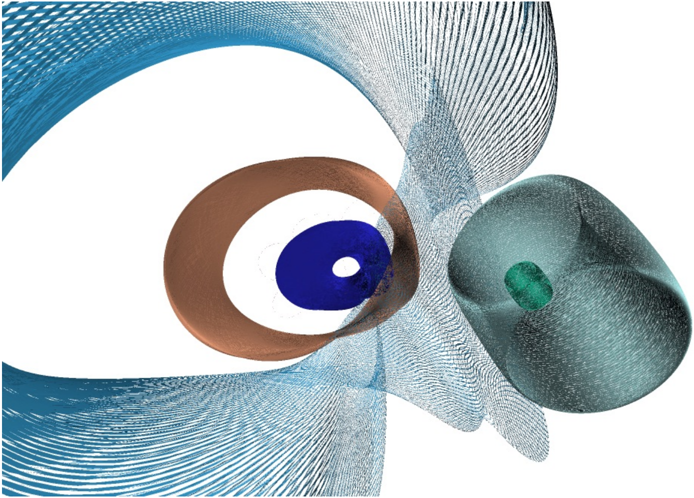
Super-Exotic Rotation Domains in Complex Surface Automorphisms



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Abstract

Automorphisms of complex surfaces can have various invariant curves. We consider rational surface automorphisms with invariant cubic curve, which have, at the same time, an invariant line, or an invariant quadratic curve, disjoint from the invariant cubic curve.

Some of them appear to have rotation domains of rank two, with two fixed points. We prove that these fixed points are linearizable to irrational rotations.

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0. Rotation domain

Fatou set

Let $f : X \to X$ be an automorphism of a complex manifold X.

A point $p \in X$ is a point of the **forward Fatou set** F_f^+ if there exists an open neighborhood U of p on which the sequence $\{f^n\}_{n\in\mathbb{N}}$ forms a normal family of holomorphic mappings from U to X.

Define the **backward Fatou set** F_f^- and the **Fatou set** F_f by

$$F_f^- = F_{f^{-1}}^+, \qquad F_f = F_f^+ \cap F_f^- .$$

Volume preserving automorphism

Suppose Ω is a Fatou component of a volume preserving automorphism f with $f(\Omega) = \Omega$. Define the set of all limits of convergent subsequences $\mathcal{G}$ by

$$\mathcal{G} = \left\{ g = \lim_{n_j \to \infty} f^{n_j} : \Omega \to \overline{\Omega} \right\}.$$

If $g = \lim_{n_j \to \infty} f^{n_j}$ is such a limit, then g must preserve volume, and thus it is locally invertible. It follows that $g : \Omega \to \Omega$.

It is known that $\mathcal{G}$ is a compact Lie group, by a theorem of H. Cartan. The connected component $\mathcal{G}_0$ of the identity must be a (real) torus.

Rank of a rotation domain

In the volume preserving Hénon map case, known result is as follows.

THEOREM (Bedford-Smilie 1991).
$\mathcal{G}_0$ is isomorphic to $\mathbb{T}^\rho$ with $\rho = 1$ or 2.

Such a domain is called a **rotation domain**, and we refer to ρ as the **rank** of the rotation domain.

Reinhardt domain

Let $D \subset \mathbb{C}^2$ be a connected open set. We say that D is a **Reinhardt domain** if $(e^{i\theta}z, e^{i\phi}w) \in D$ for all $(z, w) \in D$ and all $\theta, \phi \in \mathbb{R}$.

If Ω is a rank 2 rotation domain, then the $\mathcal{G}$-action on Ω may be conjugated to the standard linear action on $\mathbb{C}^2$.

THEOREM. (Barrett-Bedford-Dadok 1989) There are a Reinhardt domain $D \subset \mathbb{C}^2$, a linear map $L : (x, y) \mapsto (\alpha x, \beta y)$, $|\alpha| = |\beta| = 1$, and a biholomorphic map $\Phi : \Omega \to D$ such that $\Phi \circ f = L \circ \Phi$.

Volume preserving surface automorphism

Volume preserving surface automorphisms can have various kinds of rotation domains.

1. Rotation domain of rank 1.
2. Siegel disk $\cong$ Reinhardt domain $\supset \mathbb{D} \times \mathbb{D}$.
3. Exotic rotation domain $\cong$ "$\mathbb{A} \times \mathbb{D}$". (?)
4. Super-exotic rotation domain $\cong$ "$\mathbb{P} \times \mathbb{D}$". (??)
5. Ultra-exotic rotation domain $\cong$ "$\mathbb{A} \times \mathbb{A}$". (???)

REM. In case 1, there are various types, not well understood.
REM. Case 3 is numerically found without proof.
REM. Case 4 is numerically observed.
REM. Case 5 is not found yet.

REM. "$\mathbb{P} \times \mathbb{D}$" is the normal disk bundle.

Linear model

Let (x, y) and (u, v) denote two charts of a complex manifold defined by coordinate transformation as follows.

$$\begin{cases} u &= x^{-1} \\ v &= -x^2 y \end{cases} \qquad \begin{cases} x &= u^{-1} \\ y &= -u^2 v \end{cases}$$

For $\mu, \delta \in \mathbb{C}^\times$, linear maps

$$(x, y) \mapsto (\mu x, \delta \mu^{-1} y) \quad \text{and} \quad (u, v) \mapsto (\mu^{-1} u, \delta \mu v)$$

are compatible and defines an automorphism of the complex manifold.

If $|\mu| = |\delta| = 1$ and they are multiplicatively independent, this "linear map" defines an irrational rotation of rank 2.

