Julia Sets with Polyhedral Symmetries

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Julia sets with polyhedral symmetries are constructed from equivariant rational mappings of the Riemann sphere.

0. Introduction

Let \( \mathbb{C} = \mathbb{C} \cup \{\infty\} \) denote the Riemann sphere. Let us identify the Riemann sphere with the unit sphere \( S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\} \) by the stereographic projection \( z = (x_1 + ix_2)/(1 - x_3) \). Regular polyhedral groups act on the unit sphere and they induce groups of Möbius transformations of the Riemann sphere.

Let \( \Gamma \) be a group of such Möbius transformations. Rational function \( f: \mathbb{C} \to \mathbb{C} \) is said to be \( \Gamma \)-equivariant if for all \( \gamma \in \Gamma \),

\[
\gamma \circ f = f \circ \gamma
\]

holds. If \( f \) is \( \Gamma \)-equivariant, then its Julia set \( J_f \) is invariant for all \( \gamma \in \Gamma \), i.e.,

\[
\gamma(J_f) = J_f.
\]

In this note, we consider \( \Gamma \)-equivariant rational functions with super-attractive cycles. In the simplest case, the obtained function has a Julia set which is locally homeomorphic to the Sierpinski’s gasket. Other functions have generalized Sierpinski’s gaskets as their Julia sets. The construction of \( \Gamma \)-equivariant mappings is classical and some of them were known to Klein[?]. P.Doyle and C.McMullen[?] has produced a picture of one of these functions.

1. Invariant Functions and Equivariant Mappings

P.Doyle and C.McMullen[?] gave a method to produce all rational maps with given symmetries. In this section we reproduce their method. Let \( E = \mathbb{C}^2 \) be a 2-dimensional complex vector space. A point \( z \in \mathbb{C} \) corresponds to a complex line \( l = \{(x, y) \in E \mid z = x/y\} \). Let \( E^* \) denote the vector space of linear functionals of \( E \). \( E^* \) is isomorphic to \( E \) and we use coordinates \((\xi, \eta)\) on \( E^* \) such that if \( v = (\xi, \eta) \) and \( p = (x, y) \) then

\[
v(p) = \xi x + \eta y.
\]

Line \( l = \{(x, y) \in E \mid z = x/y\} \) corresponds to a line in the dual space \( E^* \) given by

\[
l^* = \{((\xi, \eta)) \in E^* \mid z = \frac{-\eta}{\xi}\}.
\]
Let $f : \mathbb{C} \to \mathbb{C}$ be a rational function of degree $d$ defined by
\[ f(z) = \frac{P(z)}{Q(z)}, \]
where $P(z)$ and $Q(z)$ are polynomials without common factor. It defines, up to a scale factor, a homogeneous polynomial map $X : E \to E$ by
\[ X(x, y) = (\tilde{P}(x, y), \tilde{Q}(x, y)), \]
where
\[ \tilde{P}(x, y) = y^d \frac{x}{y} \quad \text{and} \quad \tilde{Q}(x, y) = y^d \frac{Q(x}{y}. \]

As $E$ is a vector space, $X$ can also be regarded as a homogeneous vectorfield on $E$. Let $\Gamma \subset \text{Aut}(\mathbb{C})$ be a finite group of Möbius transformations. Let $\tilde{\Gamma} \subset \text{SL}(E)$ denote its pre-image in the group of linear transformations of determinant 1.

A vector field $X$ on $E$ is said to be invariant if there exists a character $\chi : \tilde{\Gamma} \to U(1)$, a group homomorphism of $\tilde{\Gamma}$ into the group of complex numbers with modulus 1, such that
\[ X \circ \tilde{\gamma}^{-1} = \chi(\tilde{\gamma})X \]
for all $\tilde{\gamma} \in \tilde{\Gamma}$. If the character $\chi$ is trivial, then $X$ is said to be absolutely $\tilde{\Gamma}$-invariant.

