_2}{\nu_1 x^2 - \nu_2 x + \nu_3 - y}) \right).$$

Exceptional lines

A quadratic birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ always acts by blowing up three indeterminacy points in $\mathbb{P}^2$ and blowing down the three exceptional lines joining them.

The inverse map f^{-1} is also quadratic and the images of three exceptional lines of f are the indeterminacy points of f^{-1}.

Parametrization and lines

Our parametrization $p : \mathbb{C} \to C$ of the invariant cubic curve has a nice property.

If three points $p(t_1), p(t_2), p(t_3)$ are on a line, say $\{y = ax + b\}$, then

$$t_i^3 - at_i - b = 0, \qquad i = 1, 2, 3,$$

which shows that t_1, t_2, t_3 are three roots of cubic equation $t^3 - at - b = 0$, hence $t_1 + t_2 + t_3 = 0$.

Conversely, if $t_1 + t_2 + t_3 = 0$, then t_1, t_2, t_3 are the three roots of cubic equation in t :

$$t^3 + (t_1 t_2 + t_2 t_3 + t_3 t_1)t - t_1 t_2 t_3 = 0,$$

which implies that $p(t_1), p(t_2), p(t_3)$ are on a line.

Inverse map

In order to compute the inverse map of f, we need to find the indeterminacy points of f^{-1}, which are the images of the exceptional lines of f.

Suppose the exceptional line passing through indeterminacy points $p(a_j)$ and $p(a_k)$ is mapped to $p(b_i)$, for $\{i, j, k\} = \{1, 2, 3\}$. This exceptional line intersects with C at $p(-a_j - a_k)$, which is mapped to $p(b_i)$, with

$$b_i = \lambda(-a_j - a_k + \frac{\nu_1}{3}) = \lambda(a_i - \frac{2\nu_1}{3}).$$

The dynamics of f^{-1} in the invariant curve C is

$$t \mapsto \lambda^{-1}(t - \frac{\lambda\nu_1}{3}).$$

Construction of the inverse map is similar.

Inner dynamics

Let $\tau : t \mapsto \lambda(t + \nu_1/3)$ denote the dynamics in C.
τ has a unique fixed point $t_0 = \frac{1}{3}\frac{\lambda\nu_1}{1-\lambda}$.
By linear change of variables $t = rt'$, where $r = \frac{\lambda\nu_1}{1-\lambda}$, τ is conjugate to

$$\tau' : t' \mapsto \lambda(t' + \frac{1-\lambda}{3\lambda}),$$

whose fixed point is $\frac{1}{3}$.
So, by linear change of coordinates $x = rx'$, and $y = r^3y'$, with $a_i = ra_i', i = 1, 2, 3$, birational map f has fixed point $(\frac{1}{3}, \frac{1}{27})$.

To construct surface automorphisms by blow-ups, we may suppose that f fixes $(\frac{1}{3}, \frac{1}{27})$.

Surface automorphism

We have

$$I(f) = \{p(a_1), p(a_2), p(a_3)\}$$

and

$$I(f^{-1}) = \{p(b_1), p(b_2), p(b_3)\}$$

If, for some positive integers n_1, n_2, n_3, and permutation $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$,

$$p(a_{\sigma(i)}) = f^{\circ(n_i - 1)} p(b_i), \qquad i = 1, 2, 3,$$

holds, then f lifts to a surface automorphism by blowing up $(n_1 + n_2 + n_3)$ points (provided they are all distinct)

$$p(b_i), f(p(b_i)), \cdots, f^{\circ(n_i - 1)}(p(b_i)), \qquad i = 1, 2, 3.$$

Orbit data

Positive integers (n_1, n_2, n_3) with permutation σ is said an **orbit data**.

Following Diller, we look for determinant λ and a quadratic birational transformation f, which maps C onto itself and realizes the prescribed orbit data.

conditions

In terms of inner dynamics, the conditions are as follows.

$$a_{\sigma(i)} = \lambda^{n_i - 1}(b_i - \frac{1}{3}) + \frac{1}{3}, \quad i = 1, 2, 3,$$

$$b_i = \lambda a_i + \frac{2}{3}(\lambda - 1), \qquad i = 1, 2, 3,$$

$$a_1 + a_2 + a_3 = \frac{1}{\lambda} - 1.$$

Eliminate $a_i, b_i, i = 1, 2, 3$, to obtain an equation in λ, which is a necessary condition.

Polynomial equations for orbit data n_1, n_2, n_3, σ

Necessary condition $P(\lambda) = 0$ is given by followings.