Extended **Reinhardt domain** can be defined for this linear model.

"Linearization" will be discussed later.

1. Cremona transformation

Cremona involution

A birational transformation $f : \mathbb{P}^2 \to \mathbb{P}^2$ is called a **Cremona transformation**.

Cremona involution $J : \mathbb{P}^2 \to \mathbb{P}^2$ is defined by

$$J[x : y : z] = [x^{-1} : y^{-1} : z^{-1}] = [yz : zx : xy].$$

For linear transformations $L_1, L_2 \in PGL(\mathbb{P}^2)$,

$$f = L_1 \circ J \circ L_2$$

is a quadratic birational transformation.

Cremona transformations with invariant cubic curve

A quadratic transformation $f : \mathbb{P}^2 \to \mathbb{P}^2$ always acts by blowing up three (indeterminacy) points $I(f) = \{p_1^+, p_2^+, p_3^+\}$ in $\mathbb{P}^2$ and blowing down the (exceptional) lines joining them. The inverse map f^{-1} is also a quadratic transformation and $I(f^{-1}) = \{p_1^-, p_2^-, p_3^-\}$ consists of the images of the three exceptional lines.

$$p_i^- = f(\ell(p_j^+, p_k^+)) \quad \text{for} \quad \{i, j, k\} = \{1, 2, 3\}.$$

Here, $\ell(p, q)$ denotes the line passing through p and q.

Orbit data

Suppose that for natural numbers n_1, n_2, n_3, and a permutation $\sigma : \{1,2,3\} \to \{1,2,3\}$, f satisfies

$$f^{n_i - 1}(p_i^-) = p_{\sigma(i)}^+, \qquad i = 1, 2, 3.$$

$$\ell(p_j^+, p_k^+) \to p_i^- \to f(p_i^-) \to \cdots \to p_{\sigma(i)}^+ \to \ell(p_{\sigma(j)}^-, p_{\sigma(k)}^-).$$

By blowing up in $n_1 + n_2 + n_3$ points

$$p_1^-, f(p_1^-), \cdots, f^{n_1-1}(p_1^-) = p_{\sigma(1)}^+,$$

$$p_2^-, f(p_2^-), \cdots, f^{n_2-1}(p_2^-) = p_{\sigma(2)}^+,$$

$$p_3^-, f(p_3^-), \cdots, f^{n_3-1}(p_3^-) = p_{\sigma(3)}^+,$$

f lifts to a surface automorphism.

2. Surface automorphism

Quadratic Cremona transformation

Let C be a cubic curve of one of the following :

(case C) a cuspidal cubic curve
(case L) three lines passing through a point,
(case Q) a conic and a tangent line.

THEOREM. (Diller 2011) Let orbit data n_1, n_2, n_3, $\sigma \in \Sigma_3$ be given. Except for some specific cases, there exists an automorphism f for each root of $P(\lambda)$ that is not a root of unity, which realize the orbit data, with determinant λ.

Such f is unique up to conjugacy of linear transformation preserving C.

Uehara's formula of birational transformation

Uehara(2016) obtained an explicit formula for Cremona transformations with an invariant cuspidal cubic curve.

For $\lambda \in \mathbb{C}^\times$ and $a_1, a_2, a_3 \in \mathbb{C}$ with $a_1 + a_2 + a_3 \neq 0$,

$$\begin{cases} X_C &= \lambda\left(x + \frac{\nu_1}{3} + \frac{\nu_1(y-x^3)}{\nu_1 x^2 - \nu_2 x + \nu_3 - y}\right) \\ Y_C &= \lambda^3\left((x + \frac{\nu_1}{3})^3 + y - x^3 + \frac{\nu_1(y-x^3)}{\nu_1 x^2 - \nu_2 x + \nu_3 - y}(\nu_1(x + \frac{\nu_1}{3}) - \nu_2)\right) \end{cases}$$

where $\nu_1 = a_1 + a_2 + a_3$, $\nu_2 = a_1 a_2 + a_2 a_3 + a_3 a_1$, and $\nu_3 = a_1 a_2 a_3$.

He obtained formulas also for other cases.

Characteristic polynomial

Orbit data determines the characteristic polynomial $P(\lambda)$ of $f^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$.

Bedford and Kim (2006, [BK1]) have computed explicitly for any orbit data n_1, n_2, n_3, σ.

$$P(\lambda) = \lambda^{1+\Sigma n_j} p(\frac{1}{\lambda}) + (-1)^{\mathrm{ord}\sigma} p(\lambda),$$

where

$$p(\lambda) = 1 - 2\lambda + \sum_{j=\sigma_j} \lambda^{1+n_j} + \sum_{j\neq\sigma_j} \lambda^{n_j}(1-\lambda).$$

In the following, we describe only for our cases with invariant cuspidal cubic curve $\{y = x^3\}$ and orbit data

$$(3,4,3), \sigma = (1,2), \quad \text{and} \quad (4,2,4), \sigma = (1,2).$$

We consider only volume-preserving case.

There are countably many similar cases.

For orbit data $n_1, n_2, n_3, \sigma = tr(1,2)$.

