Siegel ball and Reinhardt domain in complex Hénon dynamics

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Abstract

The Hénon map can have a locally linearizable fixed point with eigenvalues of modulus 1. The so-called "Siegel ball" can be linearized to a logarithmically convex complete Reinhardt domain. Numerical trial of linearization will be presented. (This trial was requested by E. Bedford.) Hénon map and rational automorphism of rational surface can have multiple Siegel balls. Self-anti-conjugacy of the dynamics makes the coexistence possible. Problem of coexistence of Siegel balls was also suggested by E. Bedford.

1. Siegel Ball

$(\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$ is said to satisfy a **multiplicative diophantian condition** if there are positive constants C and ν, such that

$$|\lambda_1^{k_1} \cdots \lambda_n^{k_n} - \lambda_s| \geq C|k_1 + \cdots k_n|^{-\nu}$$

holds for $s = 1, \cdots, n$, and $(k_1, \cdots, k_n) \in \mathbb{N}^n$, with $k_1 + \cdots k_n \geq 2$.

Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be holomorphic near a fixed point $O \in \mathbb{C}^n$. Let $\lambda_1, \cdots, \lambda_n$ denote the eigenvalues of df_O.

Theorem (Siegel)

If these eigenvalues satisfy a multiplicative diophantian condition, then f is holomorphically linearizable near the fixed point.

Assume that all the eigenvalues $\lambda_1, \cdots, \lambda_n$ are distict. And assume the non-resonance condition

$$\lambda^k - \lambda_s \neq 0 \quad \text{for all} \quad s = 1, \cdots, n \quad \text{and} \quad k \in \mathbb{N}^n, |k| \geq 2.$$

$$\text{For } m \geq 2, \text{ let } \quad \Omega(m) = \min_{2 \leq |k| \leq m,\ 1 \leq s \leq n} |\lambda^k - \lambda_s|.$$

Theorem (Brjuno)

$$\text{If} \qquad \sum_{j=0}^{\infty} \frac{1}{2^j} \log \frac{1}{\Omega(2^{j+1})} < \infty,$$

then f is holomorphically linearizable near the fixed point.

Theorem(Rüssmann, Raissy)

Same result holds, if f is formally linearizable and df_O is diagonalizable.

Suppose $|\lambda_s| = 1, \quad s = 1, \cdots, n$, and a multiplicative diophantian condition or the Brjuno condition holds. The maximal linearizable neighborhood of the fixed point is called a **Siegel ball**. The dynamics in the Siegel ball is holomorphically conjugate to the linear part of f at the fixed point.The image, by the conjugacy, of the Siegel ball is invariant under the linear map df_O.

Open neighborhood of the origin invariant under diagonal linear map of eigenvalues $\lambda_s,\ |\lambda_s| = 1,\ s = 1, \cdots, n$ is a

Reinhardt domain. The inverse map from the image domain to Siegel ball is holomorphic. Our Reinhardt domain must be a maximal domain of holomorphy of this inverse map. It is a logarithmically convex complete Reinhardt domain.

2. Hénon map and fixed points

Hénon map $H_{b,c} : (x, y) \mapsto (X, Y)$ is defined as

$$\begin{cases} X &= x^2 + c + by \\ Y &= x \end{cases} .$$

Fixed points, $P = (p, p)$ and $Q = (q, q)$, are given by
$p = \frac{1}{2}(1-b) + \sqrt{\frac{(1-b)^2}{4} - c}$ and $q = \frac{1}{2}(1-b) - \sqrt{\frac{(1-b)^2}{4} - c}$.

REM. $a = -c$, $d = -b$. (a, b) in Hénon's original family. d is the determinant.

Eigenvalues are given by $\lambda_P^{\pm} = p \pm \sqrt{p^2 + b}$, and $\lambda_Q^{\pm} = q \pm \sqrt{q^2 + b}$. If two eigenvalues λ_Q^+ and λ_Q^- are specified, the fixed point $Q = (q, q)$ and the parameters b and c are computed as follows.

$q = \lambda_Q^+ + \lambda_Q^-, \quad b = -\lambda_Q^+ \lambda_Q^-, \quad c = q - q^2 - bq.$

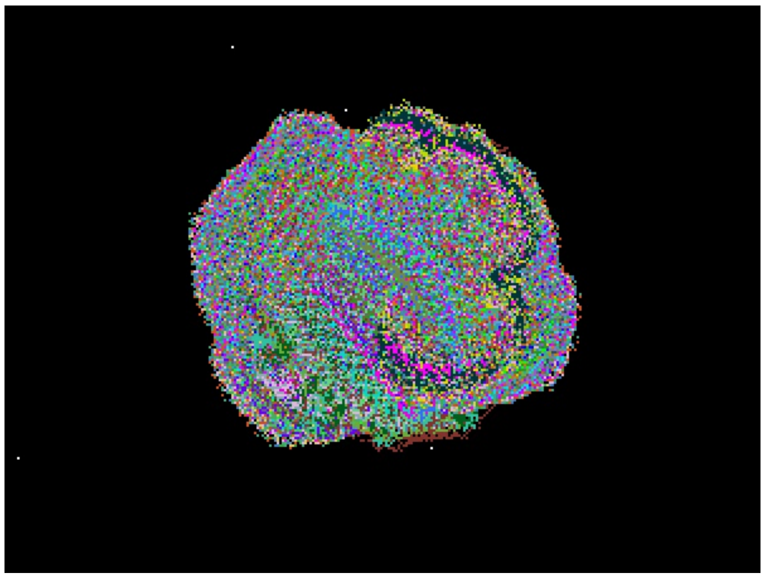
Take diophantian numbers $\theta_1, \theta_2 \in [0, 1]$ such that $\theta_1 - \theta_2$ is also diophantian. Let $\lambda_Q^+ = e^{2\pi i \theta_1}$ and $\lambda_Q^- = e^{2\pi i \theta_2}$. Compute parameters b, c, so that our Hénon map has a Siegel ball centered at fixed point Q.

