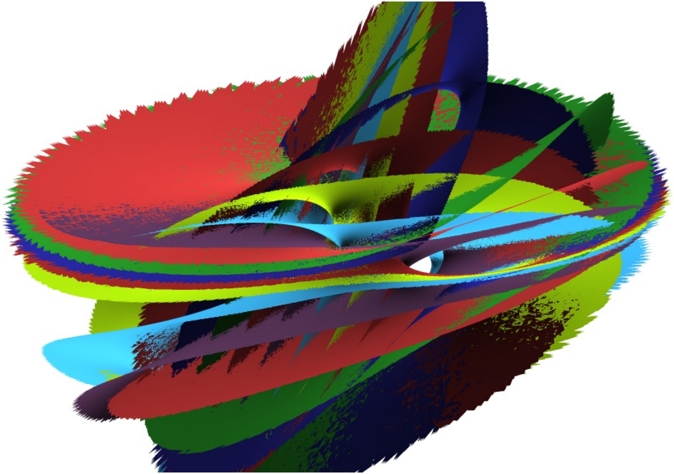
Elliptic fibrations and periodic curves in Surface Automorphisms



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Abstract

If the topological entropy of a surface automorphism is zero, then there is an invariant elliptic fibration.

We construct surface automorphisms with invariant elliptic fibration.

For some surface automorphism, all the eigenvalues, which are roots of unity, correspond to periodic cycles of curves.

The degree of the rational function that define such invariant fibration can be 3, 4, 5, 6.

Contents

0. Introduction

Dynamical degree

Let $F : S \to S$ be a bi-holomorphic automorphism of a compact Kähler surface S.

F induces cohomology homomorphism

$$F^* : H^2(S, \mathbb{Z}) \to H^2(S, \mathbb{Z}).$$

Let

$$\lambda_1 = \lim_{n \to \infty} ||(F^n)^*||^{1/n}.$$

In this note, we consider the case

$$\lambda_1 = 1, \quad \{||(F^n)^*||\}_{n \in \mathbb{N}} \text{ is unbounded.}$$

Elliptic fibration

THEOREM(Gizatullin [1980], Cantat [1999])
Assume $F \in \mathrm{Aut}(S)$, $\lambda_1 = 1$, and $\{||(F^n)^*||\}_{n\in\mathbb{N}}$ is unbounded. Then F preserves an elliptic fibration.

THEOREM(Gizatullin [1980], Bellon [1999])
Suppose that $F \in \mathrm{Aut}(S)$ preserves an elliptic fibration and $\{||(F^n)^*||\}_{n\in\mathbb{N}}$ is unbounded. Then $||(F^n)^*|| = Cn^2(1 + o(1))$ for some $C > 0$.

Theorem A

Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map. Under certain conditions, birational map induces a holomorphic automorphism $F : S \to S$ of rational surface S, which is obtained by successive blowing ups of $\mathbb{P}^2$, with projection $\pi : S \to \mathbb{P}^2$.

In this note, we construct surface automorphisms, starting from birational map f preserving a cubic curve X, such that F preserves an elliptic fibration defined by rational function $\varphi \circ \pi : S \to \mathbb{P}^1$.

Theorem A For $d = 3, 4, 5, 6$, there exist surface automorphisms preserving elliptic fibration defined by rational function of degree d.

Theorem B

If $\lambda_1 = 1$, then all the eigenvalues of F^* are roots of unity.

Existence of periodic curve of period p for F often suggests a p-th root of unity as an eigenvalue of F^*.

THEOREM B There exists a surface automorphism $F : S \to S$ with $\lambda_1 = 1$, and $\{||(F^n)^*||\}_{n \in \mathbb{N}}$ is unbounded, such that F has periodic curves of period p for each eigenvalue $\exp(2\pi i/p)$ of F^*.

Observation C

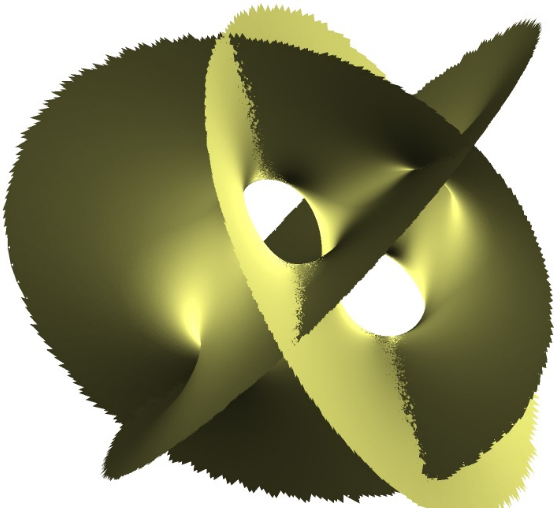
Let $\varphi \circ \pi : S \to \mathbb{P}^1$ be an elliptic fibration such that $\varphi \circ \pi \circ F = \Omega \circ \varphi \circ \pi$ for some Möbius transformation $\Omega : \mathbb{P}^1 \to \mathbb{P}^1$ with $\Omega^q = id$, $(\Omega^k \neq id.$ for $1 < k < q)$.

$$\begin{array}{ccc} S & \xrightarrow{F} & S \\ \downarrow \varphi \circ \pi & & \downarrow \varphi \circ \pi \\ \mathbb{P}^1 & \xrightarrow{\Omega} & \mathbb{P}^1. \end{array}$$

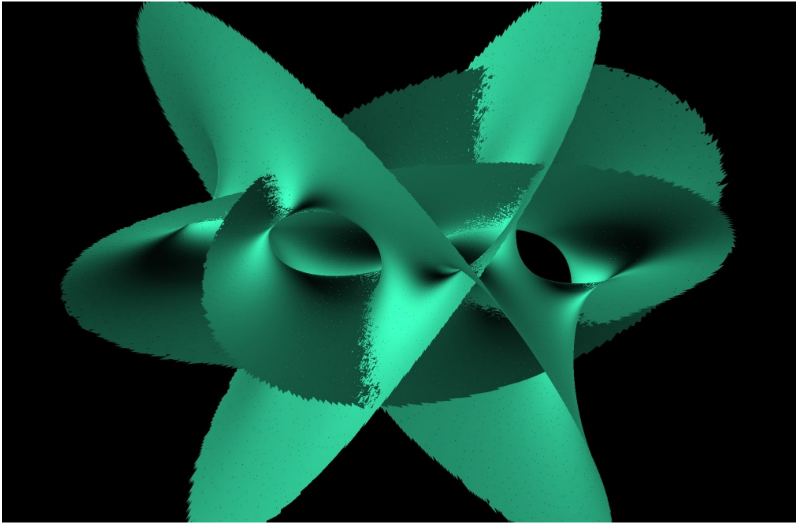
Observation C

For $q = 1, 2, 4, 5, 7$, there exist surface automorphisms preserving elliptic fibrations in the above sense.