Characteristic polynomial
(case tr) σ is a transposition $(\sigma(1) = 2, \sigma(2) = 1, \sigma(3) = 3)$.

$$P(\lambda) = (\lambda - 2)\lambda^{n_1+n_2+n_3} + \lambda^{n_1+n_2} + (\lambda - 1)(\lambda^{n_1+n_3} + \lambda^{n_2+n_3})$$

$$-(\lambda - 1)(\lambda^{n_1} + \lambda^{n_2}) + \lambda^{n_3+1} - 2\lambda + 1.$$

Picard coordinate of indeterminate points
(case tr) $\sigma = (1,2)$

$$a_i = -\frac{\lambda^{n_j-1}(\lambda^{n_i}+1)(\lambda-1)}{\lambda^{n_i+n_j}-1} + \frac{1}{3} \qquad ((i,j) = (1,2),(2,1)).$$

$$a_k = -\frac{\lambda^{n_k-1}(\lambda-1)}{\lambda^{n_k}-1} + \frac{1}{3} \qquad (k = 3).$$

Indeterminate points
(case C) cuspidal cubic curve $\{y = x^3\}$:

$$p_i^+ = (a_i, a_i^3), \quad i = 1,2,3,$$

$$p_i^- = (b_i, b_i^3), \quad i = 1,2,3.$$

Determinant and meromorphic eigen-form

In our cases of cubic curve, each component of regular part is isomorphic to $\mathbb{C}$. Automorphism $f : S \to S$ restricted to the invariant cubic curve is an "affine" map. The "derivative" $D(f|_C)$ is called the **determinant** of f.

Meromorphic $(2, 0)$-form η with pole along the invariant curve C is mapped to a scaler multiple of η.

$$f^*\eta = \lambda(f)\eta.$$

THEOREM

$$D(f|_C) = \lambda(f).$$

$\lambda(f)$ is also called the **determinant** of f. If $p \in S \setminus C$ is a periodic point of period k, then

$$\det Df_p^k = \lambda(f)^k.$$

3. Invariant curve

Invariant curve

Theorem. (Diller-Jackson-Sommese 2007)
Let $f : S \to S$ be an algebraically stable map with $\lambda_f > 1$, and suppose that $C = f(C)$ is a connected curve with $g(C) = 1$. Then by contracting finitely many curves, one may further arrange the following.

(1) $C \sim -K_S$ is an anticanonical divisor.
(2) $I(f^n) \subset C$ for every $n \in \mathbb{Z}$.
(3) Any connected curve strictly contained in C has genus zero.
(4) If W is a connected f-invariant curve not completely contained in C, then W has genus zero, is disjoint from C, and is equal to a tree of smooth rational curves, each with self-intersection -2.

Rem. Here λ_f means the first dynamical degree of f.

Nodality

With our choice of Picard coordinates, we have the following fact.

THEOREM. $3d$ (not necessarily distinct) points $p_1, \cdots, p_{3d} \in X_{reg}$ comprise the intersection of X with a curve of degree d if and only if

- each irreducible $V \subset X$ contains $d \cdot \deg V$ of the points; and
- $\sum p_j \sim 0$.

4. Invariant line

Invariant line (necessary condition)

If there exists an invariant line disjoint from the anticanonical cubic curve, it passes through three points to be blown up, one of which is an indeterminate point of the base birational map.

The sum of the Picard coordinates of the three blowup points vanishes.

The intersection of the invariant line and a component of the anticanonical curve, counted as points in $\mathbb{P}^2$, must be equal to the degree of the component.

This line necessarily contains two fixed points.

(Our automorphism has four fixed points.)

Invariant line (sufficient condition)

In our case, the anticanonical cubic curve of our surface automorphism is a cuspidal cubic curve.

THEOREM. In the case of orbit data $(n_1, n_2, 3)$ with $\sigma(3) = 3$, the surface automorphism has an invariant line passing through three blowup points $p_3^-, f(p_3^-)$, and p_3^+.

REM. In this case, the self-intersection of the strict transform of this invariant line is -2.

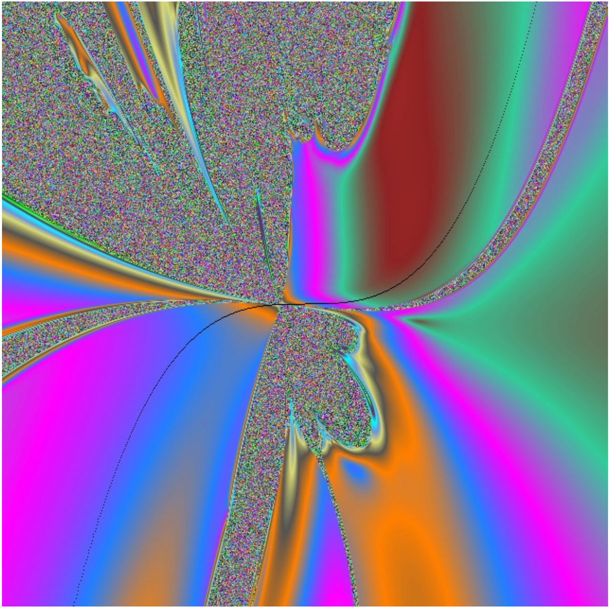
PROOF. Let $p_3^+ = (a_3, a_3^3)$, $p_3^- = (b_3, b_3^3)$, and $f(p_3^-) = (c_3, c_3^3)$. Then,

$$a_3 = -\frac{d^2(d-1)}{d^3-1} + \frac{1}{3}, \quad b_3 = -\frac{d-1}{d^3-1} + \frac{1}{3}, \quad c_3 = -\frac{d(d-1)}{d^3-1} + \frac{1}{3}.$$

Immediately we see that $a_3 + b_3 + c_3 = 0$. Hence three points $p_3^-, f(p_3^-), p_3^+$ are on a line. Let L denote this line. As L passes through the indeterminate point p_3^+, its image $f(L)$ is a line. Since $f(L)$ passes through $p_3^+ = f^2(p_3^-)$ and $f(p_3^-)$, it coincides with L.