If a point $z = (x, y) \in \mathbb{C}^2$ is in the Siegel ball, then, by setting $L = DH_{b,c}(Q)$,

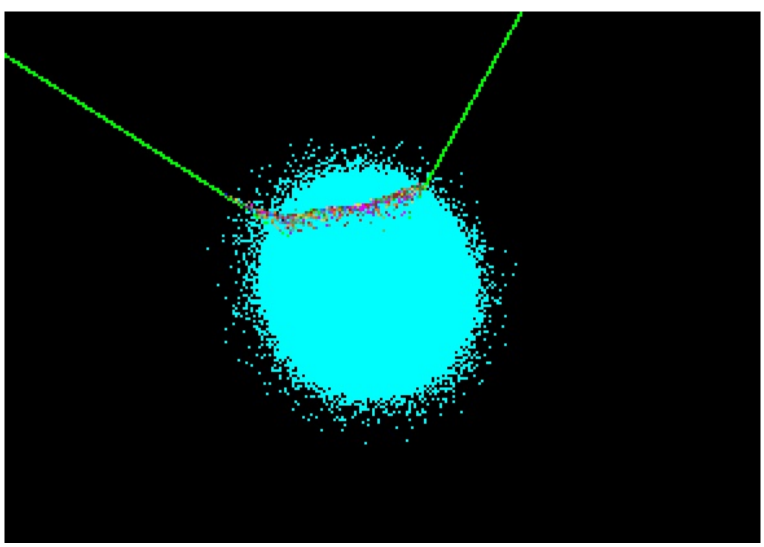
$$\Psi(z) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} L^{-k}(H_{b,c}^{\circ k}(z) - Q)$$

converges and defines the conjugacy map from Siegel ball to Reinhardt domain.