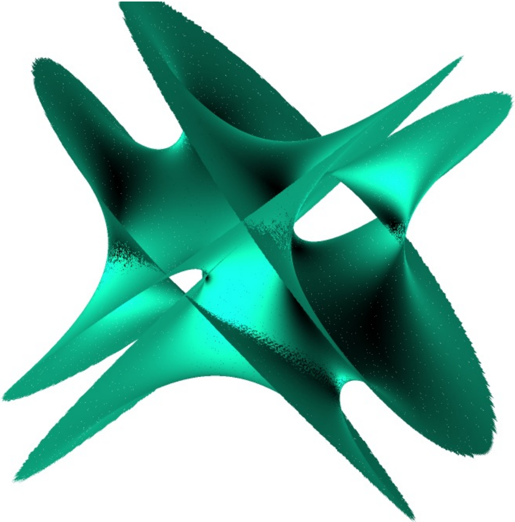
Cubic elliptic curve



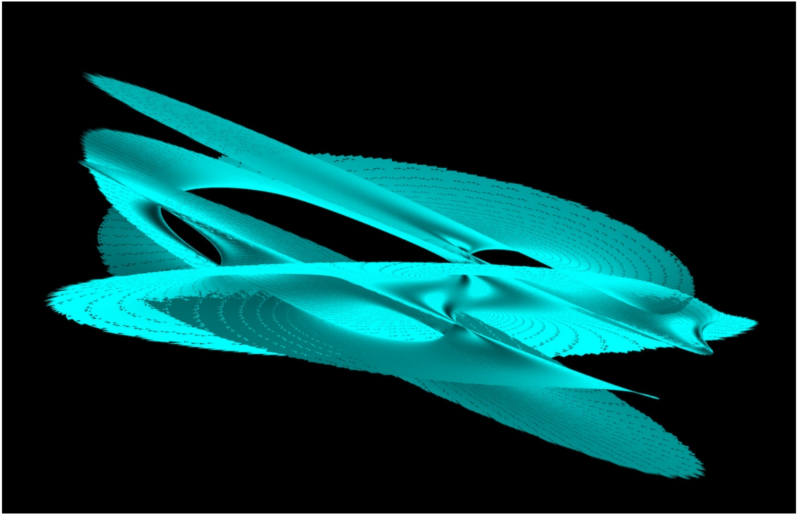
Quartic elliptic curve



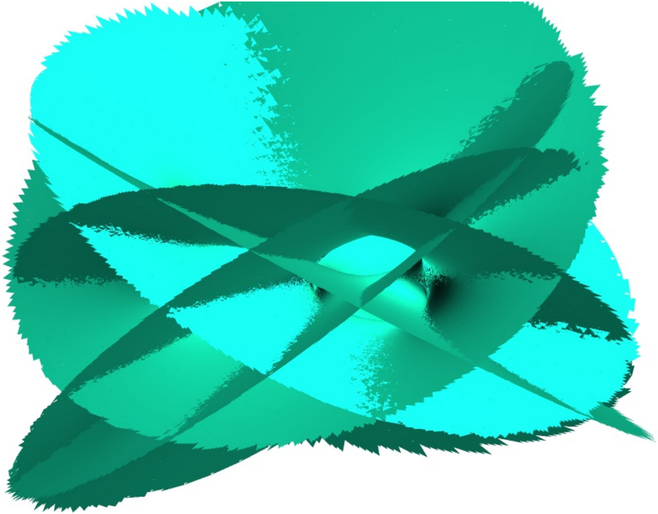
Quartic elliptic curve



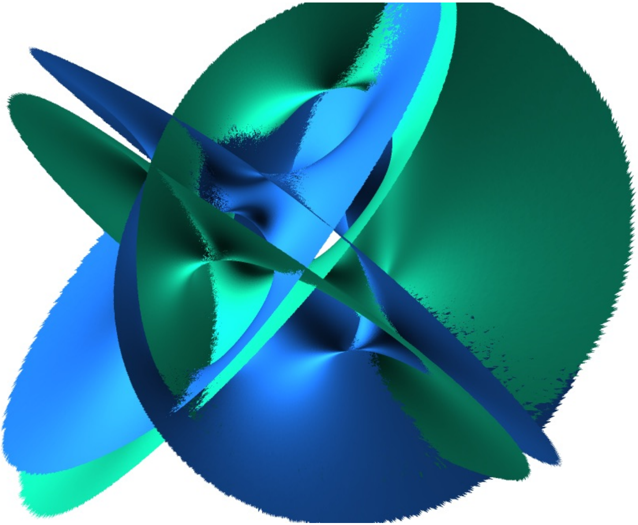
Quintic elliptic curve



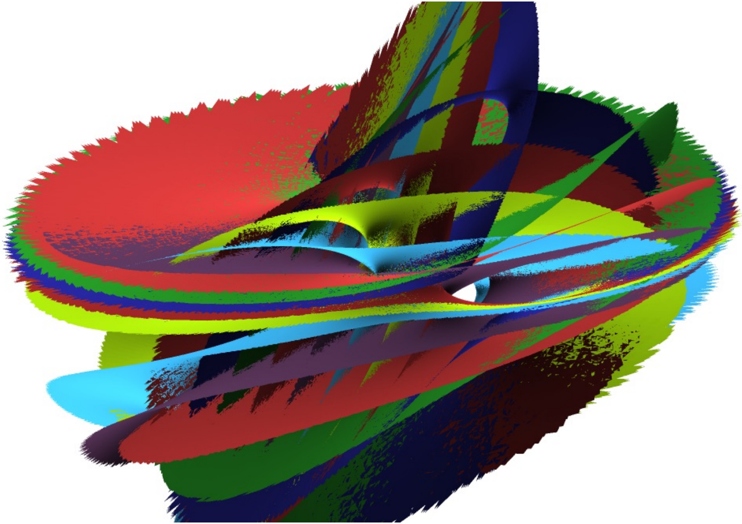
Sextic elliptic curve



Elliptic curve of period 2



Elliptic curve of period 7



1. Surface automorphism

Exceptional lines

A quadratic birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ always acts by blowing up three indeterminacy points in $\mathbb{P}^2$ and blowing down the three exceptional lines joining them.

The inverse map f^{-1} is also quadratic and the images of three exceptional lines of f are the indeterminacy points of f^{-1}.

Let

$$I(f) = \{p_1^+, p_2^+, p_3^+\}$$

and

$$I(f^{-1}) = \{p_1^-, p_2^-, p_3^-\},$$

with

$$p_i^- = f(\ell(p_j^+, p_k^+)), \quad \{i, j, k\} = \{1, 2, 3\}.$$

Orbt data

If, for some positive integers n_1, n_2, n_3, and permutation $\sigma : \{1,2,3\} \to \{1,2,3\}$,

$$p^+_{\sigma(i)} = f^{\circ(n_i-1)} p^-_i, \qquad i = 1,2,3,$$

holds, then f lifts to a surface automorphism by blowing up $(n_1 + n_2 + n_3)$ points (provided they are all distinct)

$$p^-_i, f(p^-_i), \cdots, f^{\circ(n_i-1)}(p^-_i), \qquad i = 1,2,3.$$

Exceptionally, when some of these points coincide, we need careful treatment in successive blowups.

Lift to surface automorphism

For most of given orbit data (n_1, n_2, n_3), σ, we can construct a marked blowup (S, ϕ), with a surface automorphism $F : S \to S$.

$$\begin{array}{ccc} F : S & \to & S \\ \downarrow \pi & & \downarrow \pi \\ f : \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^2 \end{array}$$

Construction of surface automorphism

1. Choose an orbit data (n_1, n_2, n_3), σ.
2. Compute the characteristic polynomial and eigenvalues.
3. Choose a cubic curve (and permutation of components).
4. Choose a Picard parametrization.
5. Choose an eigenvalue of the cohomology homomorphism.
6. Choose a compatible inner dynamics (and parameters).
7. Compute base points (with parameters).
8. Construct a marked cubic curve
9. Construct a birational map satisfying the above data.
10. Construct a marked blowup.
11. Lift the birational map to a surface automorphism.