In our case, L is disjoint from the invariant cubic curve.

Real slice (CSPt343t1)



Lefschetz formula and Atyah-Bott formula

Suppose $F : S \to S$ satisfy $\det(DF - I) \neq 0$ at all fixed points.

Lefschetz formula :

$$\sum_{F(p)=p} \mathrm{sign}(\det(DF_p - I)) \; = \; \sum_{k=0}^{4} (-1)^k \mathrm{Tr}(F_*|_{H_k(S,\mathbb{R})}).$$

Atyah-Bott formula : for $r = 0, 1, 2$,

$$\sum_{F(z)=z} \frac{\mathrm{Tr} \wedge^r DF_z}{\det(I - DF_z)} = \sum_{s=0}^{4} (-1)^s \mathrm{Tr}(F^*|_{H^{r,s}(S)}).$$

Fixed points

In our case of orbit data $(3, 4, 3), \sigma = (1, 2)$, the characteristic polynomial of $F_*|_{H_2(S,\mathbb{R})}$ is

$$P(\lambda) = \lambda^{11} - 2\lambda^{10} + \lambda^8 + \lambda^7 - \lambda^6 - \lambda^5 + \lambda^4 + \lambda^3 - 2\lambda + 1,$$

which shows

$$\mathrm{Tr}(F_*|_{H_2(S,\mathbb{R})}) = 2.$$

By Lefschetz formula, our map has 4 fixed points.

In the anticanonical cubic curve, points $[\frac{1}{3} : \frac{1}{27} : 1]$ and $[0 : 1 : 0]$ are fixed points. Other two fixed points must be on the invariant line.

Eigen values

Eigenvalues at fixed points can be directly computed from our explicit formula.

Let δ denote a root of characteristic polynomial $P(\lambda)$.
Suppose δ is non-real. δ is the determinant $\lambda(f)$ of our surface automorphism f.

At cuspidal point $[0:1:0]$, eigenvalues are δ^{-2} and δ^{-3}.
At fixed point $[\frac{1}{3}:\frac{1}{27}:1]$, eigenvalues are δ and δ^{-7}.

In the invariant line L, $f|_L : L \to L$ has two fixed points. Consider a parametrization of L with fixed points 0 and ∞. Let μ denote the multiplier of $f|_L$ at fixed point $0 \in L$, the multiplier at $\infty \in L$ is μ^{-1}.

Since the determinant of f at fixed points is δ, we see that the other eigenvalues of these fixed points are $\delta\mu^{-1}$ and $\delta\mu$, respectively.

By Atyah-Bott formula for $r = 0$:

$$\sum_{F(z)=z} \frac{\mathrm{Tr} \wedge^r DF_z}{\det(I - DF_z)} = \sum_{s=0}^{4} (-1)^s \mathrm{Tr}(F^*|_{H^{r,s}(S)}),$$

we have

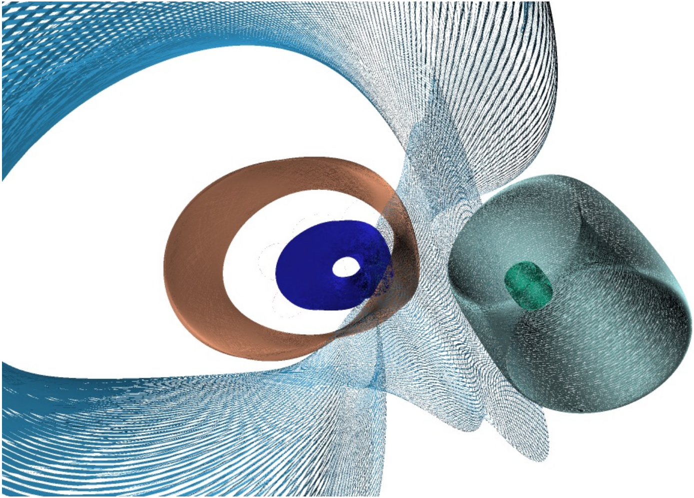
$$\frac{1}{(1-\delta^{-2})(1-\delta^{-3})} + \frac{1}{(1-\delta)(1-\delta^{-7})}$$
$$+ \frac{1}{(1-\mu)(1-\delta\mu^{-1})} + \frac{1}{(1-\mu^{-1})(1-\delta\mu)} = 1.$$

And

$$\mu + \mu^{-1} = \Delta_1 + \frac{(\Delta_2 - 2)(\Delta_5 - \Delta_2)}{\Delta_6 - \Delta_5 - \Delta_4 + \Delta_2},$$

where $\Delta_k = \delta^k + \delta^{-k} \in \mathbb{R}$.

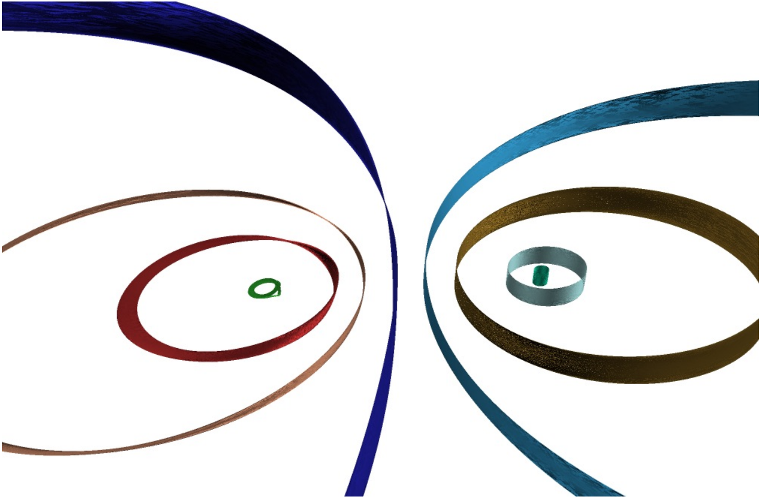
Super-exotic rotation domain (CSPt343t1SL)