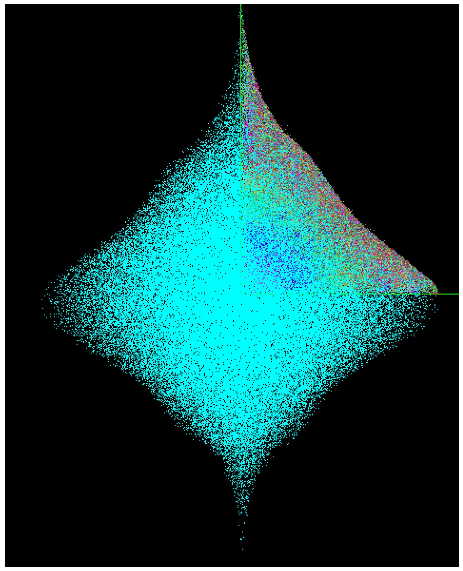
Siegel ball



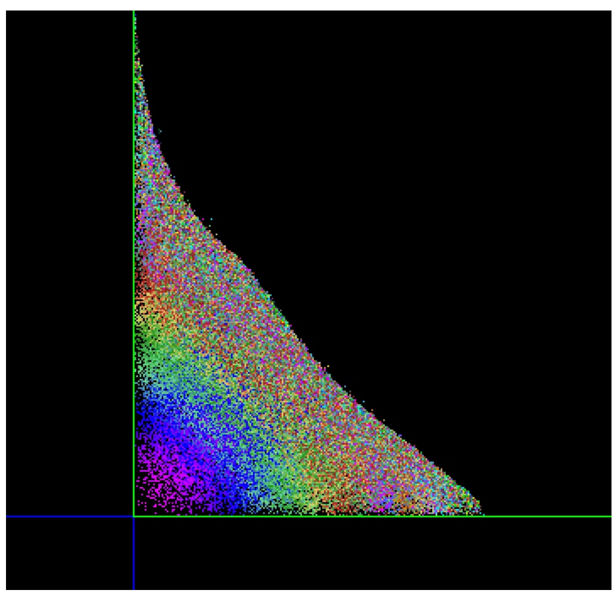
Linearlized Siegel ball



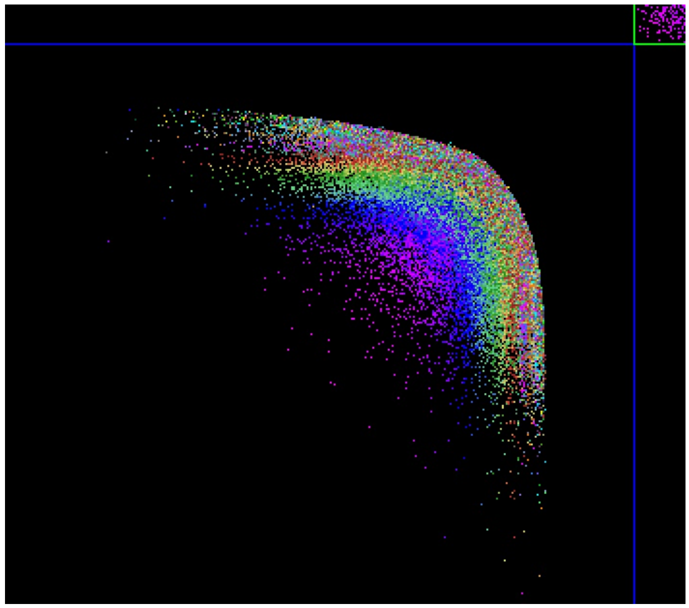
Reinhardt domain



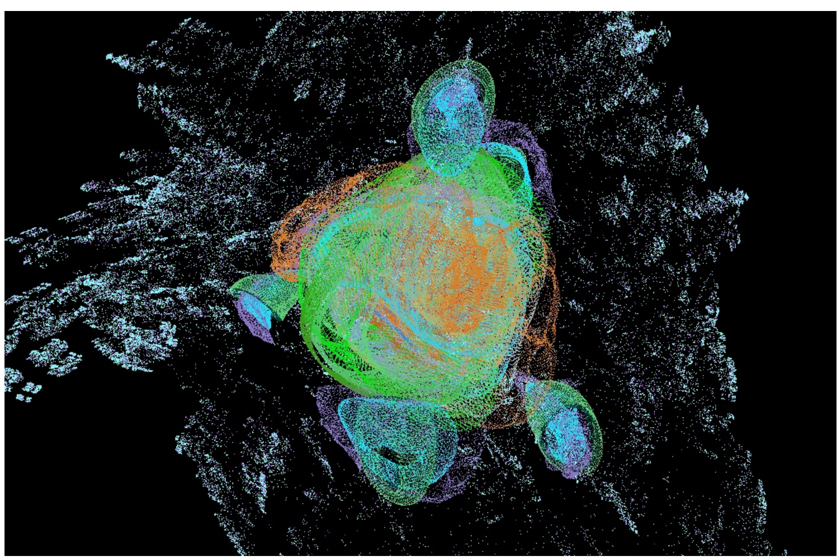
Real slice



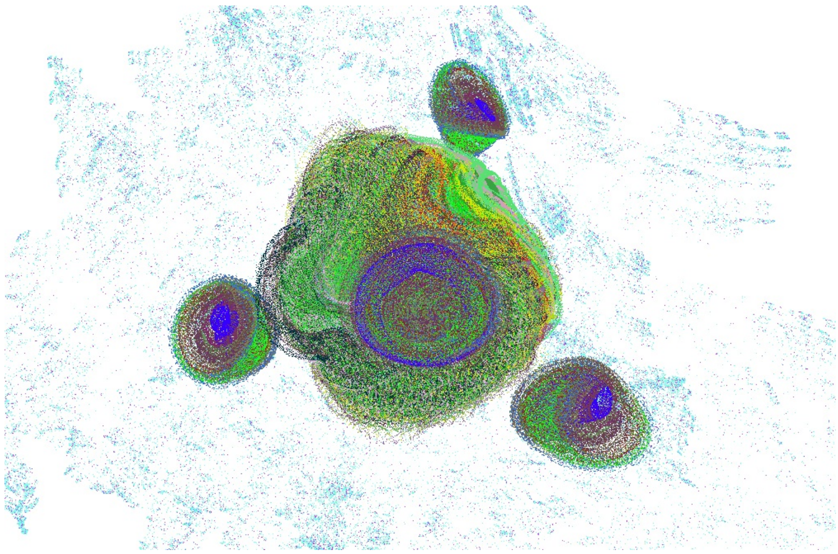
Log-log picture



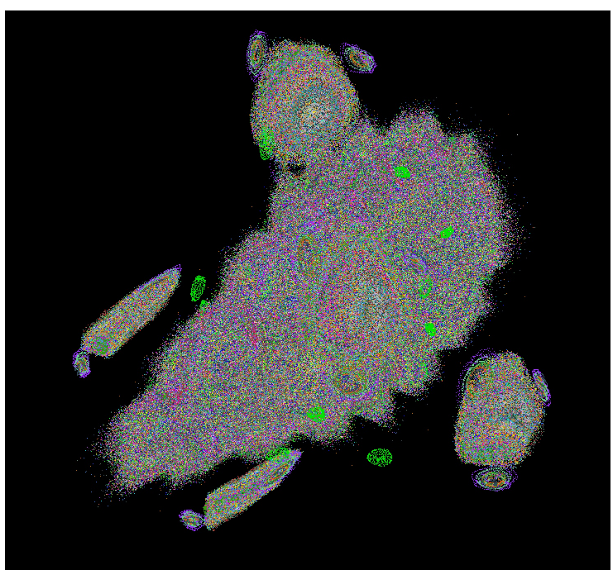
Siegel ball with satellites in Hénon map



Siegel ball with satellites in Hénon map



Siegel ball with secondary satellites



3. Swap-conjugacy and anti-conjugacy

Let us define an involution $T(x, y) = (\bar{y}, \bar{x})$. We call this map the **swap-conjugacy** map.

Rational automorphism $f : \mathbb{C}^2 \to \mathbb{C}^2$ is said to be **self-anti-conjugate**, if

$$T \circ f \circ T = f^{-1}$$

holds.

Volume-preserving Hénon map, with a Siegel ball of period 1 or 2, can be conjugated to a self-anti-conjugate automorphism:

$$h(x, y) = (y, \beta P(y) - \beta^2 x),$$

where β is a complex number satisfying $\beta\bar{\beta} = 1$, and $P(y)$ is a polynomial with real coefficients.