Picard group

Let M be a complex manifold.

A *divisor* is a linear combination $D = \sum m_j D_j$, where D_j is a hypersurface in M.

Divisors D' and D'' are *linearly equivalent* if $D' - D''$ is a divisor of a rational function.

That is, $D' - D''$ is the zero set minus pole set of some rational function.

The *Picard group* $\mathrm{Pic}(M)$ of M is the set of divisors modulo linear equivalence.

$$
\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & \mathrm{Pic}(S) & \xrightarrow{c_1} & H^2(S,\mathbb{Z}) & \longrightarrow & 0, \\
 & & & & \downarrow r & & \downarrow \iota^* & & \\
0 & \to & \mathrm{Pic}_0(X) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{\mathrm{deg}} & H^2(X,\mathbb{Z}) & \longrightarrow & 0.
\end{array}
$$

X : cuspidal cubic, three lines through a point, quadric with a tangent line

$$\mathrm{Pic}_0(X) \simeq \mathbb{C},$$

X : nodal cubic (one, two, or three modes)

$$\mathrm{Pic}_0(X) \simeq \mathbb{C}/\mathbb{Z},$$

X : elliptic cubic

$$\mathrm{Pic}_0(X) \simeq \mathbb{C}/\Lambda.$$

3. Rational curves

Genus formula

Let $Y \subset S$ denote the strict transform of $X \subset \mathbb{P}^2$
The canonical class

$$K_S = \phi(k_n) \in H^2(S, \mathbb{Z})$$

is given by diviser $-Y$. And $Y \subset S$ is the anticanonical curve.

(Hirzebruch-Riemann-Roch) (simplest case)
The genus of a rational curve $C \subset S$ is given by *genus formula*

$$g(C) = \frac{1}{2} C \cdot (C + K_S) + 1.$$

$g(C)$ is called the *arithmetic genus* of C. If C is a smooth curve, it is the genus of C as a Riemann surface.

$$g(C) = \frac{1}{2} C \cdot (C + K_S) + 1.$$

If $C \cap Y = \phi$, then $C \cdot K_S = 0$.

If $C \cdot K_S = 0$ and $C^2 = -2$, then $g(C) = 0$.

If $C \cdot K_S = 0$ and $C^2 = 0$, then $g(C) = 1$.

Rational curves

Suppose that the projection $\kappa : X_{\mathrm{reg}} \to \mathrm{Pic}_0(X)$ is chosen so that $\sum_{V \subset X} \deg(V) \cdot 0_V$ is the diviser cut out by a line in $\mathbb{P}^2$. Following is a classical theorem.

THEOREM 3d (not necessarily distinct) points $p_1, \cdots, p_{3d} \in X_{\mathrm{reg}}$ comprise the intersection of X with a curve of degree d if and only if

each irreducible $V \subset X$ contains $d \deg(V)$ of the points;

and $\sum p_j \sim 0$.

Degree map

The pullback $\iota^*: H^2(S,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ by the embedding map $\iota: X \hookrightarrow S$ can be decomposed as

$$H^2(S,\mathbb{Z}) \overset{c_1^{-1}}{\simeq} \mathrm{Pic}(S) \xrightarrow{r} \mathrm{Pic}(Y) \simeq \mathrm{Pic}(X) \xrightarrow{\deg} H^2(X,\mathbb{Z}).$$

Let $V_1, \cdots, V_k$ be the irreducible components of the cubic curve $X \subset \mathbb{P}^2$.

Suppose $C \subset S$ be the strict transform in S of rational curve $\pi(C) \subset \mathbb{P}^2$ of degree d. Then

$$\pi(C) \cdot V_i = d \deg(V_i), \qquad i = 1, \cdots, k,$$

and

$$\iota^*(H) = \sum \deg(V_i) V_i.$$

Summing up, if C is the cohomology class of a rational curve, then $C \in \mathrm{Ker}(\iota^*)$.

Picard sum

Next let $\rho_0 : \mathrm{Ker}(\iota^*) \to \mathrm{Pic}_0(X)$ be defined by:

$$\mathrm{Ker}(\iota^*) \hookrightarrow H^2(S, \mathbb{Z}) \overset{c_1^{-1}}{\simeq} \mathrm{Pic}(S) \overset{r}{\longrightarrow} \mathrm{Pic}_0(X).$$

$\rho_0(C)$ gives the sum of Picard parameters of points in $\pi(C) \cap X$, (with multiplicities).

Hence $\rho_0(C) \sim 0 \in \mathrm{Pic}_0(X)$, if C is a rational curve.

PROPOSITION. If $C \in H^2(S, \mathbb{Z})$ satisfies :

$$C \cdot K_S = 0, \quad C^2 = -2, \quad C \cdot e_0 \geq 0,$$

$$\iota^*(C) = 0 \in H^2(X, \mathbb{Z}), \quad \rho_0(C) \sim 0 \in \mathrm{Pic}_0(X),$$

then $C \in H^2(S, \mathbb{Z})$ is represented by a curve of arithmetic genus 0.

PROPOSITION If $(F^p)^*(C) = C$, with C as in the preceding proposition, then C is a periodic curve of period p.

3. Elliptic fibration

Class of a generic fiber

Suppose surface automorphism $F : S \to S$ satisfies

$$\lambda_1 = 1, \quad \{||(F^n)^*||\} \quad \text{is unbounded.}$$

PROPOSITION Up to positive multiple, there is a unique nef class $\theta \in H^{1,1}(S)$ such that $F^*\theta = \theta$. Moreover $\theta^2 = 0$, $\theta \cdot K_S = 0$, and can assume that $\theta \in H^2(S, \mathbb{Z})$.

REM. The invariant class is obtained by

$$\theta = \lim_{n\to\infty} \frac{(F^n)^*\omega}{||(F^n)^*\omega||}.$$

for some Kähler class ω.

construction of elliptic fibration

1. Choose an orbit data $(n_1, n_2, n_3), \sigma$.
   (such that step 3 can be executed.)
2. Compute the characteristic polynomial $P(\lambda)$ of F^*.
   ($P(\lambda)$ can be computed only from the orbit data.)
3. Check that all zeros of $P(\lambda)$ are roots of unity, and that F^* has a Jordan block of size 3 for eigenvalue 1.
4. Choose appropriate cubic curve and parameters and compute the Picard coordinates of blowup points.
5. Construct a marked blowup and the automorphism $F : S \to S$.
6. Find the class of a generic fiber θ.
7. Find periodic classes of arithmetic genus 0.
8. Find polynomials for cycles of periodic curves.
9. Verify that $\sum C_i = \theta$ for cycles of periodic curves.
10. Construct a rational function $\varphi : S \to \mathbb{P}^1$.

Characteristic polynomial

Orbit data determines the characteristic polynomial $P(\lambda)$ of $F^* : H^2(S,\mathbb{Z}) \to H^2(S,\mathbb{Z})$.