In our case, $\delta_1 = 0.598563149\cdots + 0.801075625\cdots i$,

$$\mu_1 + \mu_1^{-1} = 1.599348676\cdots.$$

Super-exotic rotation domain (CSPt343t1SL)



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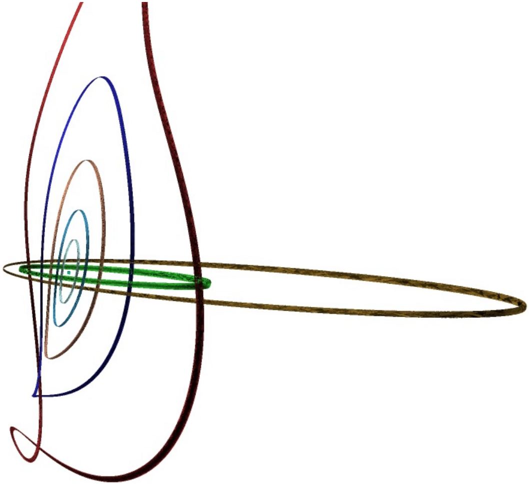

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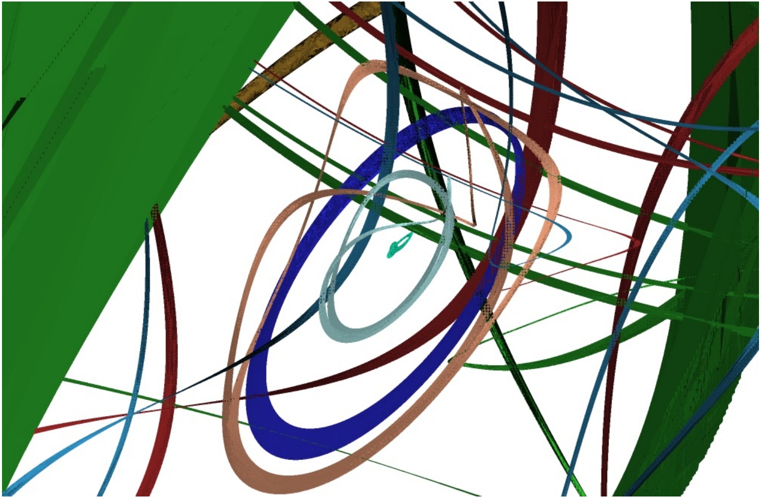

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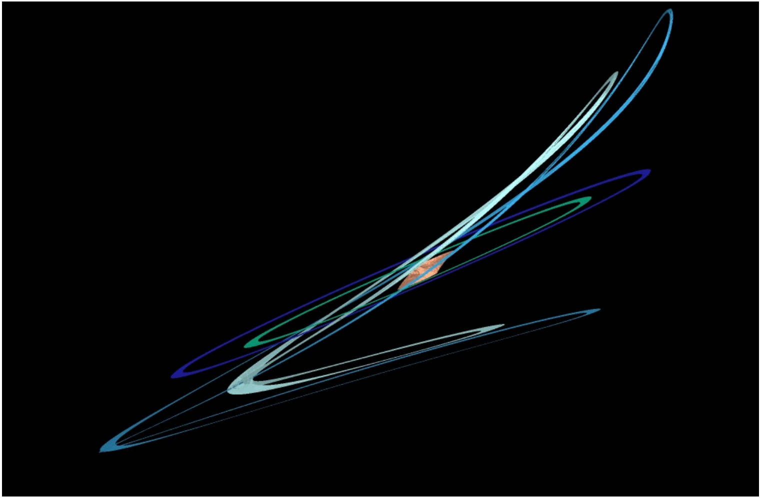


Linearizability

Definition Let $(\lambda_1, \cdots, \lambda_n)$ be nonzero complex numbers. We say (λ_i) are **multiplicatively independent** if the only solution to

$$\lambda_1^{k_1} \ldots \lambda_n^{k_n} = 1$$

with $k = (k_1, \cdots, k_n) \in \mathbb{Z}^n$ is $k = 0$. The numbers (λ_i) are **jointly Diophantine** if there exist $C, M > 0$ such that for all integral exponents $(k_1, \cdots, k_n) \in \mathbb{Z}^n$, not all zero, we have

$$|\lambda_1^{k_1} \ldots \lambda_n^{k_n} - 1| > C(\max |k_i|)^{-M} > 0.$$

Theorem(McMullen, 2002) Let X be a complex n-manifold, and let $f : X \to X$ be a holomorphic map fixing $p \in X$. If the eigenvalues (λ_i) of Df_p are algebraic, multipicatively independent and satisfy $|\lambda_i| = 1$, then f has a Siegel disk at p.

Rem. Here, Siegel disk means a neighborhood of the fixed point that is isomorphic to a polydisk with irrational rotation.

Fel'dman's ineqality to Diophantian condition

For algebraic numbers $(\lambda_i), (k_i)$, not all zero,

$$|k_0 2\pi i + k_1 \log \lambda_1 + \cdots + k_n \log \lambda_n| > \exp(-M(d + \log H)),$$

where d is the degree of the field $\mathbb{Q}[k_1, \cdots, k_n, \lambda_1, \cdots, \lambda_n]$, $M = M(\lambda_i, d)$ is a constant depending only on (λ_i) and d, $H = \max H(k_i)$, and the *height* $H(k) = \sum |a_j|$ if $p(k) = 0$ where $p(x) = \sum_0^s a_j x^j$ is an irreducible polynomial with relatively prime coefficients $a_j \in \mathbb{Z}$.