Proposition 1.1. Homogeneous vector field $X$ on $E$ is $\tilde{\Gamma}$-invariant if and only if its corresponding rational function $f : \mathbb{C} \to \mathbb{C}$ satisfies
\[ f \circ \gamma = f \quad \text{for all } \gamma \in \Gamma. \]

By an isomorphism between $E$ and $E^*$:
\[ E \ni (\xi, \eta) \longleftrightarrow -\eta dx + \xi dy \in E^*, \]
the vectorfield $X = (\tilde{P}(x, y), \tilde{Q}(x, y))$ corresponds to a 1-form
\[ \theta = -\tilde{Q}(x, y)dx + \tilde{P}(x, y)dy. \]
Hence, rational function $f(z)$ corresponds to a homogeneous 1-form $\theta : E \to E^*$ up to a scale factor. To the identity map $z \mapsto z$ of the Riemann sphere corresponds the “identity” vectorfield $E \to E$
\[ (x, y) \mapsto (x, y), \]
more precisely,
\[ (x, y) \mapsto x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}. \]
And the 1-form $\lambda : E \to E^*$ corresponding to this vector field is given by
\[ \lambda = -ydx + xdy. \]
A vector field $X$ on $E$ is said to be $\tilde{\Gamma}$-equivariant if there exists a character
\[ \chi : \tilde{\Gamma} \to U(1) \]
such that

\[ \tilde{\gamma} \circ X \circ \tilde{\gamma}^{-1} = \chi(\tilde{\gamma})X \]

holds for all \( \tilde{\gamma} \in \tilde{\Gamma} \). If the character \( \chi \) is trivial, \( X \) is said to be absolutely \( \tilde{\Gamma} \)-equivariant.

Proposition 1.2. Vector field \( X \) is \( \tilde{\Gamma} \)-equivariant if and only if its corresponding rational function \( f : \tilde{\mathbb{C}} \to \tilde{\mathbb{C}} \) is \( \Gamma \)-equivariant, i.e.,

\[ \gamma \circ f = f \circ \gamma \quad \text{for all } \gamma \in \Gamma. \]

A homogeneous 1-form \( \theta \) is said to be \( \tilde{\Gamma} \)-equivariant if there exists a character \( \chi : \tilde{\Gamma} \to U(1) \) such that

\[ \theta \circ \tilde{\gamma}^{-1} = \chi(\tilde{\gamma})\tilde{\gamma}^*\theta \quad \text{for all } \tilde{\gamma} \in \tilde{\Gamma}, \]

where \( \tilde{\gamma}^*\theta \) is a 1-form \( E \to E^* \) defined by

\[ (\tilde{\gamma}^*\theta(x,y))(\xi, \eta) = \theta(x,y)(\tilde{\gamma}(\xi, \eta)). \]

Homogeneous 1-form \( \theta \) is said to be absolutely \( \tilde{\Gamma} \)-equivariant if the character \( \chi : \tilde{\Gamma} \to U(1) \) is trivial.

Proposition 1.3. Vectorfield \( X = E \to E \) is \( \tilde{\Gamma} \)-equivariant if and only if its corresponding 1-form \( \theta : E \to E^* \) is equivariant.

Proof. Let \( \tilde{\gamma} \in SL(E) \). Then

\[ \tilde{\gamma} \left( \begin{array}{c} \tilde{P}(\tilde{\gamma}^{-1}(x,y)) \\ \tilde{Q}(\tilde{\gamma}^{-1}(x,y)) \end{array} \right) = \chi(\tilde{\gamma}) \left( \begin{array}{c} \tilde{P}(x,y) \\ \tilde{Q}(x,y) \end{array} \right) \]

implies

\[ (-\tilde{Q}(\tilde{\gamma}^{-1}(x,y)), \tilde{P}(\tilde{\gamma}^{-1}(x,y))) = \chi(\tilde{\gamma})(-\tilde{Q}(x,y), \tilde{P}(x,y))\tilde{\gamma}. \]

Note that 1-form \( \lambda = -ydx + xdy \) is absolutely \( SL(E) \)-equivariant. Let \( \omega = dx \wedge dy \) be the volume form. Then \( d\lambda = 2\omega \). Obviously, \( \omega \) is also absolutely \( SL(E) \)-equivariant.

Theorem 1.4. (P.Doylae and C.McMullen)

A homogeneous 1-form \( \theta \) is \( \tilde{\Gamma} \)-equivariant if and only if

\[ \theta = f(x,y)\lambda + dg(x,y) \]

where \( f \) and \( g \) are \( \tilde{\Gamma} \)-invariant homogeneous polynomials with the same character \( \chi : \tilde{\Gamma} \to U(1) \) and \( \deg(g) = \deg(f) + 2 \).

Proof. Suppose \( \theta \) is \( \tilde{\Gamma} \)-equivariant. Then \( d\theta = h(x,y)\omega \) is a \( \tilde{\Gamma} \)-equivariant 2-form. Since \( \omega \) is absolutely equivariant, \( h(x,y) \) is a homogeneous polynomial which is \( \tilde{\Gamma} \)-invariant with the same character as \( \theta \). Let \( f(x,y) = h(x,y)/(\deg(h) + 2) \). Then \( d(f\lambda) = h(x,y)\omega \) holds and \( \theta - f(x,y)\lambda \) is a closed 1-form. Integrating this closed form along lines from the origin, we get a unique homogeneous primitive function \( g(x,y) \). By uniqueness, \( g(x,y) \) is \( \tilde{\Gamma} \)-invariant with the same character as \( \theta \). The converse is clear. The condition on degrees assures that the sum is homogeneous.