Proposition

$h(x, y)$ is self-anti-conjugate.

Proof

By a direct computation, we see innediately

$$T \circ h \circ T(x, y) \;=\; T \circ h(\bar{y}, \bar{x})$$

$$= \; T(\bar{x}, \beta P(\bar{x}) - \beta^2 \bar{y}) \;=\; (\bar{\beta}\overline{P(\bar{x})} - \bar{\beta}^2 y, x),$$

$$\text{and} \;\; h^{-1}(x, y) = (\beta^{-1} P(x) - \beta^{-2} y, x).$$

Hence $T \circ h \circ T = h^{-1}$ holds if $\overline{P(\bar{x})} = P(x)$ and $\bar{\beta} = \beta^{-1}$.

Proposition

If the classical Hénon map has a Siegel ball around a fixed point, or has a cycle of Siegel balls around periodic points of period 2, then it is conjugate to a self-anti-conjugate map.

Proof

Suppose a fixed point of Hénon map $H(x,y) = (y, y^2+c+bx)$ has a Siegel ball with eigenvalues $\beta\lambda$ and $\beta\bar{\lambda}$, $|\beta| = |\lambda| = 1$. We see $-b = \det(DH) = \beta^2$. Then the Jacobin matrix at the fixed point (q,q) is

$$DH_{(q,q)} = \beta \begin{pmatrix} 0 & \bar{\beta} \\ -\beta & 2\bar{\beta}q \end{pmatrix}.$$

We require the trace of this matrix to be real, we set $q = t\beta$, $t \in \mathbb{R}$. Then $(t\beta)^2 - (1+\beta^2)t\beta + c = 0$ must hold and $c = -\alpha b$, with $\alpha = t^2 - (\beta + \bar{\beta})t \in \mathbb{R}$.

The conjugacy from Hénon map is given by $x' = \bar{\beta}x, y' = \bar{\beta}y$, with $b = -\beta^2, c = -\alpha b$ and $P(z) = z^2 + \alpha, \alpha \in \mathbb{R}$.

If periodic point of period 2 has a cycle of Siegel balls, the Hénon map is conjugate to our self-anti-conjugate map.

Let (p,q) and (q,p) be the periodic points of period two, which satisfy $q = p^2 + c + bq, \quad p = q^2 + c + bp, \quad p \neq q.$ We have $p + q = b - 1, \quad pq = (b-1)^2 + c,$ and $|b| = 1.$

The Jacobian matrix of the 2-cycle is given by

$$D(H^2)_{(p,q)} = b \begin{pmatrix} 1 & \frac{2p}{b} \\ \frac{2q}{b} & 1 + \frac{4pq}{b} \end{pmatrix}.$$

We require the trace to be real.

$$2 + \frac{4pq}{b} = 4(b + \bar{b}) + 4c\bar{b} \in \mathbb{R}.$$

Let $c = -\alpha b$ with $\alpha \in \mathbb{R}$.

Conjugacy map to self-anti-conjugate map is same as in the previous case.

4. Self-anti-conjugate orbit

The following arguements hold if $P(\bar{z}) = \overline{P(z)}$.

Forward orbit of an initial point is anti-conjugate to the backward orbit of the swap-conjugate inititial point. If $z_0 = (x_0, y_0)$ and $w_0 = T(z_0)$. Then $h^n(z_0) = T(h^{-n}(w_0))$ for n = 1, 2, ... Especially, if initial point is **self-swap-conjugate**, say $T(z_0) = z_0$, then $z_n = T(z_{-n})$.

If initial point is mapped to its swap-conjugate point, say $z_1 = h(z_0) = T(z_0)$, then $z_n = T(z_{-n+1})$, for n=2,3,... We will call this pair z_0 and z_1 a **swap-conjugate pair**.

Suppose periodic orbit $z_0, z_1, \cdots, z_{p-1}$ of h contains a self-swap-conjugate point, say $T(z_0) = z_0$. Then we have $T(z_k) = z_{p-k}, k = 1, 2, \cdots$.

If p is even, then $z_{p/2}$ is a self-swap-conjugate point.

If p is odd, then $z_{(p-1)/2}$ and $z_{(p+1)/2}$ is a swap-conjugate pair.

Suppose periodic orbit contains a swap-conjugate pair. Then we have another swap-conjugate-pair if the period is even.

We have a self-swap-conjugate point if the period is odd.

Theorem H

If periodic orbit $z_0, z_1, \cdots, z_{p-1}$ of h contains a self-swap-conjugate point or a swap-conjugate pair, then the Jacobian matrix of the cycle is of the form

$$D(h^p)_{z_0} = \beta^p A,$$

where $\det(A) = 1$ and $\mathrm{trace}(A) \in \mathbb{R}$.

We need some prelimiaries of anti-linear algebra. Proof of this theorem will be given below. For $2 \times 2-$matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define its **anti-conjugate matrix** $A^!$ by,

$$A^! = \begin{pmatrix} \bar{a} & -\bar{c} \\ -\bar{b} & \bar{d} \end{pmatrix}.$$

Anti-conjugacy is an involution. We see immediately the followings.

$$A^{!!} = A, \quad (AB)^! = B^! A^!, \quad \beta^! = \bar{\beta} \ \ \text{(as scalar matrix).}$$

We say A is **self-anti-conjugate** if $A^! = A$. Clearly, if A is self-anti-conjugate, then $a, d \in \mathbb{R}$, $b = -\bar{c}$, and both $\det(A)$ and $\mathrm{trace}(A)$ are real. The following propositions are easily verified.