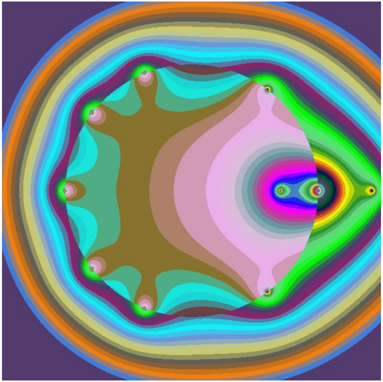
For $k_i \in \mathbb{Z}$, we have $H = \max |k_i|$, and M depends only on (λ_i), therefore

$$\exp(-M(d + \log H)) = C(\max |k_i|)^{-M}$$

for $C > 0$.

[Fe] H.I. Fel'dman, An improvement of the estimate of a linear form in the logarithms of algebraic numbers, Math. USSR Sb. **6** (1968), 393-406.

Roots of characteristic polynomial (CSPt343)



In our case, $\delta_1 = 0.598563149\cdots + 0.801075625\cdots i$,

$$\mu_1 + \mu_1^{-1} = 1.599348676\cdots.$$

Real dissipative case, $\delta_0 = 0.701751792\cdots$,

$$\mu_0 + \mu_0^{-1} = -0.926139081\cdots.$$

Other roots

$$\delta_2 = -0.373734227\cdots + 0.927535836\cdots i,$$
$$\mu_2 + \mu_2^{-1} = -1.64026646\cdots.$$
$$\delta_3 = -0.788207452\cdots + 0.615409630\cdots i,$$
$$\mu_3 + \mu_3^{-1} = -3.695849616\cdots.$$

Note that in the δ_3 case, $|\mu_3 + \mu_3^{-1}| > 2$, so,

$$|\delta_3| = 1 \qquad \text{and} \qquad |\mu_3| \neq 1.$$

Hence, μ_3 is not a root of unity, and so is its Galois conjugates μ_0, μ_1, μ_2.

Multiplicative independence

Proposition In our case, μ_1 and $\nu_1 = \delta_1 \mu_1^{-1}$ are algebraic and multiplicatively independent.

Proof δ_1 is a root of a Salem polynomial. It is algebraic and not a root of unity. μ_1 is algebraic and not a root of unity.

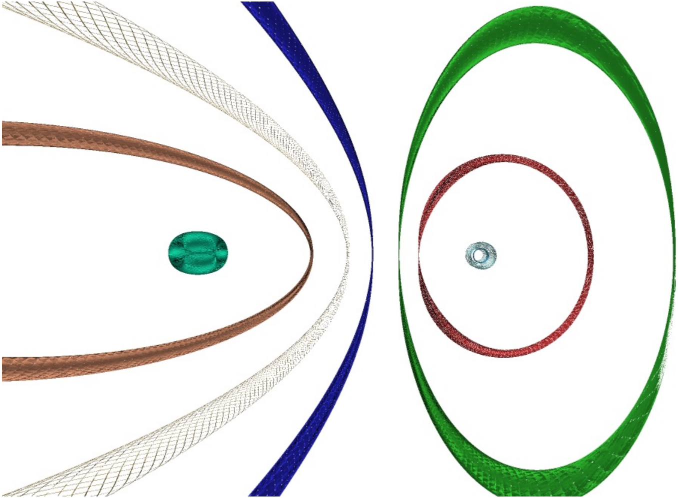
δ_0 is a Galois-conjugate of δ_1. And μ_0, $\nu_0 = \delta_0 \mu_0^{-1}$ corresponds to μ_1, ν_1 by the conjugacy.

Suppose $\mu_1^j \nu_1^k = 1$. Then $\mu_0^j \nu_0^k = 1$ holds, too.

Since $|\mu_0| = 1$ and $|\nu_0| = |\delta_0| < 1$, we see that $k = 0$. And as μ_0 is not a root of unity, we conclude $j = 0$.

Proposition Our surface automorphism has Siegel disks (in the sense of McMullen) at two fixed points in the invariant line L.

Super-exotic rotation domain (CSPt343t2SL)



Remark

Observe the proof above for the dissipative case of δ_0. The corresponding surface automorphism is dissipative. We proved the existence of an attracting Riemann sphere with irrational rotation. This cannot exist for automorphisms of $\mathbb{C}^2$.

For dissipative Hénon map case:

PROPOSITION(Bedford and Smilie, 1991) Let Ω be a connected component of $\mathrm{int}\, K^+$ that is recurrent and has period m. Then one of the following occurs:

(i) There is an attracting fixed point $p \in \Omega$ for f^m, and Ω is the basin of p under f^m.

(ii) There is a retraction $\rho : \Omega \to \Omega$ onto a smooth subvariety $\mathcal{D} = \rho(\Omega)$ that is invariant under f^m. Further, $\mathcal{D}$ is either a Siegel disk or a Herman ring.

THEOREM(Fornæss-Sibony, 1995). Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a Hénon mapping.. Let Ω be a recurrent Fatou component in the interior of K^+. Assume $f(\Omega) = \Omega$. Then Ω is one of the following types.

(i) There is a fixed attracting point $p \in \Omega$ and Ω is biholomorphic to $\mathbb{C}^2$.