2. Rational Functions with Tetrahedral Symmetry
In this section, we consider the tetrahedral group acting on the Riemann sphere. We identify the unit sphere $S^2 \subset \mathbb{R}^3$ and the Riemann sphere $\mathbb{C}$ by the stereographic projection. The Riemann sphere and the complex projective space $\mathbb{P}E$ are identified by $z = x/y$.

Consider a regular tetrahedron with their vertices at

$$V_4 = \{0, \sqrt{2}, \sqrt{2} \omega, \sqrt{2} \omega^2\},$$

where $\omega = (-1 + \sqrt{3}i)/2$ is a cubic root of 1. Homogeneous function

$$h_4(x, y) = x^4 - 2\sqrt{2}xy^3$$

vanishes at these vertices. The centers of the faces of the tetrahedron

$$V'_4 = \{\infty, -\frac{1}{\sqrt{2}}, -\frac{\omega^2}{\sqrt{2}}, -\frac{\omega}{\sqrt{2}}\},$$

defines

$$h'_4(x, y) = 2\sqrt{2}x^3y + y^4.$$

The midpoints of the edges of the tetrahedron are given by

$$V_6 = \left\{-\frac{1 \pm \sqrt{3}}{\sqrt{2}} \omega^k \mid k = 0, 1, 2\right\}$$

and they define

$$h_6(x, y) = x^6 + 5\sqrt{2}x^3y^3 - y^6.$$

Let

$$h_8(x, y) = h_4h'_4 = 2\sqrt{2}x^7y - 7x^4y^4 - 2\sqrt{2}xy^7.$$

The tetrahedral group is isomorphic to the alternative group $A_4$. Let $\tilde{A}_4$ denote its pre-image in $SL(E)$. Define $\tilde{g}_1, \tilde{g}_2 \in \tilde{A}_4$ by

$$\tilde{g}_1 = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad \tilde{g}_2 = \frac{i}{\sqrt{3}} \begin{pmatrix} -1 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}.$$

These elements and $-I : (x, y) \mapsto (-x, -y)$ generate the group $\tilde{A}_4$.

Homogeneous function $h_4$ is $\tilde{A}_4$-invariant with character satisfying

$$\chi(\tilde{g}_1) = \omega$$

and $h'_4$ has character satisfying

$$\chi(\tilde{g}_1) = \omega^2.$$

Homogeneous functions $h_6, h_8, h'_4$ and $(h'_4)^3$ are absolutely $\tilde{A}_4$-invariant. They have a relation

$$h_6^2 = h'_4^3 + (h'_4)^3.$$

Let

$$h_{12} = h'_4 - (h'_4)^3 = x^{12} - 22\sqrt{2}x^9y^3 - 22\sqrt{2}x^3y^9 - y^{12}.$$
Let $R^A_4$ denote the ring of absolutely $\tilde{A}_4$-invariant polynomials. This ring is generated by $h_6, h_8$ and $h_{12}$. There is a relation between these generators:

$$h_6^4 = h_{12}^2 + 4h_8^3.$$ 

Since all the orbits of the $A_4$-action on the Riemann sphere consist of even number of points, $R^A_4$ is a graded ring and can be decomposed as

$$R^A_4 = \bigoplus_{k=0}^{\infty} R^A_{2k},$$

where $R^A_{2k}$ denotes the vector space of absolutely $\tilde{A}_4$-invariant homogeneous polynomials of degree $2k$.

**Proposition 2.1.**

$$\dim R^A_{2k} = \frac{1}{12} \{2k + 1 + 3(-1)^k + 8(1 - (k \text{ mod } 3))\}.$$ 

**Proof.** This formula can be obtained by directly computing

$$\dim R^A_{2k} = \frac{1}{|A_4|} \sum_{g \in A_4} \text{trace} \rho_{2k}(g),$$

where $\rho_{2k} : A_4 \to GL(R_{2k})$ is the representation of the tetrahedral group on the linear space of homogeneous polynomials of degree $2k$.