Proposition

$BB^!$ is self-anti-conjugate for 2×2-matrix B.

Proposition

If A is self-anti-conjugate, then $BAB^!$ is self-anti-conjugate for 2×2-matrix B.

Suppose $z_0, z_1, \cdots, z_{p-1}$ be a periodic cycle of our Hénon map $h(x, y) = (y, \beta P(y) - \beta^2 x)$, where $z_k = (x_k, y_k)$.

Let us assume that the period $p = 2q + 1$ is odd, and z_0 is self-swap-conjugate, so that $T(z_0) = z_0$, $\quad T(z_q) = z_{q+1}$. The derivative of h at $z = (x, y)$ is given by

$$Dh_z = \begin{pmatrix} 0 & 1 \\ -\beta^2 & \beta P'(y) \end{pmatrix} = \beta \begin{pmatrix} 0 & \bar{\beta} \\ -\beta & P'(y) \end{pmatrix}.$$

At the swap-conjugate pair $z_q = (x_q, y_q)$ and $z_{q+1} = (x_{q+1}, y_{q+1}) = T(x_q, y_q)$, we have

$$\bar{y}_q = x_{q+1} = y_q = \bar{x}_{q+1} \in \mathbb{R}, \quad \text{and} \quad y_{q+1} = \bar{x}_q.$$

Hence,

$$Dh_{z_q} = \beta \begin{pmatrix} 0 & \bar{\beta} \\ -\beta & P'(y_q) \end{pmatrix},$$

and by setting $Dh_{z_q} = \beta A_q$, A_q is a self-anti-conjugate matrix.

Let us consider the derivative Dh_{z_0} at self-swap-conjugate point, and $Dh_{z_{p-1}}$ at its pre-image. We have $z_0 = h(z_{p-1})$, $T(z_0) = z_0$, $z_1 = h(z_0)$, $T(z_1) = z_{p-1}$. Note that $y_{p-1} = \bar{x}_1$ and $x_1 = y_0$, hence $y_{p-1} = \bar{y}_0$.

We compute the derivative of $h \circ h$ at z_{p-1}, as follows.

$$Dh_{z_0} = \beta \begin{pmatrix} 0 & \bar{\beta} \\ -\beta & P'(y_0) \end{pmatrix}, \quad Dh_{z_{p-1}} = \beta \begin{pmatrix} 0 & \bar{\beta} \\ -\beta & P'(y_{p-1}) \end{pmatrix}.$$

Hence, by setting $Dh_{z_0} = \beta B_0$, $Dh_{z_{p-1}} = \beta B_0^!$, we see that $D(h \circ h)_{z_{p-1}} = \beta^2 B_0 B_0^!$, and $A_0 = B_0 B_0^!$ is self-anti-conjugate.

Let z_k and z_{p-k} be swap-conjugate points, *i.e.* $z_{p-k} = T(z_k)$. Note that $z_{p-k-1} = T(z_{k+1})$, and $y_{p-k-1} = x_{p-k} = \bar{y}_k$. Compute the derivatives Dh_{z_k} and $Dh_{z_{p-k-1}}$ as follows.

$$Dh_{z_k} = \beta \begin{pmatrix} 0 & \bar{\beta} \\ -\beta & P'(y_k) \end{pmatrix}, \quad Dh_{z_{p-k-1}} = \beta \begin{pmatrix} 0 & \bar{\beta} \\ -\beta & P'(y_{p-k-1}) \end{pmatrix}.$$

By setting $Dh_{z_k} = \beta B_k$, we have $Dh_{z_{p-k-1}} = \beta B_k^!$.

Now, we try to compose the derivatives along the periodic orbit. Suppose $z_0, z_1, \cdots, z_{p-1}$ be a periodic cycle of our Hénon map $h(x, y) = (y, \beta P(y) - \beta^2 x)$, where $z_k = (x_k, y_k)$. And assume that the period $p = 2q + 1$ is odd, and z_0 is self-swap-conjugate.

Proposition

For $k = 1, \cdots, q$, derivative of h^{2k} at z_{p-k} is of the form

$$D(h^{2k})_{z_{p-k}} = \beta^{2k} A_{k-1},$$

where A_{k-1} is a self-anti-conjugate matrix, *i.e.* $A_{k-1}^! = A_{k-1}$.

Proof

Let $Dh_{z_0} = \beta B_0$, then $Dh_{z_{p-1}} = \beta B_0^!$. Set $A_0 = B_0 B_0^!$, then $A_0^! = A_0$ and $D(h^2)_{z_{p-1}} = \beta^2 A_0$. Now assume $D(h^{2k})_{z_{p-k}} = \beta^{2k} A_{k-1}$ and $A_{k-1}^! = A_{k-1}$. Then by setting $A_k = B_k A_{k-1} B_k^!$, we have $A_k^! = A_k$ and

$$D(h^{2(k+1)})_{z_{p-(k+1)}} = Dh_{z_k} D(h^{2k})_{z_{p-k}} Dh_{z_{p-(k+1)}}$$

$$= \beta B_k \beta^{2k} A_{k-1} \beta B_k^! = \beta^{2(k+1)} B_k A_{k-1} B_k^! = \beta^{2(k+1)} A_k.$$