(ii) There exists a Riemann surface $\tilde{\Sigma}$ which is a closed complex submanifold in Ω such that $d(f^n(X), \tilde{\Sigma}) \to 0$ for any compact X in Ω. The Riemann surface $\tilde{\Sigma}$ is biholomorphic to either a disk, a punctured disk or an annulus, and $f|\tilde{\Sigma}$ is conjugate to an irrational rotation.

(iii) The domain Ω is a Siegel domain and all convergent subsequence of (f^n) converge to an automophism of Ω.

Extended Reinhardt domain ?

We proved the existence of a rotation set (domain?) containing a Riemann sphere and Siegel disks.

In $\mathbb{C}^2$ or in $\mathbb{P}^2$, following theory can be applied, but it is not applicable for surface automorphisms.

If Ω is a rank 2 rotation domain, then the $\mathcal{G}$-action on Ω may be conjugated to the standard linear action on $\mathbb{C}^2$.

THEOREM. (Barrettt-Bedford-Dadok 1989) There are a Reinhardt domain $D \subset \mathbb{C}^2$, a linear map $L : (x, y) \mapsto (\alpha x, \beta y)$, $|\alpha| = |\beta| = 1$, and a biholomorphic map $\Phi : \Omega \to D$ such that $\Phi \circ f = L \circ \Phi$.

[BBD] D. Barrett, E. Bedford and J. Dadok, $\mathbb{T}^n$-actions on holomorphically separable complex manifolds. Math. Z. 202(1989), no. 1, 65-82.

Other cases

There are countably many cases with such rotation behavior of rank 2.

Surface automorphisms with invariant quadric curves can have such rotation domains, too. The proof is almost same as for the invariant line case.

There are countably many surface automorphisms for this case, too.

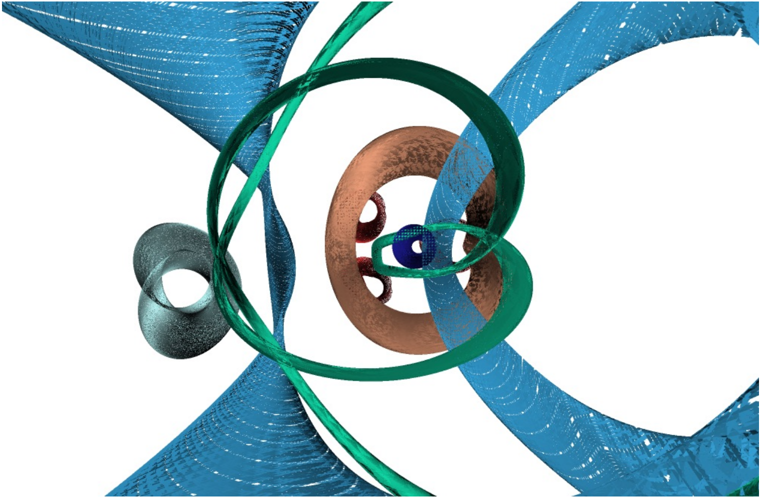
Typical case is :

Invariant cuspidal curve,

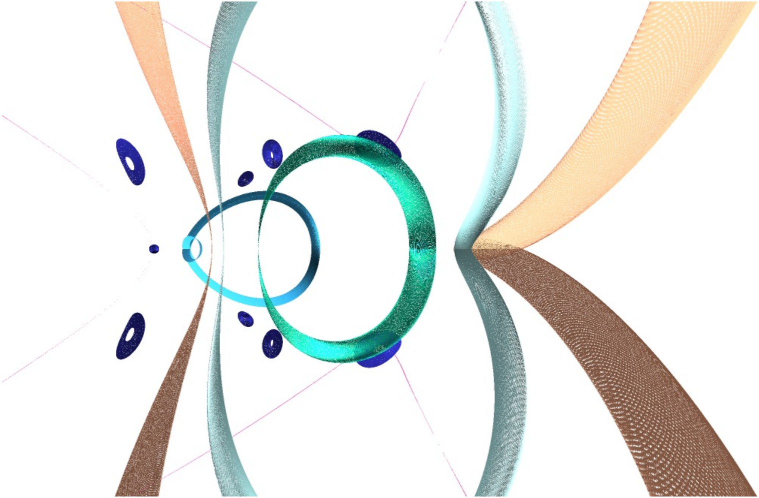
Orbit data $(2, 4, 4), \sigma = (1, 2)$.

Eigenvalues t_2, t_3.

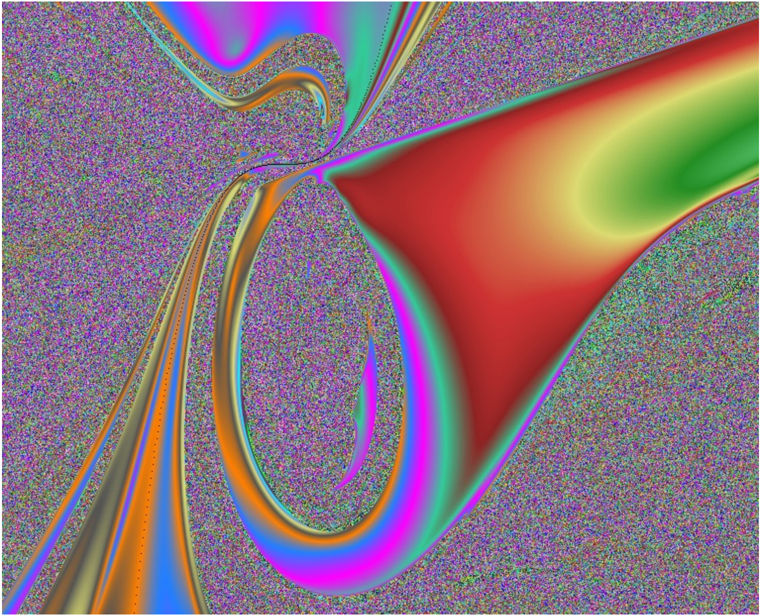
Super-exotic rotation domain (CSPt244t2SQ0)



Super-exotic rotation domain (CSPt244t3Q)



Super-exotic rotation domain (CSPt244t3R)



5. Invariant conic

Invariant conic (necessary condition)

If there is an invariant quadratic curve, disjoint from the anticanonical cubic curve, it must pass trough 6 points to be blown up, two of which are indeterminate points of the base birational map.