(case id) $\sigma = id$.

$$P(\lambda) = (\lambda - 2)\lambda^{n_1+n_2+n_3} + \lambda^{n_1+n_2} + \lambda^{n_2+n_3} + \lambda^{n_3+n_1}$$

$$- \lambda^{n_1+1} - \lambda^{n_2+1} - \lambda^{n_3+1} + 2\lambda - 1.$$

(case tr) σ is a transposition $(\sigma(1) = 2, \sigma(2) = 1, \sigma(3) = 3)$.

$$P(\lambda) = (\lambda - 2)\lambda^{n_1+n_2+n_3} + \lambda^{n_1+n_2} + (\lambda - 1)(\lambda^{n_1+n_3} + \lambda^{n_2+n_3})$$

$$-(\lambda - 1)(\lambda^{n_1} + \lambda^{n_2}) + \lambda^{n_3+1} - 2\lambda + 1.$$

(case cy) σ is a cyclic permutation $(\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1)$.

$$P(\lambda) = (\lambda - 2)\lambda^{n_1+n_2+n_3} + (\lambda - 1)(\lambda^{n_1+n_2} + \lambda^{n_2+n_3} + \lambda^{n_3+n_1})$$

$$+(\lambda - 1)(\lambda^{n_1} + \lambda^{n_2} + \lambda^{n_3}) + 2\lambda - 1.$$

Picard coordinate of indeterminate points

(case id) $\sigma = id$.

$$a_i = -\frac{\lambda^{n_i-1}(\lambda-1)}{\lambda^{n_i}-1} + \frac{1}{3} \qquad (i = 1, 2, 3).$$

(case tr) $\sigma = (1, 2)$

$$a_i = -\frac{\lambda^{n_j-1}(\lambda^{n_i}+1)(\lambda-1)}{\lambda^{n_i+n_j}-1} + \frac{1}{3} \qquad ((i, j) = (1, 2), (2, 1)).$$

$$a_k = -\frac{\lambda^{n_k-1}(\lambda-1)}{\lambda^{n_k}-1} + \frac{1}{3} \qquad (k = 3).$$

(case cy) $\sigma = (1, 2, 3)$

$$a_i = -\frac{\lambda^{n_k-1}(\lambda^{n_j}(\lambda^{n_i}+1)+1)(\lambda-1)}{\lambda^{n_i+n_j+n_k}-1} + \frac{1}{3}$$

$$((i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)).$$

Characteristic polynomial

Orbit data determines the characteristic polynomial $P(\lambda)$ of $f^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$.

Bedford and Kim [BK1] have computed explicitly for any orbit data n_1, n_2, n_3, σ.

$$P(\lambda) = \lambda^{1+\Sigma n_j} p(\frac{1}{\lambda}) + (-1)^{\text{ord}\sigma} p(\lambda),$$

where

$$p(\lambda) = 1 - 2\lambda + \sum_{j=\sigma_j} \lambda^{1+n_j} + \sum_{j \neq \sigma_j} \lambda^{n_j}(1-\lambda).$$

The polynomial $P(\lambda)$ obtained as a necessary condition and the characteristic polynomial $P(\lambda)$ coincide (not by chance).

5. Examples

Pictures above are for orbit data $(n_1, n_2, n_3) = (4, 4, 4)$ with $\sigma = id$.

(case id) $\sigma = id$.

$$P(\lambda) = (\lambda - 2)\lambda^{n_1+n_2+n_3} + \lambda^{n_1+n_2} + \lambda^{n_2+n_3} + \lambda^{n_3+n_1} - \lambda^{n_1+1} - \lambda^{n_2+1} - \lambda^{n_3+1} + 2\lambda - 1.$$

(case id) $\sigma = id$.

$$a_i = -\frac{\lambda^{n_i-1}(\lambda - 1)}{\lambda^{n_i} - 1} + \frac{1}{3} \qquad (i = 1, 2, 3).$$

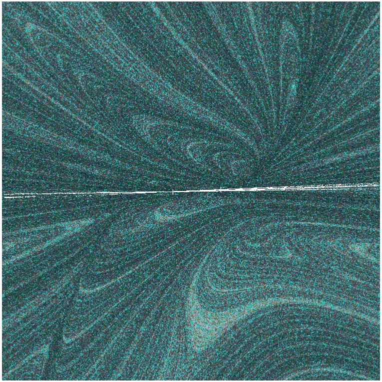
The characteristic polynomial is as follows.

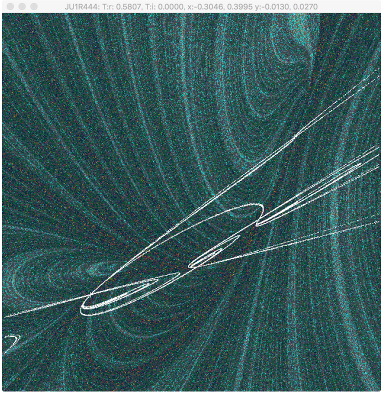
$$P(\lambda) = (\lambda - 1)(\lambda^4 - 1)^2(\lambda^4 - \lambda^3 - \lambda^2 - \lambda + 1).$$

The inner coordinates of indeterminate points are :