Bedford and Kim [BK1] have computed explicitly for any orbit data n_1, n_2, n_3, σ.

$$P(\lambda) = \lambda^{1+\Sigma n_j} p(\frac{1}{\lambda}) + (-1)^{\mathrm{ord}\sigma} p(\lambda),$$

where

$$p(\lambda) = 1 - 2\lambda + \sum_{j=\sigma_j} \lambda^{1+n_j} + \sum_{j\neq\sigma_j} \lambda^{n_j}(1-\lambda).$$

Conditions for the existence of quadratic transformation is given by :

THEOREM (Diller[2011]) Let $\tau : X \to X$ be an automorphism with multiplier a and translation b_V, $V \subset X \subset \mathbb{P}^2$. Given points $p_1^+, p_2^+, p_3^+ \in X$, there exists a quadratic transformation $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ properly fixing X with $f|_X = \tau$ and $I(f) = \{p_1^+, p_2^+, p_3^+\}$ if and only if

(1) For each irreducible $V \subset X$, we have

$\sharp\{j : p_j^+ \in V \cap X_{\mathrm{reg}}\} = 2\deg(V) - \deg(\tau(V))$ and

$\sharp\{j : p_j^- \in V\} = 2\deg(V) - \deg(\tau^{-1}(V))$.

In particular $I(f) \subset X_{\mathrm{reg}}$.

(2) $\sum p_j^+ \sim a^{-1}\sum_{V\subset X} \deg(V) b_V \not\sim 0$.

The transformation f is unique when it exists and the points of indeterminacy $p_j^- \in I(f^{-1})$ then satisfy the following.

(3) Given $j \in \{1,2,3\}$, let L be the line defined by the two points $I(f) \setminus \{p_j^+\}$, and let $V \subset X$ be the irreducible component containing the third point in $X \cap L$.
Then $p_j^- \in \tau(V)$.

(4) For each $j \in \{1,2,3\}$, we have
$p_j^- - ap_j^+ \sim b_j - \sum_{V \subset X} \deg(V) b_V$, where b_j is the translation for the component containing p_j^+.

Sketch of the proof of THEOREM A

Recall

THEOREM A For $d = 3, 4, 5, 6$, there exist surface automorphisms preserving elliptic fibration defined by rational function of degree d.

Proof is done by finding examples. Here, we construct an example for $d = 5$.

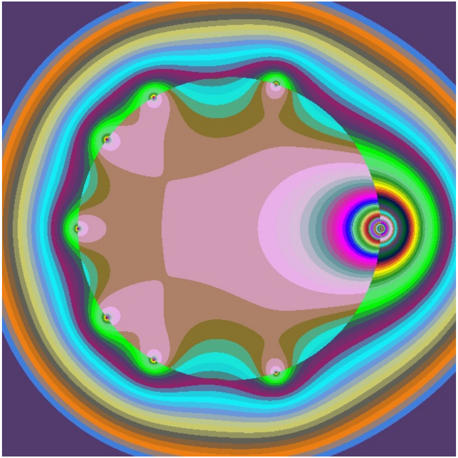
Choose orbit data $(n_1, n_2, n_3) = (1, 4, 5)$ with transposition $\sigma = (1, 2)$.

The characteristic polynomial for this orbit data is :

$$P(\lambda) = (\lambda - 1)(\lambda^2 - 1)(\lambda^3 - 1)(\lambda^5 - 1).$$

Factorization of $P(\lambda)$

$$P(\lambda) = \lambda^{11} - \lambda^{10} - \lambda^9 + \lambda^7 + \lambda^4 - \lambda^2 - \lambda + 1.$$



Choose a quadric, $\mathcal{Q} = \{xy = 1\}$, and a tangent line, $\mathcal{L} = \{x = 0\}$, as our cubic curve $X = \mathcal{Q} \cup \mathcal{L}$.

Let the Picard parametrization be defined as follows.

$$p_{\mathcal{Q}}(t) = (t^{-1}, t), \quad p_{\mathcal{L}}(t) = (0, -t).$$

Let $\beta = b_{\mathcal{Q}}$ and $\gamma = b_{\mathcal{L}}$ be the translation in the quadric and the tangent line, respectively. We set the multiplier $a = 1$.

Specify the inner dynamics $\tau : X \to X$ is as follows.

$$\tau(\mathcal{Q}) = \tau(\mathcal{Q}), \quad \tau(\mathcal{L}) = \tau(\mathcal{L}),$$

$$\tau(z) \sim z + \beta, \qquad (z \in \mathcal{Q}),$$

$$\tau(z) \sim z + \gamma, \qquad (z \in \mathcal{L}).$$

Compute the Picard coordinates of the indeterminate points $p_1^+, p_2^+ \in \mathcal{Q}, p_3^+ \in \mathcal{L}$, and $p_1^-, p_2^- \in \mathcal{Q}, p_3^- \in \mathcal{L}$ in $\mathrm{Pic}_0(X)$, which satisfy the following conditions.

$$p_2^+ \sim p_1^- + (n_1 - 1)\beta, \quad p_1^+ \sim p_2^- + (n_2 - 1)\beta, \quad p_3^+ \sim p_3^- + (n_3 - 1)\gamma,$$

$$p_1^- \sim p_1^+ - \beta - \gamma, \quad p_2^- \sim p_2^+ - \beta - \gamma, \quad p_3^- \sim p_3^+ - 2\beta,$$

and

$$p_1^+ + p_2^+ + p_3^+ \sim 2\beta + \gamma \not\sim 0.$$

Here, $\sim$ stands for equality in $\mathrm{Pic}_0(X) \cong \mathbb{C}$.

For this system of equations to have a solution, it is necessary that:

$$\frac{4}{n_1 + n_2} + \frac{1}{n_3} = 1.$$

By choosing a parameter $s \in \mathbb{C} \setminus \{0\}$,

$$\beta \sim \frac{2s}{n_1 + n_2}, \quad \gamma \sim \frac{s}{n_3}.$$

This parameter s can be normalized to 1 by a change of Picard coordinates in $\mathrm{Pic}_0(X)$.

In our case, $\beta \sim \frac{2}{5}$ in the quadric, and $\gamma \sim \frac{1}{5}$ in the tangent line.

And by setting $p_1^+ \sim \alpha$, with $\alpha \in \mathbb{C}$, we have

$$p_1^- = p_2^+ \sim \alpha - 0.6, \quad p_2^- \sim \alpha - 1.2,$$

$$p_3^+ \sim 1.6 - 2\alpha, \quad p_3^- \sim 0.8 - 2\alpha.$$

This parameter α can be chosen as desired by a change of Picard coordinates.

Construct a surface automorphism, $F : S \to S$, with these data and

$$\tau : X \to X, \quad \tau(\mathcal{Q}) = \tau(\mathcal{Q}), \quad \tau(\mathcal{L}) = \tau(\mathcal{L}),$$

$$\tau(z) \sim z + \beta, \qquad (z \in \mathcal{Q}),$$

$$\tau(z) \sim z + \gamma, \qquad (z \in \mathcal{L}).$$

Birational map f

Let $a_1 = p_1^+$, $a_2 = p_2^+$, $a_3 = p_3^+$, $b = \beta$, $c = \gamma$. Then the quadratic transformation $f : (x, y) \mapsto (X, Y)$ is given by the following.