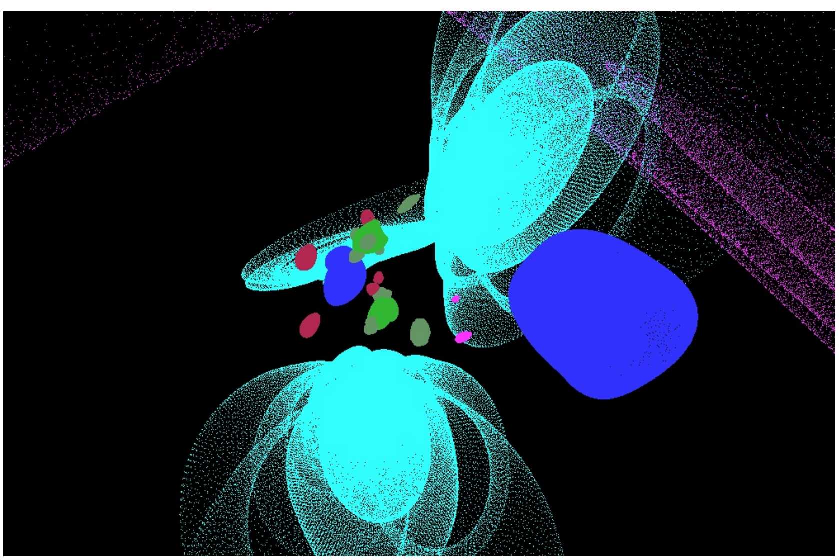
Proof of Theorem H

As proved in the previous proposition, if a periodic cycle contains a self-anti-conjugate point, say $z_0 = T(z_0)$, and period $p = 2q+1$ is odd, then the Jacobian matrix of the cycle is of the form

$$D(h^p)_{z_q} = \beta^{2q} A_{q-1} \beta A_q = \beta^p A_{q-1} A_q.$$

Here, A_{q-1} and A_q are self-aniti-conjugate matrices. Set $A = A_{q-1} A_q$. As is easily verified, $\det(A) = 1$, and $\mathrm{trace}(A) \in \mathbb{R}$. Other cases of self-anti-conjugate cycles can be proved similarly.

Siegel balls in birational automorphism



5. Rational automorphism of complex surface

Here, we notice that similar results hold for some rational automorphisms of complex surface. Rational automorphism,

$$f(x,y)=(y,\frac{y+\alpha}{x+i\beta}+i\beta)$$

is self-anti-conjugate if α and β are real.

More generally, rational automorphism

$$f(x,y)=(y,\frac{P(y)}{x+i\beta}+i\beta)$$

is self-anti-conjugate if β is real and $\overline{P(\bar{x})}=P(x)$.

The self-anti-conjugacy of rational automorphism can be verified immediately as follows.

$$T\circ f\circ T(x,y)=T\circ f(\bar{y},\bar{x})$$
$$=T(\bar{x},\frac{P(\bar{x})}{\bar{y}+i\beta}+i\beta)=(\frac{\overline{P(\bar{x})}}{y-i\beta}-i\beta,x),$$

and

$$f^{-1}(x,y)=(\frac{P(x)}{y-i\beta}-i\beta,x).$$

Hence $T\circ f\circ T=f^{-1}$ is satisfied. We have a theorem for self-anti-conjugate reational automorphisms.

Theorem R

If periodic orbit $z_0,z_1,\cdots,z_{p-1}$ of self-anti-conjugate birational automorphism, f, contains a self-swap-conjugate point or a swap-conjugate pair, then the Jacobian matrix of the cycle is of the form

$$D(h^p)_{z_0}=\lambda A,\quad \lambda=\prod_{k=0}^{p-1}\frac{|x_k+i\beta|}{x_k+i\beta},$$

where $\quad z_k = (x_k, y_k), \quad \det(A) = 1 \quad$ and $\quad \mathrm{trace}(A) \in \mathbb{R}$.

Suppose z_0 is self-swap-conjugate and the period $p = 2q+1$ is odd. For $k = 0, \cdots, p-1$,

$$Df_{z_k} = \frac{1}{x_k + i\beta} \begin{pmatrix} 0 & x_k + i\beta \\ -y_{k+1} + i\beta & P'(y_k) \end{pmatrix}.$$

As z_0 is self swap-conjugate, $\quad T(z_0) = z_0, \quad T(z_1) = z_{p-1}$, $x_1 = y_0 = \bar{x}_0 = \bar{y}_{p-1}, \quad$ and $\quad y_1 = \bar{x}_{p-1}$.

$$Df_{z_0} = \frac{1}{x_0 + i\beta} \begin{pmatrix} 0 & x_0 + i\beta \\ -y_1 + i\beta & P'(y_0) \end{pmatrix},$$

$$Df_{z_{p-1}} = \frac{1}{x_{p-1} + i\beta} \begin{pmatrix} 0 & x_{p-1} + i\beta \\ -y_0 + i\beta & P'(y_{p-1}) \end{pmatrix}.$$

By setting

$$Df_{z_0} = \frac{1}{x_0 + i\beta} B_0,$$

we have

$$Df_{z_{p-1}} = \frac{1}{x_{p-1} + i\beta} B_0^{!}$$

and $\quad A_0 = B_0 B_0^{!} \quad$ is self-anti-conjugate.