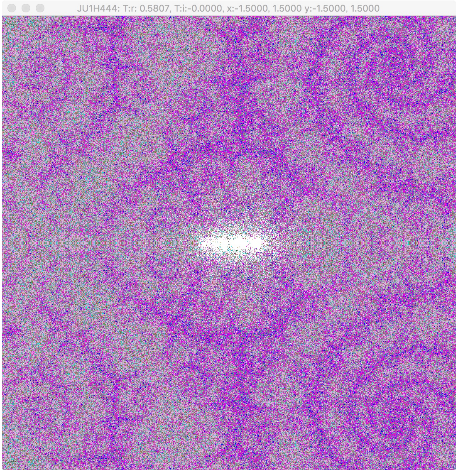
$$a = a_1 = a_2 = a_3 = \frac{\lambda^3(\lambda - 1)}{1 - \lambda^4} + \frac{1}{3}.$$

Real slice (forward orbit)

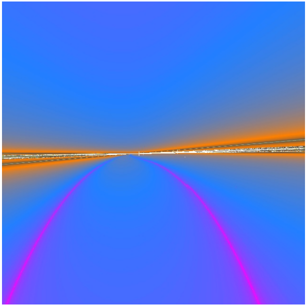


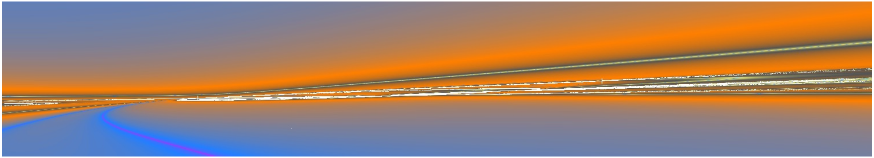


Horizontal slice(forward orbit)

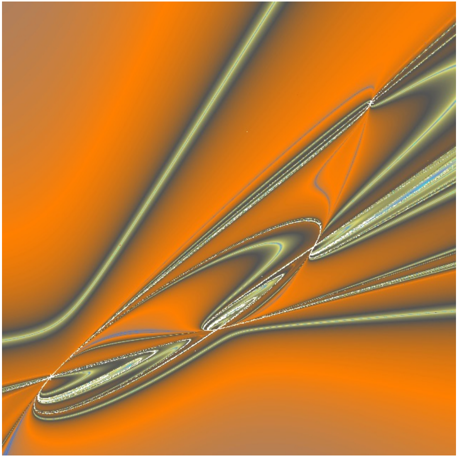


Real slice(backward orbit)

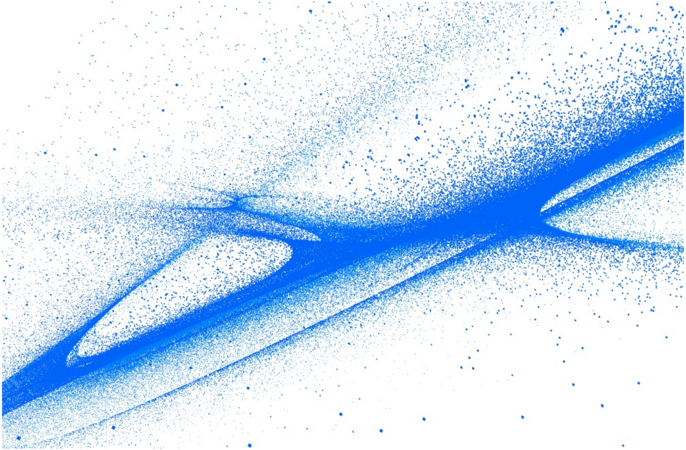




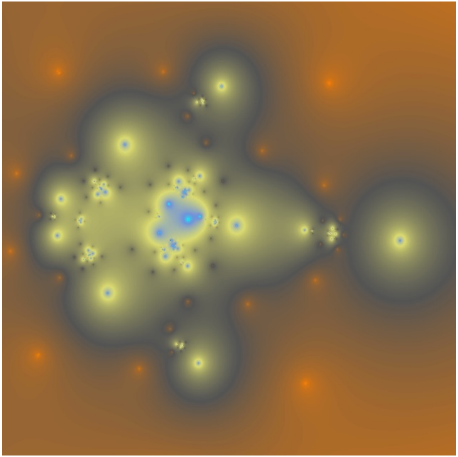
Real slice(backwardorbit)