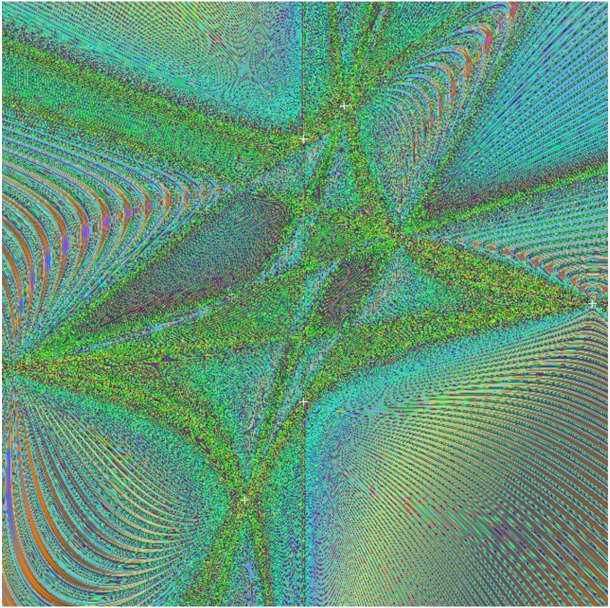
$$X = \frac{x(a_1 a_2 x + y - a_1 - a_2)}{(\beta x + 1)(a_1 a_2 x + y - a_1 - a_2) - \nu_1(xy - 1)},$$

$$Y = \frac{(\beta^2 x + 2\beta + y)(a_1 a_2 x + y - a_1 - a_2) + \nu_1(a_3 - 2\beta)(xy - 1)}{(\beta x + 1)(a_1 a_2 x + y - a_1 - a_2) - \nu_1(xy - 1)}$$

where

$$\nu_1 = a_1 + a_2 + a_3 = 2\beta + \gamma.$$

Real slice



Now, let $A_1 \in H^2(S,\mathbb{Z})$ denote the cohomology class of the exceptional fiber $[\pi^{-1}(p_2^+)]$. Let $B_i = [\pi^{-1}(f^{1-i}(p_1^+))]$, $i = 1,2,3,4$, and $C_i = [\pi^{-1}(f^{1-i}(p_3^+))]$, $i = 1,2,3,4,5$.

Let $H \in H^2(S,\mathbb{Z})$ denote the class of a generic line $[\pi^{-1}(L)]$.

A basis of $H^2(S,\mathbb{Z})$ is given by classes

$$H, A_1, B_1, B_2, B_3, B_4, C_1, C_2, C_3, C_4, C_5.$$

Automorphism $F^* : H^2(S,\mathbb{Z}) \to H^2(S,\mathbb{Z})$ acts as follows.

$$H \mapsto 2H - A_1 - B_1 - C_1,$$

$$A_1 \mapsto H - A_1 - C_1,$$

$$B_1 \mapsto B_2 \mapsto B_3 \mapsto B_4 \mapsto H - B_1 - C_1,$$

$$C_1 \mapsto C_2 \mapsto C_3 \mapsto C_4 \mapsto C_5 \mapsto H - A_1 - B_1.$$

The characteristic polynomial of F^* is

$$P(\lambda) = (\lambda - 1)(\lambda^2 - 1)(\lambda^3 - 1)(\lambda^5 - 1).$$

Eigenvalue $\lambda = 1$ has multiplicity 4.

The eigenspace of F^* for eigenvalue 1 is two-dimensional and spanned by

$$\mathcal{Q} = 2H - A_1 - B_1 - B_2 - B_3 - B_4,$$

$$\mathcal{L} = H - C_1 - C_2 - C_3 - C_4 - C_5.$$

The anticanonical class is $-K_S = \mathcal{Q} + \mathcal{L}$.

The eigenvector θ, which must be the eigenvector in the Jordan block of size 3, is the unique class (up to scalar) with $\theta^2 = 0$.

$$\theta = 2\mathcal{Q} + \mathcal{L}.$$

Set $\theta = q\mathcal{Q} + r\mathcal{L}$. Then as $\mathcal{Q}^2 = -1$, $\mathcal{L}^2 = -4$, $\mathcal{Q} \cdot \mathcal{L} = 2$,

$$\theta^2 = (q\mathcal{Q} + r\mathcal{L})^2 = -(q - 2r)^2.$$

Setting $q = 2, r = 1$, we have

$$\theta = 2\mathcal{Q}+\mathcal{L} = 5H-2A_1-2B_1-2B_2-2B_3-2B_4-C_1-C_2-C_3-C_4-C_5.$$

Periodic curve of period 2

The eigenspace of F^* for eigenvalue -1 is one-dimensional and spanned by

$$W = H - 2A_1 - C_1 + C_2 - C_3 + C_4 - C_5.$$

Periodic vector of period 2 is a linear combination of $\mathcal{Q}, \mathcal{L}$, and W.

Especially, the vectors sum up to θ, as they form a singular fiber of the elliptic fibration.

So we try to find such vector in the subspace spanned by θ and W.

$$\theta = 5H - 2A_1 - 2B_1 - 2B_2 - 2B_3 - 2B_4 - C_1 - C_2 - C_3 - C_4 - C_5.$$

$$W = H - 2A_1 - C_1 + C_2 - C_3 + C_4 - C_5.$$

Considering parities and the positivity, we take vectors

$$U = \frac{1}{2}(\theta + W) = 3H - 2A_1 - B_1 - B_2 - B_3 - B_4 - C_1 - C_3 - C_5,$$

$$V = \frac{1}{2}(\theta - W) = 2H - B_1 - B_2 - B_3 - B_4 - C_2 - C_4.$$

We see that $U^2 = -2$, $V^2 = -2$, and $U \cdot V = 2$.

Periodic positive root

Among the linear combinations of U and V, U and V are the only positive root of self intersection -2. For,

$$(uU + vV)^2 = -2(u - v)^2, \quad 3u + 2v \geq 0.$$

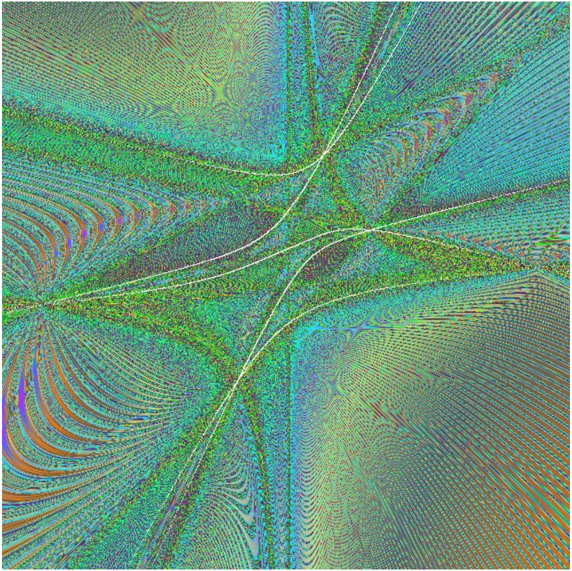
We find

$$F^*(U) = V, \quad F^*(V) = U,$$
$$\mathcal{Q} \cdot U = 0, \quad \mathcal{L} \cdot U = 0,$$
$$\mathcal{Q} \cdot V = 0, \quad \mathcal{L} \cdot V = 0,$$
$$\rho_0(U) = 0, \quad \rho_0(V) = 0,$$

and

$$U + V = \theta.$$

Periodic curve of period 2



Picard coordinates

Recall
$A_1 = [\pi^{-1}(p_2^+)]$.
$B_i = [\pi^{-1}(f^{1-i}(p_1^+))]$, $i = 1, 2, 3, 4$,
$C_i = [\pi^{-1}(f^{1-i}(p_3^+))]$, $i = 1, 2, 3, 4, 5$.

and

$$p_1^+ \sim \alpha,$$

$$p_1^- = p_2^+ \sim \alpha - 0.6, \quad p_2^- \sim \alpha - 1.2,$$

$$p_3^+ \sim 1.6 - 2\alpha, \quad p_3^- \sim 0.8 - 2\alpha.$$

Picard coordinates are as follows.