The sum of the Picard coordinates of these 6 blowup points vanish.

The number of blowup points in each component of the anticanonical curve must be 2 times the degree of the component.

The invariant quadratic curve contains two fixed points.

Invariant quadratic curve

Suppose the anticanonical curbic curve of surface automorphism is a cuspidal cubic curve.

THEOREM. In the case of orbit data $(2, 4, n)$ with transposition (1,2), the surface automorphism has an invariant quadratic curve passing through six blowup points $p_1^+, p_1^-, p_2^+, p_2^-, f(p_2^-), f^2(p_2^-)$.

PROOF. Quadratic curve is mapped to a quadratic curve by Cremona transformation if the quadratic curve passes through exactly two indeterminate points. If there exists a quadratic curve passing through these 6 points, its image by f is a quadratic curve, since p_1^+ and p_2^+ are indeterminate points. Points $p_2^+ = f(p_1^-)$, $p_1^+ = f^3(p_2^-)$, $f(p_2^-)$, $f^2(p_2^-)$ are in the image quadratic curve. The line passing through p_1^+ and p_3^+, which contains another point in the quadratic curve, is mapped to p_1^-. Hence p_1^- is in the image of the quadratic curve. Similarly, p_2^- is in the image, too. The image quadratic curve must be the same quadratic curve, since 6 points determine the quadratic curve.

So, we only need to prove the existence of a quadratic curve passing through the 6 points.

Let

$$a_1 = -\frac{d(d^4+1)(d-1)}{d^6-1} + \frac{1}{3}, \quad a_2 = -\frac{d^3(d^2+1)(d-1)}{d^6-1} + \frac{1}{3},$$

$$b_1 = -\frac{(d^2+1)(d-1)}{d^6-1} + \frac{1}{3}, \quad b_2 = -\frac{(d^4+1)(d-1)}{d^6-1} + \frac{1}{3},$$

$$c_1 = -\frac{d(d^2+1)(d-1)}{d^6-1} + \frac{1}{3}, \quad c_2 = -\frac{d^2(d^2+1)(d-1)}{d^6-1} + \frac{1}{3}.$$

These are the x-coordinates of the blowup points.

$$p_1^+ = (a_1, a_1^3), \quad p_1^- = (b_1, b_1^3),$$
$$p_2^+ = (a_2, a_2^3), \quad p_2^- = (b_2, b_2^3),$$
$$f(p_2^-) = (c_1, c_1^3), \quad f^2(p_2^-) = (c_2, c_2^3).$$

Immediately, we see that

$$a_1 + a_2 + b_1 + b_2 + c_1 + c_2 = 0.$$

Consider polynomial of degree 6 :

$$P(z) = (z - a_1)(z - a_2)(z - b_1)(z - b_2)(z - c_1)(z - c_2)$$
$$= z^6 + A_4 z^4 + A_3 z^3 + A_2 z^2 + A_1 z + A_0.$$

Let $Q(x, y)$ be a quadratic polynomial defined by

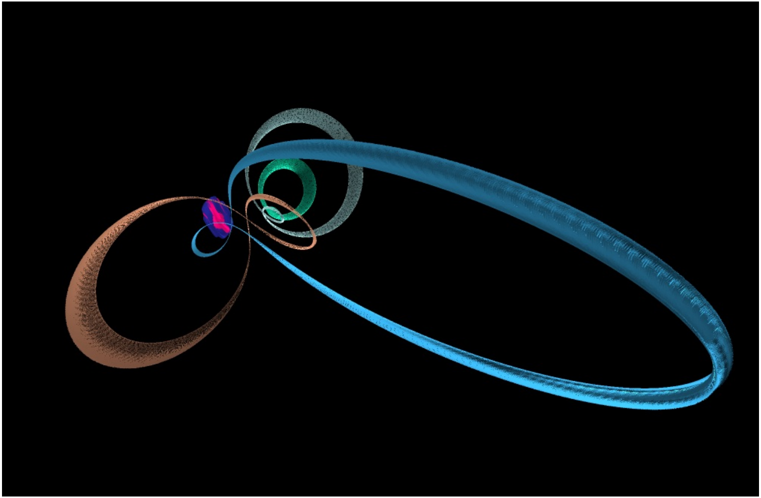
$$Q(x, y) = y^2 + A_4 xy + A_3 y + A_2 x^2 + A_1 x + A_0.$$

The 6 points $p_1^+, p_1^-, p_2^+, p_2^-, f(p_2^-), f^2(p_2^-)$ satisfy $Q(x, y) = 0$. Hence the quadratic curve $Q(x, y) = 0$ passes through these 6 points.

We conclude that quadratic curve $\{Q(x, y) = 0\}$ is invariant under f.

REM. The strict transform of this quadratic curve has self-intersection -2.

Super-exotic rotation domain (QLt155t1)



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