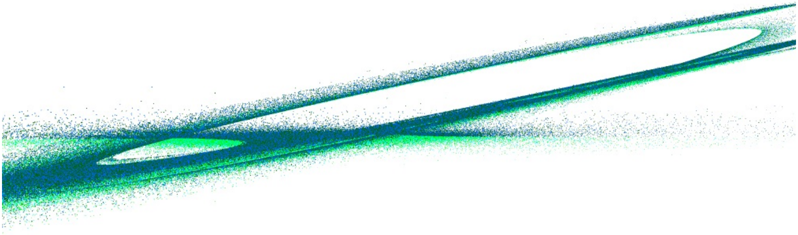
Plot of an orbit



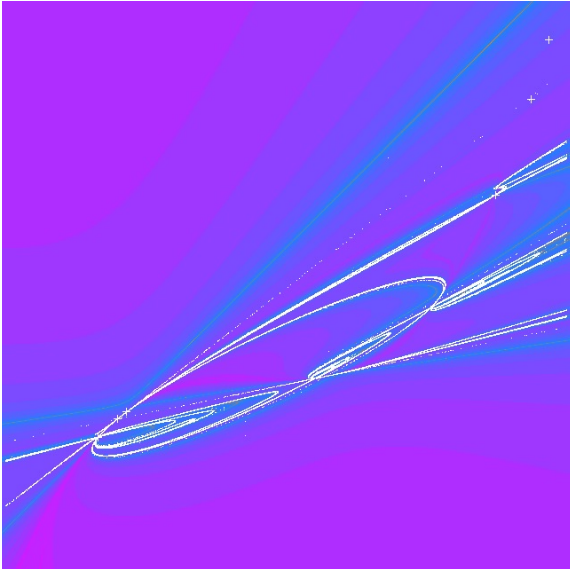
Vertical slice(backward orbit)



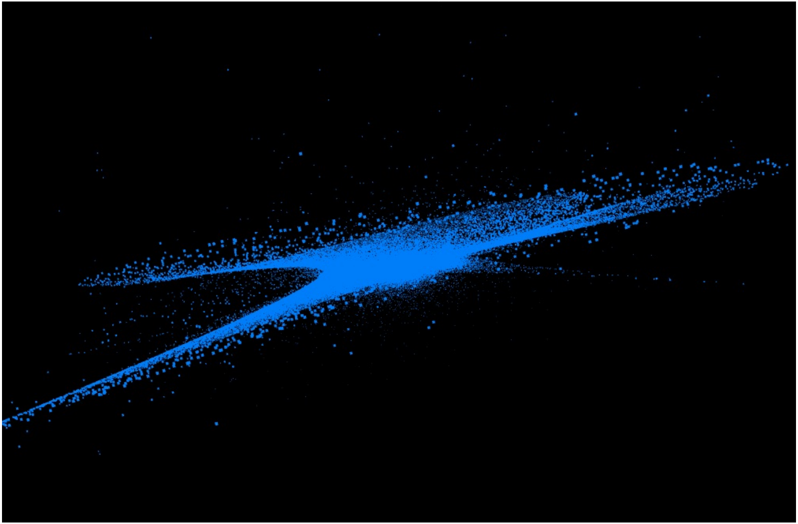
Other example : (4,5,6), id.