$$\rho_0(A_1) = \alpha - 0.6,$$

$$\rho_0(B_i) = \alpha + 0.4(1 - i), \quad i = 1, 2, 3, 4,$$

$$\rho_0(C_i) = 1.6 - 2\alpha + 0.2(1 - i). \quad i = 1, 2, 3, 4, 5.$$

$$\rho_0(H) = 0.$$

Lines

In our case, positive roots representable by a line are

$$H - A_1 - B_2 - C_4, \quad H - B_1 - B_3 - C_5, \quad H - B_2 - B_4 - C_1$$

$$H - A_1 - B_3 - C_2, \quad H - B_1 - B_4 - C_3,$$

and

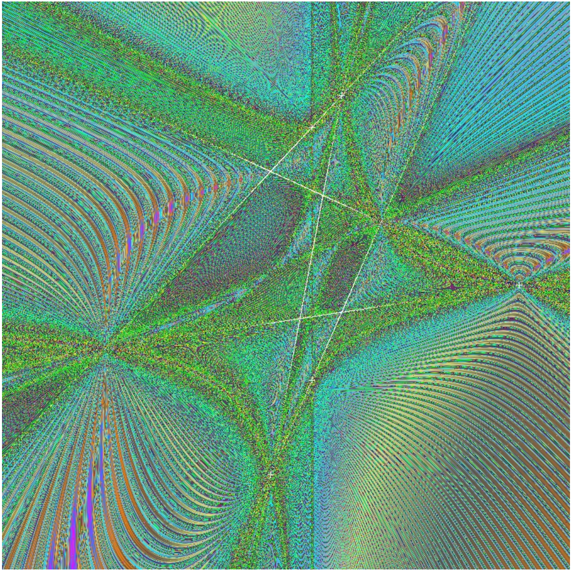
$$H - B_2 - B_3 - C_3.$$

Periodic curve of period 5

And periodic curve of period 5 is found as follows.

$$
\begin{aligned}
& H - B_2 - B_4 - C_1 \\
\mapsto\ & H - A_1 - B_3 - C_2 \\
\mapsto\ & H - B_1 - B_4 - C_3 \\
\mapsto\ & H - A_1 - B_2 - C_4 \\
\mapsto\ & H - B_1 - B_3 - C_5 \\
\mapsto\ & H - B_2 - B_4 - C_1
\end{aligned}
$$

Periodic curve of period 5



Periodic curve of period 3

Periodic curve of period 3 is found as follows.

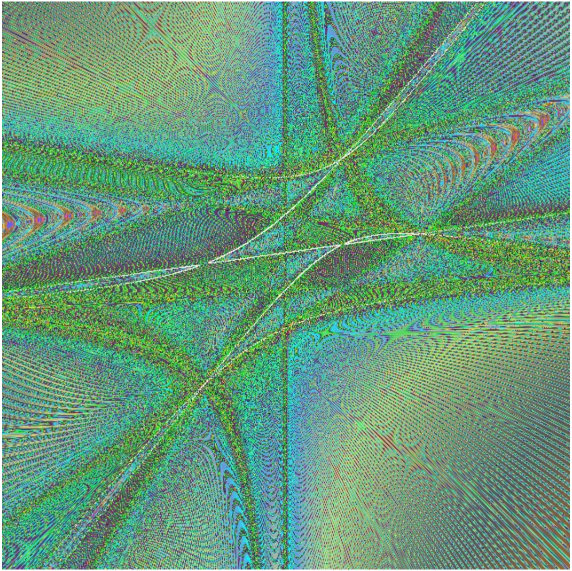
$$H - B_2 - B_3 - C_3$$

$$\mapsto 2H - A_1 - B_1 - B_3 - B_4 - C_1 - C_4$$

$$\mapsto 2H - A_1 - B_1 - B_2 - B_4 - C_2 - C_5$$

$$\mapsto H - B_2 - B_3 - C_3.$$

Periodic curve of period 3



Invariant rational function

Let P_5 denote the polynomial of degree 5, which defines the quintic curve consisting of the five lines of period 5.

Since these five lines are mapped to the five lines,

$$P_5 \circ f = Q_5 \cdot P_5,$$

where Q_5 is a product of equations of exceptional lines, with multiplicities counted according to the multiplicities of P_5 at the blowdown points, determined from θ. Q_5 is a polynomial of degree 5.

Similarly, for periodic curve of period 2 or 3, we have

$$P_2 \circ f = Q_2 \cdot P_2, \qquad P_3 \circ f = Q_3 \cdot P_3.$$

Here, Q_2, Q_3, Q_5 have the same diviser. Hemce, $\varphi = P_5/P_3$ is invariant under f. This invariant function lifts to $\varphi \circ \pi : S \to \mathbb{P}^1$.

Theorems

We gave an example of a surface automorphism with an invariant elliptic fibration defined by a rational function of degree 5.

For other cases of d, we have similar examples.

Finding periodic curves is a good exercise, and left to the reader.

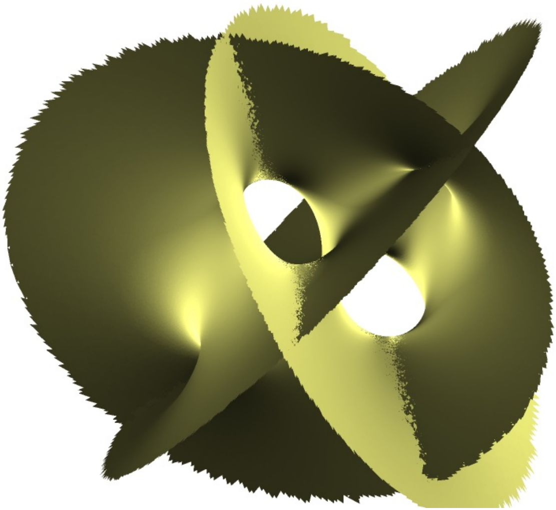
For $d = 3$, most cases with $n_1 + n_2 + n_3 = 9$.

For $d = 4$, orbit data $(1, 7, 2)$, $\sigma = (1, 2)$ transposition, with invariant quadric and an invariant tangent line as cubic curve.

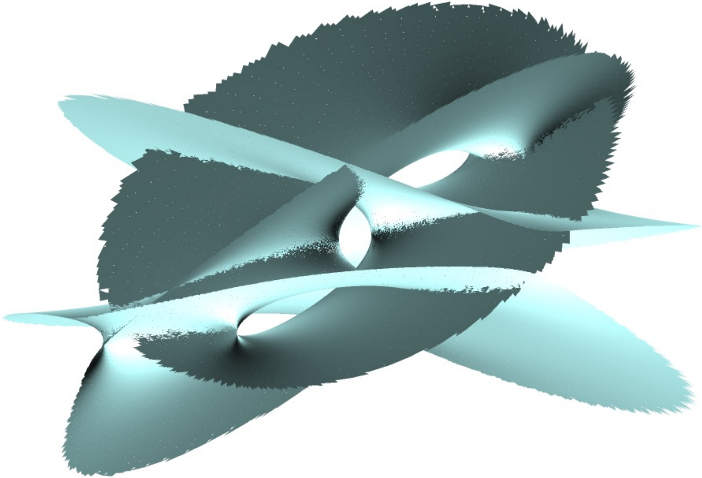
For $d = 6$, orbit data $(2, 3, 6)$, $\sigma = id.$, with three invariant lines passing through a point.

Our example furnishes the proof of Theorem B.