As a corollary, we have:

**Proposition 2.2.**

$$R^A_0 = \mathbb{C}, \quad R^A_2 = 0, \quad R^A_4 = 0,$$

$$R^A_6 = Ch_6 = R^A_0 h_6, \quad R^A_8 = Ch_8 = R^A_0 h_8,$$

$$R^A_{10} = 0, \quad R^A_{12} = Ch_{12} + Ch_6^2 = R^A_0 h_{12} + R^A_0 h_6,$$

$$R^A_{14} = Ch_6 h_8 = R^A_8 h_8 = R^A_0 h_6 h_8, \quad R^A_{16} = Ch_8^2 = R^A_8 h_8.$$

$$R^A_{18} = Ch_6 h_{12} + Ch_6^3 = R^A_0 h_{12} h_6,$$

$$R^A_{20} = Ch_8 h_{12} + Ch_8 h_6^2 = R^A_0 h_{12} h_8, \quad R^A_{22} = Ch_6 h_8^2 = R^A_8 h_6 h_8 = R^A_{14} h_8,$$

$$R^A_{24} = Ch_{12}^2 + Ch_{12} h_6^2 + Ch_6^4.$$ 

In general, $R^A_{2k+12} = R^A_{2k} h_{12} + Ch_6^\alpha h_8^\beta$ holds for some $\alpha$ and $\beta$ with $6\alpha + 8\beta = 2k + 12$, $h_6^\alpha h_8^\beta \notin R^A_{2k} h_{12}$.

Equivariant mappings are constructed by Theorem 1.4. If $f(x, y)$ and $g(x, y)$ are $\tilde{\Gamma}$-invariant homogeneous polynomials with the same character and if $\deg(g) = \deg(f) + 2$, then

$$\theta = f(x, y) \lambda + dg(x, y)$$

defines a $\tilde{\Gamma}$-equivariant 1-form. We denote the $\Gamma$-equivariant rational mapping corresponding to $\theta$ by

$$\psi(f, g) : \tilde{\mathbb{C}} \to \tilde{\mathbb{C}}.$$ 

$\psi(f, g)(z)$ is obtained by setting $z = x/y$ in
\[
\frac{xf(x, y) + \frac{\partial g}{\partial y}(x, y)}{yf(x, y) - \frac{\partial g}{\partial x}(x, y)}.
\]

Note that the numerator and the denominator may have a common factor and the degree of the obtained mapping \(\psi(f, g)\) may be smaller than \(\text{deg}(g) - 1\).

\(A_4\)-equivariant maps are computed as follows. \(A_4\)-invariant homogeneous functions with character \(\chi(\tilde{g}_1) = \omega\) are the functions \(h_4, (\tilde{h}_4')^2\) and absolutely \(\tilde{A}_4\)-invariant homogeneous functions multiplied by one of these two functions. For \(\tilde{A}_4\)-invariant homogeneous functions with character \(\chi(\tilde{g}_1) = \omega^2\), we have absolutely \(\tilde{A}_4\)-invariant homogeneous functions multiplied by \(\tilde{h}_4\) or \(\tilde{h}_4^2\). Hence, for these functions we obtain, for example,

\[
\psi(0, h_4^k) = \frac{3z}{\sqrt{2}z^3 - 1}, \quad k = 1, 2, \ldots
\]

and

\[
\psi(0, (\tilde{h}_4')^k) = -\frac{z^3 + \sqrt{2}}{3z^2}, \quad k = 1, 2, \ldots
\]

From absolutely \(\tilde{A}_4\)-invariant homogeneous functions, we obtain, for example, the following \(A_4\)-equivariant mappings:

\[
\psi(0, h_6^k) = -\frac{5z^3 - \sqrt{2}}{z^2(\sqrt{2}z^3 + 5)}, \quad k = 1, 2, \ldots
\]

\[
\psi(0, h_8^k) = -\frac{z(z^6 - 7\sqrt{2}z^3 - 7)}{7z^6 - 7\sqrt{2}z^3 - 1}, \quad k = 1, 2, \ldots
\]

\[
\psi(0, h_{12}^k) = \frac{11z^9 + 33z^3 + \sqrt{2}}{z^2(\sqrt{2}z^9 - 33z^6 - 11)}, \quad k = 1, 2, \ldots
\]