As we supposed, z_q and z_{q+1} is an anti-conjugate pair satisfying

$$T(z_q) = z_{p-q} = z_{q+1} = f(z_q).$$

$$y_{q+1} = \bar{x}_q, \quad \bar{y}_q = x_{q+1} = y_q.$$

$$Df_{z_q} = \frac{1}{x_q + i\beta} \begin{pmatrix} 0 & x_q + i\beta \\ -y_{q+1} + i\beta & P'(y_q) \end{pmatrix}.$$

Set $\quad Df_{z_q} = \frac{1}{x_q + i\beta} A_q, \quad$ then $\quad A_q \quad$ is self-anti-conjugate.

As in the Hénon map case, suppoose $z_0, z_1, \cdots, z_{p-1}$ be a periodic cycle of our birational automorphism f. And assume the period $p = 2q+1$ is odd, and z_0 is self-swap-conjugate.

Proposition

For $k = 1, \cdots, q$, derivative of f^{2k} at z_{p-k} is of the form

$$D(f^{2k})_{z_{p-k}} = \left(\prod_{j=0}^{k-1} \frac{1}{(x_j + i\beta)(x_{p-j-1} + i\beta)} \right) A_{k-1}$$

where A_{k-1} is a self-anti-conjugate matrix, *i.e.* $A^!_{k-1} = A_{k-1}$.

Proof of proposition

For $k = 1$,

$$Df_{z_0} Df_{z_{p-1}} = \frac{1}{(x_0 + i\beta)(x_{p-1} + i\beta)} A_0,$$

as shown in the computation above.

Now, aasume that

$$D(f^{2k})_{z_{p-k}} = \left(\prod_{j=0}^{k-1} \frac{1}{(x_j + i\beta)(x_{p-j-1} + i\beta)} \right) A_{k-1}$$

holds with self-anti-conjugate matrix A_{k-1}. Then

$$D(f^{2(k+1)})_{z_{p-(k+1)}} = Df_{z_k} D(f^{2k})_{z_{p-k}} Df_{z_{p-(k+1)}}$$

$$= \frac{1}{x_k + i\beta} B_k \left(\prod_{j=0}^{k-1} \frac{1}{(x_j + i\beta)(x_{p-j-1} + i\beta)} \right) A_{k-1} \frac{1}{x_{p-(k+1)} + i\beta} B^!_k$$

$$= \left(\prod_{j=0}^{k} \frac{1}{(x_j + i\beta)(x_{p-j-1} + i\beta)} \right) B_k A_{k-1} B^!_k.$$

By setting $A_k = B_k A_{k-1} B^!_k$, A_k is self-anti-conjugate.

Proof of Theorm R The Jacobian matriox of the periodic cycle is given by

$$D(f^p)_{z_q} = D(f^{2q})_{z_{p-q}} Df_{z_q}$$

$$= \left(\prod_{j=0}^{q} \frac{1}{(x_j + i\beta)(x_{p-j-1} + i\beta)}\right) A_{q-1} \frac{1}{x_q + i\beta} A_q$$

$$= \left(\prod_{j=0}^{p-1} \frac{1}{x_j + i\beta}\right) A_{q-1} A_q.$$

Note that $\det(A_{q-1}A_q) \in \mathbb{R}$ and $\operatorname{trace}(A_{q-1}A_q) \in \mathbb{R}$, since A_{q-1} and A_q are self-anti-conjugate.

Now, consider the determinant of the Jacobian matrix.

$$\det(Df_{z_k}) = \frac{y_{k+1} - i\beta}{x_k + i\beta}.$$

$$\det(D(f^p)_{z_q}) = \prod_{k=0}^{p-1} \frac{y_{k+1} - i\beta}{x_k + i\beta} = \prod_{k=0}^{p-1} \frac{\bar{x}_{p-k-1} - i\beta}{x_k + i\beta} = \prod_{k=0}^{p-1} \frac{\bar{x}_k - i\beta}{x_k + i\beta}.$$

Then $|\det(D(f^p)_{z_q})| = 1$. Hence, by setting

$$\lambda = \prod_{k=0}^{p-1} \frac{|x_k + i\beta|}{x_k + i\beta}, \qquad A = \left(\prod_{k=0}^{p-1} \frac{1}{|x_k + i\beta|}\right) A_{q-1} A_q,$$

we obtain

$$D(f^p)_{z_q} = \lambda A,$$

with $|\lambda| = 1$ and $\det(A) = 1$, $\operatorname{trace}(A) \in \mathbb{R}$.

Other cases of self-anti-conjugate periodic cycles are proved smimilarly.

Bedford and Kim studied rotation domains for a surface automorphism $f_{a,b}(x,y) = (y, (y+a)/(x+b))$.

Proposition

If $f_{a,b}$ has a siegel ball around a fixed point, then the automorphism is conjugte to our self-anti-conjugate automorphism.