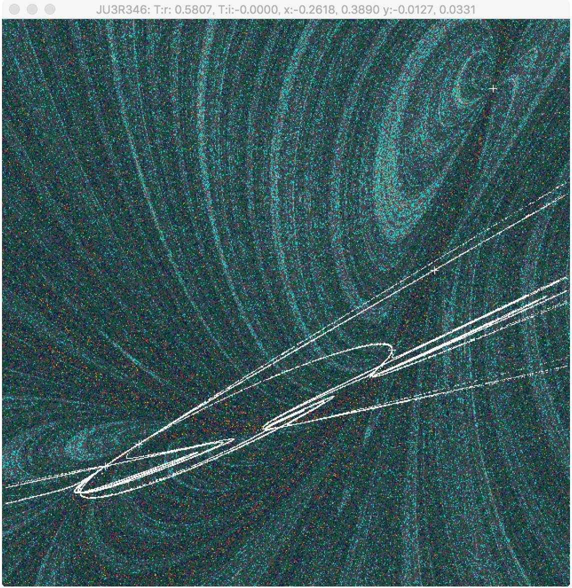
(4,5,6), *id*., real slice



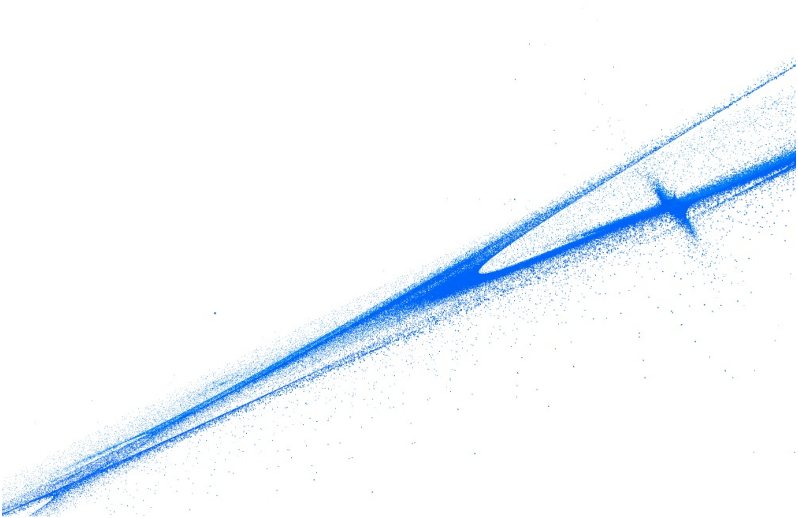
(3,4,6),cyclic



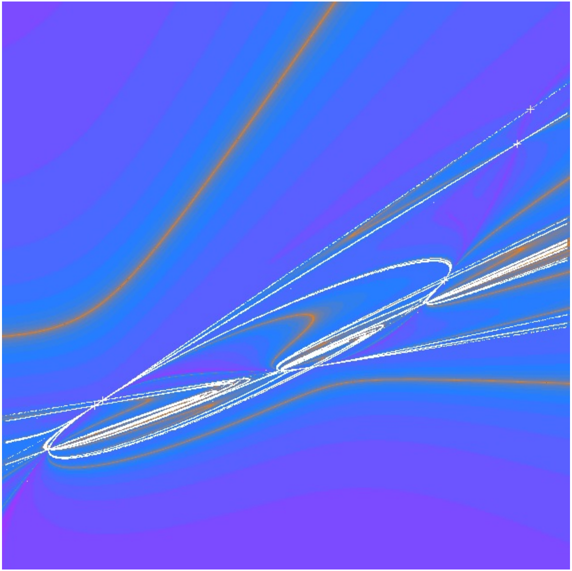
(3,4,6),cyclic, real slice



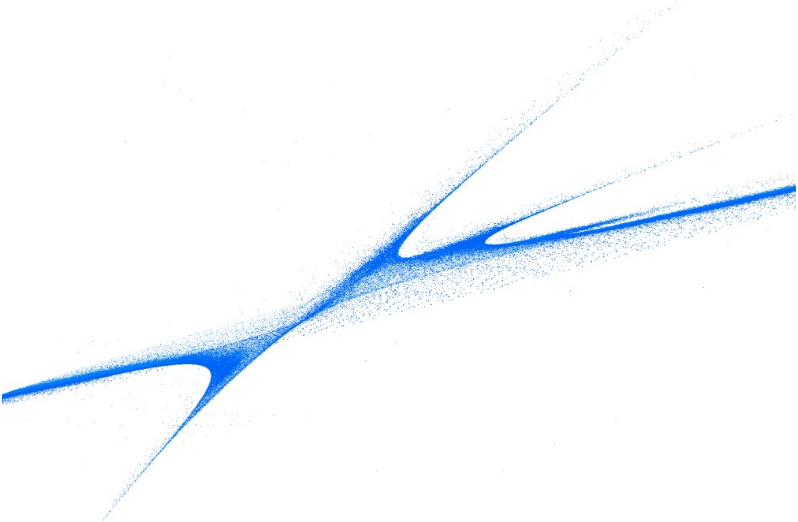
(3,4,4), transposition (1,2)



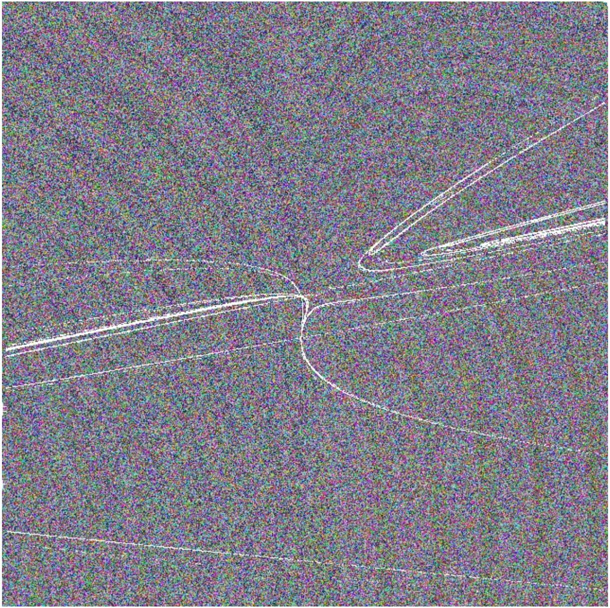
(3,4,4), transposition (1,2), real slice



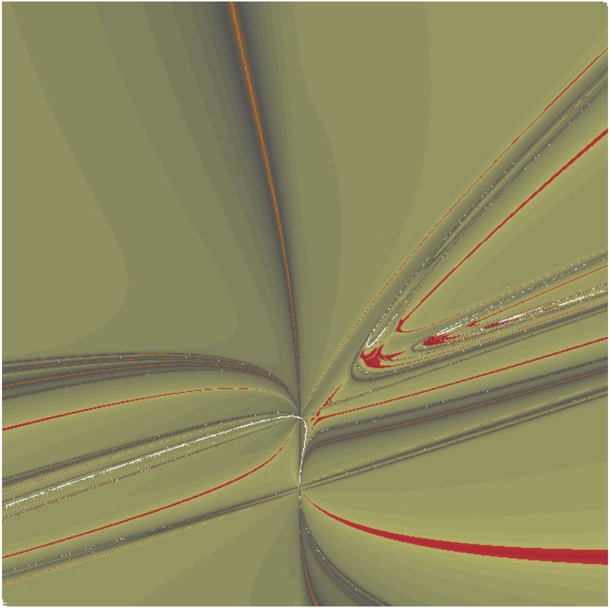
QLc355 case



QLc355 case, real slice



QLc355 case, real slice(backward)



6. ND2 case

case ND2

The case ND2 is treated as follows.