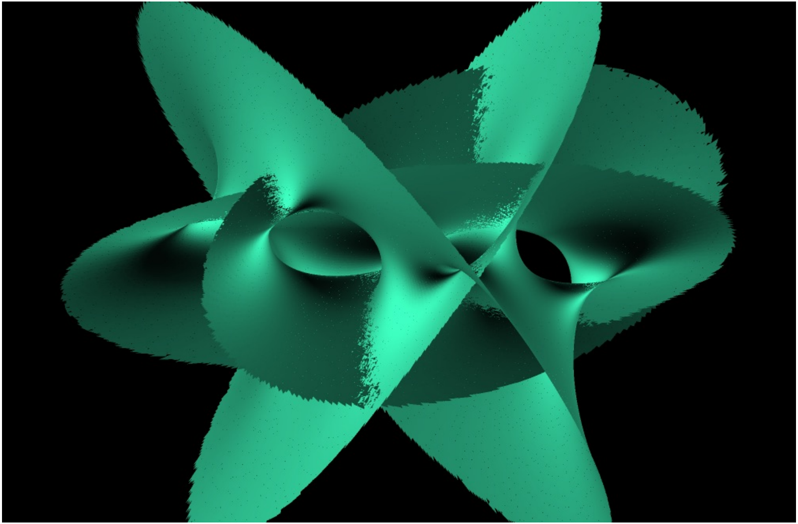
Cuspidal cubic, $(1, 5, 3)$, $\sigma = tr(1, 2)$



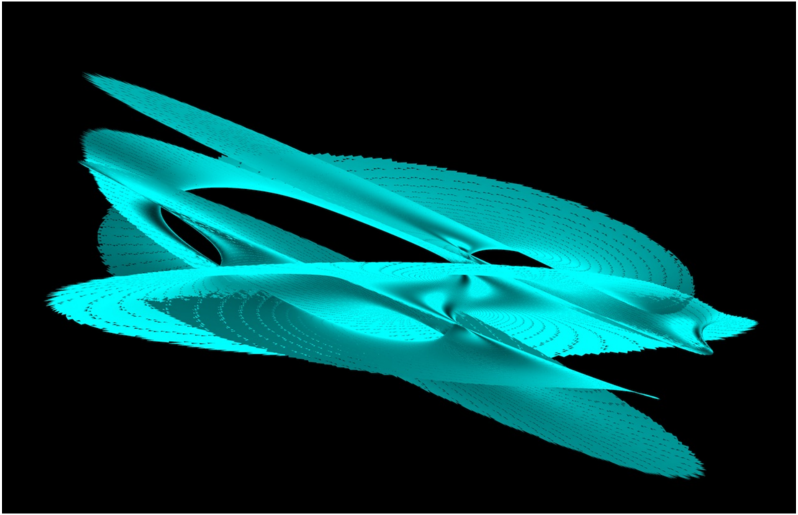
Quadric and a tangent line, $(1, 7, 2)$, $\sigma = tr(1, 2)$.



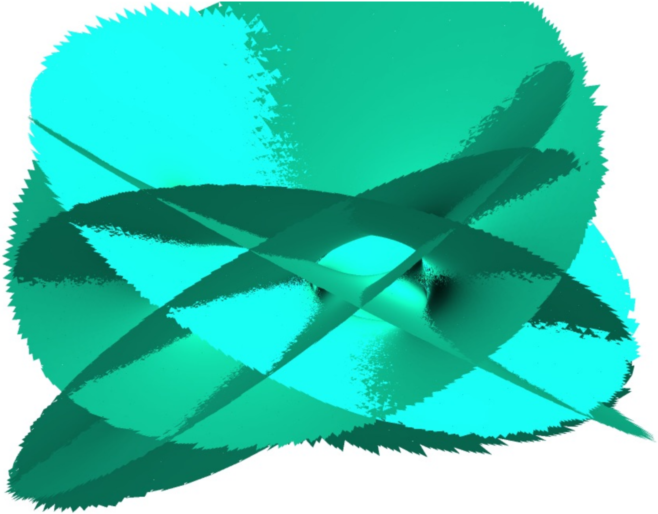
Quadric and a tangent line, $(4, 4, 2)$, $\sigma = id$.



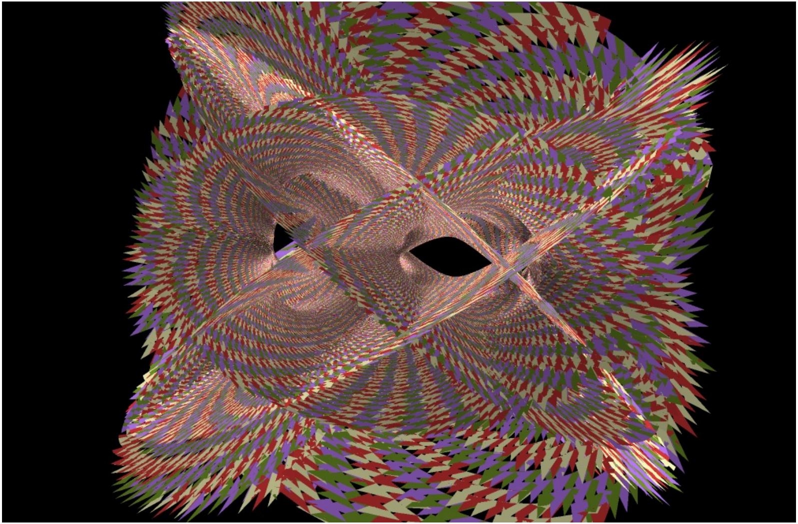
Quadric and a tangent line, $(1,4,5)$, $\sigma = tr(1,2)$



Three lines through a point, $(2, 3, 6)$,$\sigma = id$.



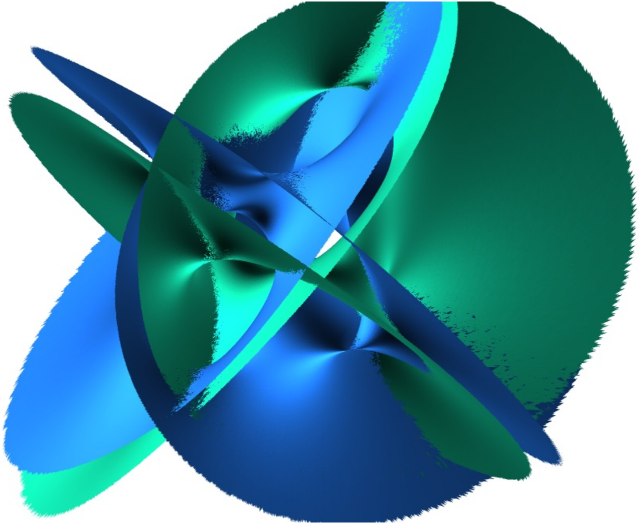
Three lines through a point, id, $(2,3,6)$, $\sigma = id$.