Proof of Proposition

The fixed point (p,p) satisfies $p(p+b) = p+a$. Assume that the eigenvalues of $Df_{a,b}$ at the fixed point are $\lambda\mu$ and $\lambda\bar{\mu}$, $|\lambda| = |\mu| = 1$. Then

$$\det(Df_{a,b})_{(p,p)} = \frac{p+a}{(p+b)^2} = \lambda^2.$$

By eliminating a, we obtain

$$p = \frac{b\lambda^2}{1-\lambda^2}, \quad p+b = \frac{b}{1-\lambda^2}.$$

The differential at the fixed point is

$$D(f_{a,b})_{(p,p)} = \begin{pmatrix} 0 & 1 \\ -\frac{p+a}{(p+b)^2} & \frac{1}{p+b} \end{pmatrix} = \lambda \begin{pmatrix} 0 & \bar{\lambda} \\ -\lambda & \frac{(\bar{\lambda}-\lambda)}{b} \end{pmatrix}.$$

Hence we require

$$\frac{(\bar{\lambda}-\lambda)}{b} \in \mathbb{R}.$$

We set $b = 2i\beta$, $\beta \in \mathbb{R}$, and $\lambda = \cos\theta + i\sin\theta$. Then we have

$$p = \frac{2i\beta\lambda^2}{1-\lambda^2} = -i\beta - \frac{\cos\theta}{\sin\theta}\beta, \quad \text{and}$$

$$a = p(p+b) - p = \frac{\beta^2}{\sin^2\theta} + \frac{\cos\theta}{\sin\theta}\beta + i\beta.$$

As

$$\alpha = \frac{\beta^2}{\sin^2\theta} + \frac{\cos\theta}{\sin\theta}\beta \in \mathbb{R},$$

set $a = \alpha + i\beta$. Then $x' = x + i\beta, y' = y + i\beta$ gives the conjugacy from $f_{a,b}(x,y) = (y, (y+a)/(x+b))$ to our self-anti-conjugate map

$$f(x', y') = (y', \frac{y' + \alpha}{x' + i\beta} + i\beta).$$

Proposition

If $f_{a,b}$ has a cycle of siegel balls around periodic point of period 2, then the automorphism is conjugte to our self-anti-conjugate automorphism.

Proof

Suppose 2-cycle of $f_{a,b}$ has Siegel balls. Let (p,q) and (q,p) denote the periodic point of period 2.

$$p = \frac{q+a}{p+b}, \quad q = \frac{p+a}{q+b},$$

and p, q are two roots of $x^2 + (b+1)x + b + 1 - a = 0$. Hence, $pq = b + 1 - a$ and $p + q = -b - 1$.

$$\det(D(f^2_{a,b})_{(p,q)}) = \frac{p+a}{(q+b)^2}\frac{q+a}{(p+b)^2} = \frac{1-a+b}{1-a}.$$

Now, assume eigenvalues of the cycle are $\lambda\mu$ and $\lambda\bar{\mu}$, $|\lambda| = |\mu| = 1$. Then

$$\det(D(f^2_{a,b})_{(p,q)}) = \frac{1-a+b}{1-a} = \lambda^2.$$

Compute the Jacobian matrix of the 2-cycle,

$$D(f^2_{a,b})_{(p,q)} = \frac{1}{(p+b)(q+b)}\begin{pmatrix} -q(p+b) & p+b \\ -q & -p(q+b)+1 \end{pmatrix}$$

As $(p+b)(q+b) = 1-a = \frac{b\bar{\lambda}}{\lambda-\bar{\lambda}}$, by setting

$$D(f_{a,b}^2)_{(p,q)} = \lambda A,$$

We have

$$\det A = 1, \quad \text{trace } A = \frac{\lambda-\bar{\lambda}}{b}(2a-1+b^2-b).$$

Eliminate a by using $1-a = \frac{b\bar{\lambda}}{\lambda-\bar{\lambda}}$. And by setting $\lambda = \cos\theta + i\sin\theta$, we get

$$\text{trace } A = -2\cos\theta + 2i\sin\theta \ (b+\frac{1}{b}).$$

As eigenvalues of A are μ and $\bar{\mu}$, we require the trace to be real. We conclude that b is pure imaginary.

Let $b = 2i\beta$. Then

$$a = 1 - \frac{b\bar{\lambda}}{\lambda-\bar{\lambda}} = 1 - \frac{2i\beta(\cos\theta - i\sin\theta)}{2i\sin\theta} = 1 - \frac{\cos\theta}{\sin\theta}\beta + i\beta.$$

Let $\alpha = 1 - \frac{\cos\theta}{\sin\theta}\beta$, and get $a = \alpha + i\beta$.

Conjugacy to self-anti-conjugate map is same as in the case of Siegel ball around a fixed point.

Proposition

If self-swap-conjugate point, $z_0 = (x_0, y_0)$, is a peiodic point, of period p, of a self-anti-conjugate map g, then the Jacobian matrix of the cycle is of the following form.

$$D(g^p)_{z_0} = \lambda A,$$

where, $|\lambda| = 1$, $\det(A) = 1$, and $\text{trace}(A) \in \mathbb{R}$.

Proof

As g is self-anti-conjugate,

$$T \circ g^p \circ T = g^{-p}.$$

And as z_0 is self-swap-conjugate,

$$T(z_0) = z_0.$$

Hence

$$T \circ D(g^p)_{T(z_0)} \circ T = D(g^{-p})_{z_0}.$$

We have

$$T \circ D(g^p)_{z_0} \circ T = (D(g^p)_{z_0})^{-1}.$$

Set

$$D(g^p)_{z_0} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then we have

$$\begin{pmatrix} \bar{d} & \bar{c} \\ \bar{b} & \bar{a} \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

And

$$ad - bc = \frac{d}{\bar{d}} = \frac{a}{\bar{a}}.$$

Set $a = r\lambda$ and $d = s\lambda$ with $r, s \in \mathbb{R}$ and $|\lambda| = 1$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \lambda \begin{pmatrix} r & -b\bar{\lambda} \\ -c\bar{\lambda} & s \end{pmatrix}.$$

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