Take parametrization in curve $\{z(xy - z^2) = 0\}$ as follows
Let $t \in \mathbb{C}/\mathbb{Z}$.

$$p_Q(t) = (e^{2\pi it}, e^{-2\pi it}) \in Q = \{xy = 1\}.$$

$$p_L(t) = [1 : -e^{2\pi it} : 0] \in L = \text{line at infinity}.$$

Let $p_j^+ \in \mathbb{C}/\mathbb{Z}$, and set $A_j = e^{2\pi i p_j^+}$, $j = 1, 2, 3$.

For translation b in the hyperbola Q, set $B = e^{2\pi ib}$.
For translation c in the line at infinity L, set $C = e^{2\pi ic}$.

case ND2

We construct birational map $f : (x, y) \mapsto (X, Y)$, as follows. As the line at infinity is mapped to itself, the denominator must be of degree 1 defining the line passing through the indeterminacy points $p_Q(p_1^+) = (A_1, A_1^{-1})$ and $p_Q(p_2^+) = (A_2, A_2^{-1})$, the denominator can be set to

$$f_3(x, y) = x - A_1 - A_2 + A_1 A_2 y.$$

The dynamics in the hyperbola $\{xy = 1\}$ is $(x, y) \mapsto (Bx, B^{-1}y)$. Let numerators be

$$f_1(x, y) = B((x - A_1)(x - A_2) + P(xy - 1)),$$

$$f_2(x, y) = B^{-1}((1 - A_1 y)(1 - A_2 y) + Q(xy - 1)),$$

for some $P, Q \in \mathbb{C}$. ($X = \frac{f_1}{f_3}, Y = \frac{f_2}{f_3}$)

case ND2

As the dynamics in the line at infinity is $z \mapsto Cz$, with $z = y/x$,

$$Z = \lim_{x,y\to\infty} Y/X = A_1A_2B^{-2}P^{-1}z\frac{z + QA_1^{-1}A_2^{-1}}{z + P^{-1}},$$

gives $P = A_3^{-1}$ and $Q = A_1A_2A_3$. (Used $B^2C = A_1A_2A_3$.)

The Cremona transformation $F : (x, y) \mapsto (X, Y)$ is given by

$$X = B\frac{x^2 - (A_1 + A_2)x + A_1A_2 + A_3^{-1}(xy - 1)}{x - A_1 - A_2 + A_1A_2y},$$

$$Y = B^{-1}\frac{A_1A_2y^2 - (A_1 + A_2)y + 1 + A_1A_2A_3(xy - 1)}{x - A_1 - A_2 + A_1A_2y}.$$

Orbit data for ND2

Relation between parameters of indeterminate points and translation b and c :

$$p_1^+ + p_2^+ + p_3^+ \equiv 2b + c \mod 1.$$

Relation between parameters of indeteminate points :

$$p_1^- \equiv p_1^+ - b - c, \quad p_2^- \equiv p_2^+ - b - c, \quad p_3^- \equiv p_3^+ - 2b \mod 1.$$

For orbit data $(n_1, n_2, n_3), \sigma$, parameters must satisfy followings.

$$p_{\sigma(j)}^+ \equiv p_1^- + (n_j - 1)b \mod 1, \quad j = 1, 2,$$

$$p_3^+ \equiv p_3^- + (n_3 - 1)c \mod 1.$$

Here, σ is either *id*. or transposition $(1, 2)$.

(ND2) transposition case

For integers m_1, m_2, m_3, and a complex number s, we get

$$b \equiv \frac{(n_3 - 1)(m_1 + m_2) + 2m_3}{(n_1 + n_2 - 4)(n_3 - 1) - 4} \mod 1,$$

$$c \equiv \frac{2(m_1 + m_2) + (n_1 + n_2 - 4)m_3}{(n_1 + n_2 - 4)(n_3 - 1) - 4} \mod 1,$$

and

$$p_1^+ \equiv \frac{n_2 - 1}{2} b + s - \frac{m_1}{2} \mod 1,$$

$$p_2^+ \equiv \frac{n_1 - 1}{2} b + s + \frac{m_2}{2} \mod 1,$$

$$p_3^+ \equiv b - 2s \mod 1.$$

Parameter s gives choice of coordinates. When $s = 0$, the map has symmetries. It is reversible by the complex conjugation, and it is symmetric with respect to the conjugate diagonal. It is also reversible by swapping involution $(x, y) \mapsto (y, x)$.

(ND2) case $\sigma = id$.