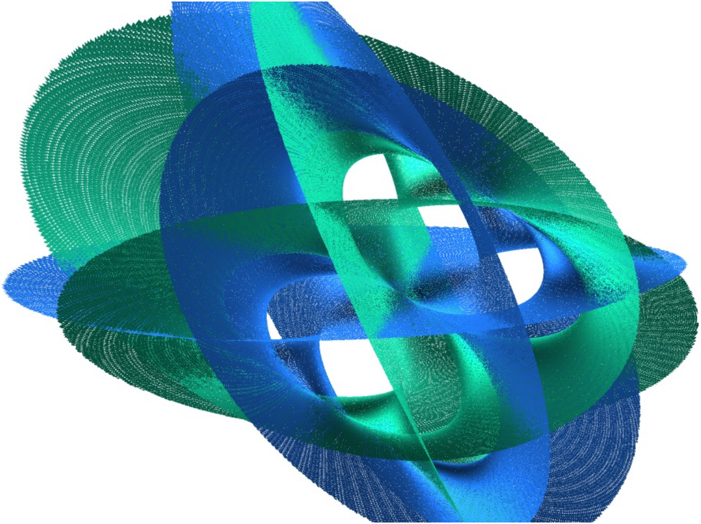
And OBSERVATION C...

Case of $q = 1$ is classical.
A case for $q = 5$, orbit data $(1, 1, 7), cy.$ is proved in [BK1].

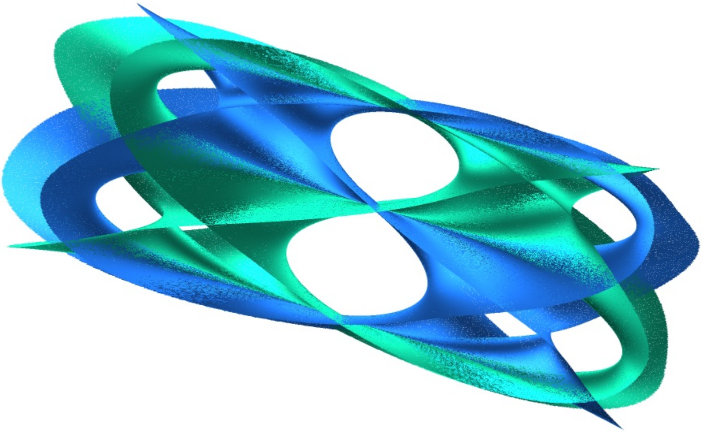
Nodal cubic with two components, id., $(1,5,3)$, $\sigma = tr(1,2)$, $m = (1,0,2)$



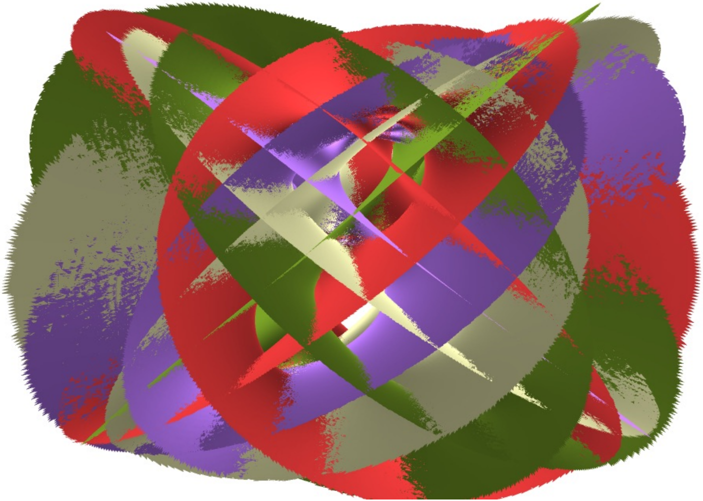
Three lines through a point, transposition, $(1,5,3)$, $\sigma = tr(1,2)$



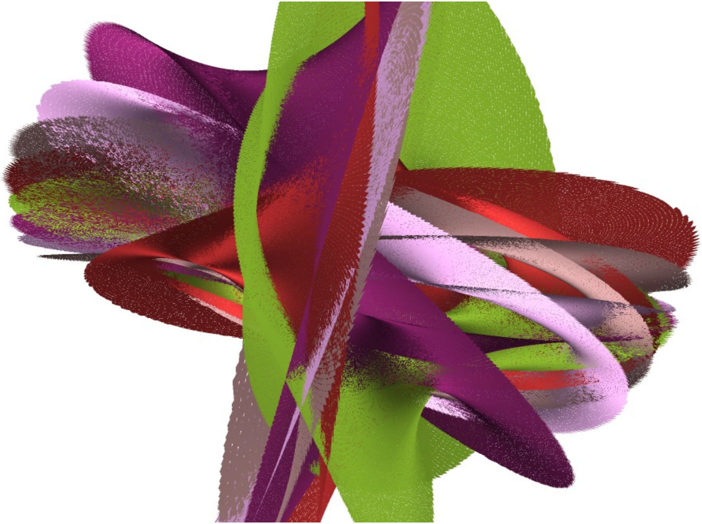
Three lines through a point, transposition, $(3, 3, 3)$, $\sigma = tr(1, 2)$



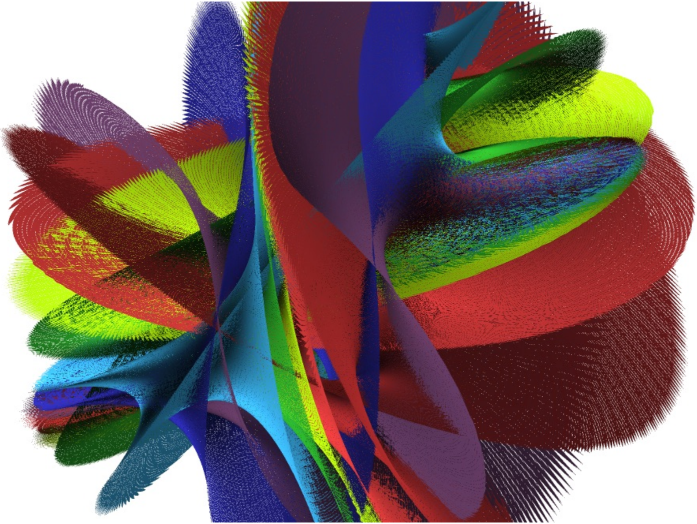
Three lines through a point, cyclic, $d = i$, $(1, 5, 3)$, $\sigma = tr$.



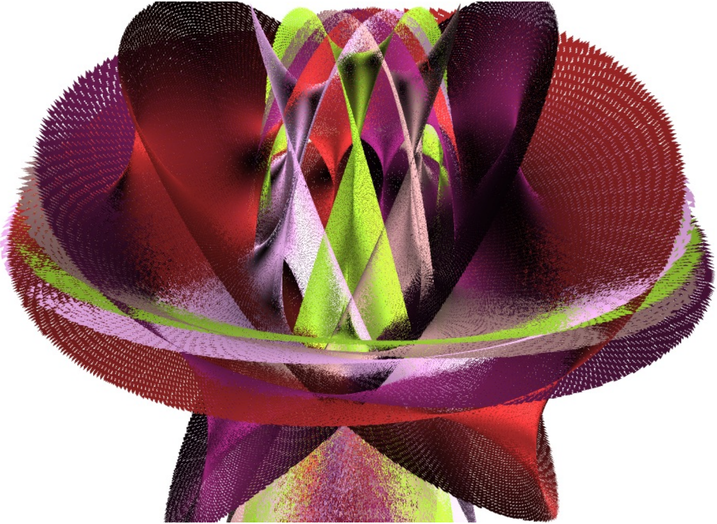
Quadric and a tangent line, id, $d = e^{4\pi i/5}$,(2, 4, 3), $\sigma = tr$.



Quadric and tangent line, tr., $d = e^{2\pi i/7}$,(1, 3, 5), $\sigma = cy$.



Quadric and tangent line, tr, $d = e^{2\pi i/5}$, $(1, 1, 7)$,cy.



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