In the case of $\sigma = id.$, we need $n_1 = n_2$.
For $m_1, m_3 \in \mathbb{Z}$, $\ell \in \mathbb{Z}$ and $\zeta_1, \zeta_2 \in \mathbb{C}$, we get

$$b \equiv \frac{(n_3 - 1)m_1 + m_3}{(n_1 - 2)(n_3 - 1) - 2} \mod 1,$$

$$c \equiv \frac{2m_1 + (n_1 - 2)m_3}{(n_1 - 2)(n_3 - 1) - 2} \mod 1,$$

and

$$p_1^+ \equiv \frac{2b + c + \ell}{3} + \zeta_1 + \zeta_2 \mod 1,$$

$$p_2^+ \equiv \frac{2b + c + \ell}{3} + \zeta_1 - \zeta_2 \mod 1,$$

$$p_3^+ \equiv \frac{2b + c + \ell}{3} - 2\zeta_1 \mod 1.$$

Parameters ζ_1, ζ_2 gives choice of coordinates.

example

In the case of orbit data $(n_1, n_2, n_3) = (4, 3, 5)$, $\sigma = (1, 2)$, and $(m_1, m_2, m_3) = (1, 1, 1)$, with $s = 0$, we have

$$b \equiv \frac{1}{4}, \quad c \equiv \frac{7}{8},$$

$$p_1^+ \equiv \frac{3}{4}, \quad p_2^+ \equiv \frac{3}{8}, \quad p_3^+ \equiv \frac{1}{4}.$$

And

$$p_1^- \equiv \frac{5}{8}, \quad p_2^- \equiv \frac{1}{4}, \quad p_3^- \equiv \frac{3}{4}.$$

To observe the symmetries of the Cremona transformation, it can be rewritten as follows.

$$X = B\left(x + \frac{(A_3^{-1} - A_1A_2)(xy-1)}{x - (A_1 + A_2) + A_1A_2y}\right),$$

$$Y = B^{-1}\left(y + \frac{(A_3 - A_1^{-1}A_2^{-1})(xy-1)}{y - (A_1^{-1} + A_2^{-1}) + A_1^{-1}A_2^{-1}x}\right).$$

When $s = 0$,

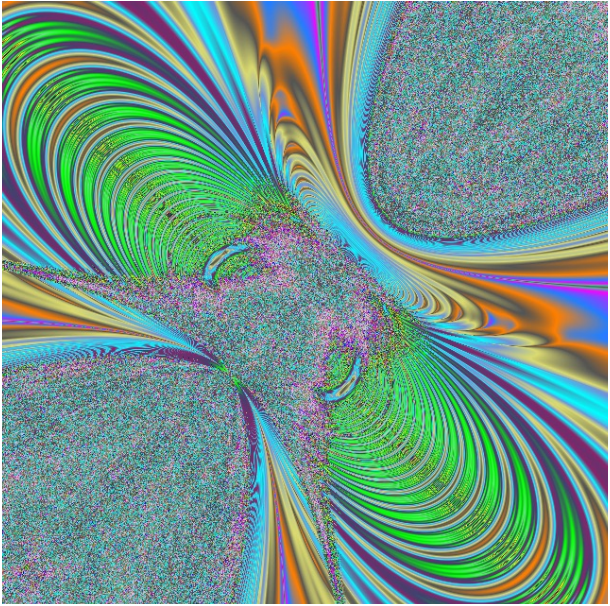
$$p_1^- \equiv -p_2^+, \quad p_2^- \equiv -p_1^+, \quad p_3^- \equiv -p_3^+,$$

we see

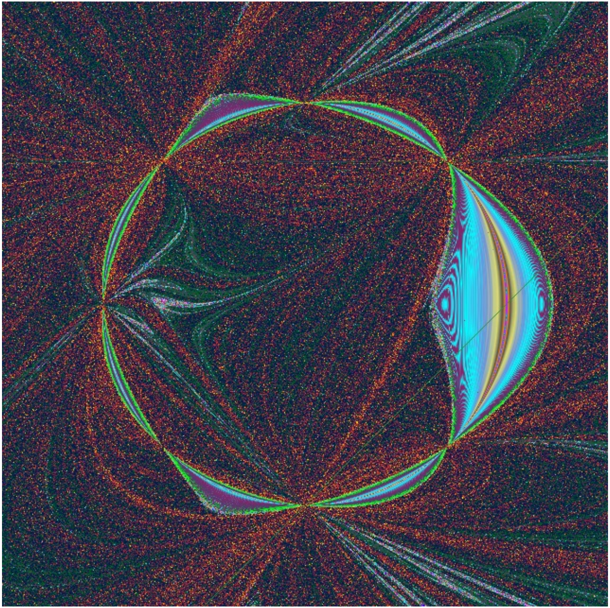
$$\overline{f} = f^{-1} = S \circ f \circ S, \qquad T \circ f \circ T = f,$$

where $S : (x, y) \mapsto (y, x)$, $T : (x, y) \mapsto (\bar{y}, \bar{x})$, are involutions. Therefore, $f : (x, y) \mapsto (X, Y)$ is reversible with respect to involution S, and involution by the complex conjugation. It is symmetric with respect to involution T.

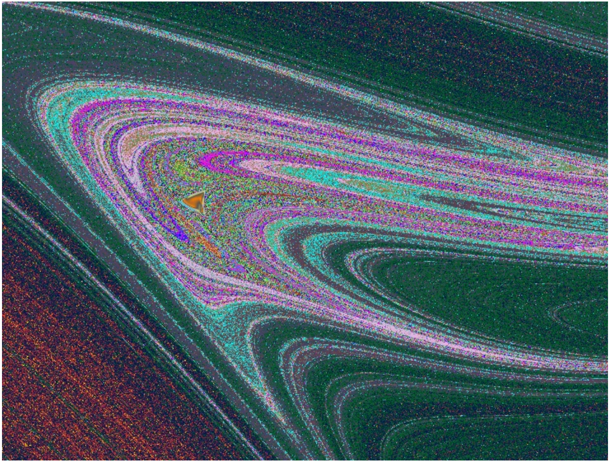
Real slice for ND2map (4,3,5),$\sigma = (1, 2)$



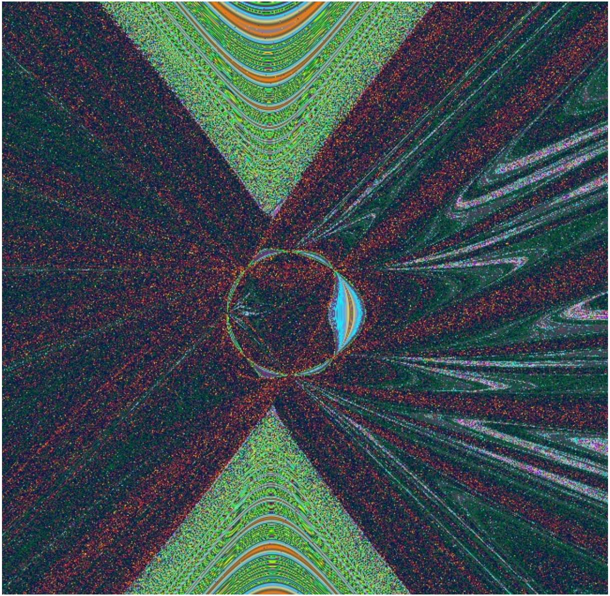
Conjugate diagonal slice for ND2map (4,3,5),$\sigma = (1, 2)$



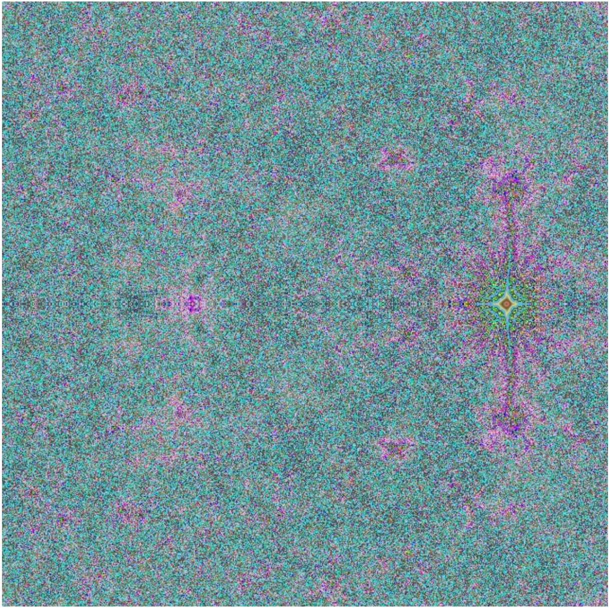
Conjugate diagonal slice for ND2map (4,3,5),$\sigma = (1, 2)$, some part



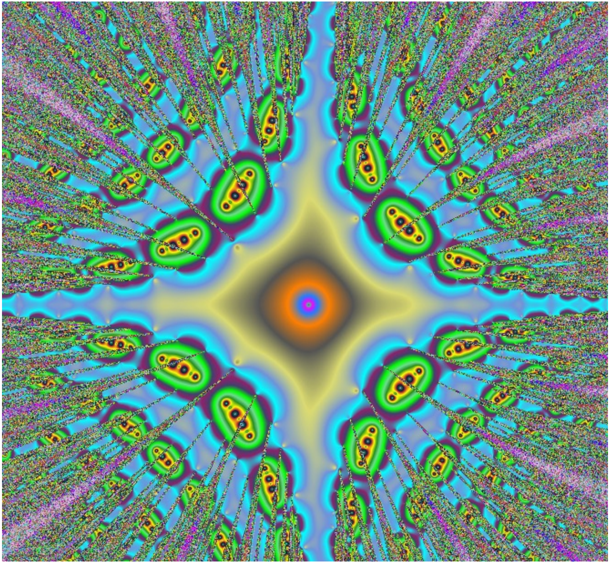
Conjugate diagonal slice for ND2map (4,3,5),$\sigma = (1, 2)$, zoomed out



Diagonal slice for ND2map (4,3,5),$\sigma = (1, 2)$



Diagonal slice for ND2map (4,3,5),$\sigma = (1, 2)$, zoomed in



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