If a rational function \(\varphi : \mathbb{C} \to \mathbb{C}\) is \(A_4\)-equivariant, then orbits of the \(A_4\)-action on the Riemann sphere are mapped onto some \(A_4\)-orbits. The vertices \(V_4\) of the tetrahedron are mapped either onto themselves or onto the centers of the faces \(V_4'\). The midpoints of the edges of the tetrahedron \(V_6\) are mapped onto midpoints of edges, since they are the only points whose orbits consist of six points. The intersection of the tetrahedron and its dual tetrahedron is a regular octahedron. Homogeneous equation \(h_{12} = 0\) defines the set of midpoints of the edges of this octahedron

\[
V_{12} = \{\pm i, \pm \frac{\sqrt{3} \pm i}{2}, (\sqrt{2} \pm \sqrt{3})\omega^k\},
\]

where \(k = 0, 1, 2\). Centers of the faces of the octahedron

\[
V_8 = \{0, \infty, \sqrt{2}\omega^k, -\frac{\omega^k}{\sqrt{2}}\}
\]

are defined by \(h_8 = 0\), and the vertices are given by \(h_6 = 0\). The homogeneous equation \(h_8 = 0\) defines also the vertices of the cube which is the dual of the regular octahedron. Similarly, \(h_6 = 0\) defines the centers of the faces of this regular cube.

By looking at the equivariance, we see that the mapping \(\varphi\) restricted to the set of vertices or the set of the centers of the faces is either the identity map or the antipodal
map $z \mapsto -1/\bar{z}$. If $\varphi$ has critical points at these special points whose orbits consist of numbers smaller than the order of the polyhedral group, they are: super-attractive fixed points, super-attractive cycles of period two, or mapped into the antipodal unstable fixed points. The last of these cases occur when $\varphi$ is $A_4$-equivariant with critical points at $V_4$ or $V'_4$, mapping these points into their antipodal point and these antipodal points are unstable fixed points. In this case,

$$\tilde{\varphi} = -\frac{1}{\varphi(z)}$$

gives a rational function which has super-attractive fixed points on the real line and super-attractive cycles of period two. The restriction of $\tilde{\varphi}$ to these critical points is the conjugation $z \mapsto \bar{z}$. $\tilde{\varphi}$ is not $A_4$-equivariant ($-1/\bar{\varphi}(z)$ is $A_4$-equivariant but not complex analytic).

Now, let us consider $A_4$-equivariant maps which have super-attractive basins. We can verify the following facts.

Formulas of these functions are as follows.

$$\psi(2\sqrt{2}h_6, h_8) = \psi(0, h'_4)$$
$$\psi(-2\sqrt{2}h_6, h_8) = \psi(0, h_4)$$

3. Rational functions with octahedral symmetry

Six points $V_6$ of the Riemann sphere defines a regular octahedron. Points of $V_6$ are its vertices. The centers of the faces correspond to the points of $V_8 = V_4 \cup V'_4$ defined by $h_8 = 0$. The midpoints of the edges $V_{12}$ are given by $h_{12} = 0$. The polyhedral group acting on the
regular octahedron is isomorphic to the symmetric group $S_4$. Let $\tilde{S}_4 \subset SL(E)$ denote its pre-image.

Homogeneous functions $h_6$ and $h_{12}$ are $\tilde{S}_4$-invariant with character

$$\chi(\tilde{g}_3) = -1,$$

where

$$\tilde{g}_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$ 

$\tilde{g}_3$ is the generator of $\tilde{S}_4/\tilde{A}_4 \cong \mathbb{Z}_2$. Homogeneous functions $h_4$ and $h'_4$ are not $\tilde{S}_4$-invariant.

Homogeneous function $h_8$ is absolutely $\tilde{S}_4$-invariant.

The ring of absolutely $\tilde{S}_4$-invariant polynomials have only terms of even degree. The octahedral group $S_4$ can be considered to act on the polynomials with even degree terms. We denote this ring by $R^{S_4}$ and decompose it into homogeneous parts:

$$R^{S_4} = \bigoplus_{k=0}^{\infty} R^{S_4}_{2k}.$$ 

$R^{S_4}$ is generated by $h_8, h_6^2,$ and $h_6 h_{12}$. They have a relation

$$h_6^4 = h_{12}^2 + 4 h_8^3.$$ 

Proposition 3.1.

$$\dim R^{S_4}_{2k} = \frac{1}{24} (2k + 1 + 9(-1)^k + 8(1 - (k \text{ mod } 3)) + 6(-1)^{k/2})$$

Proposition 3.2. $\tilde{S}_4$-invariant functions with character $\chi(\tilde{g}_3) = -1$ form a vector space $R^{S_4} h_6 + R^{S_4} h_{12}$.

Since $A_4 \subset S_4$, $\tilde{S}_4$-invariant homogeneous polynomials are also $\tilde{A}_4$-invariant. And $S_4$-equivariant mappings are also $A_4$-equivariant mappings. Among $S_4$-equivariant mappings, we have, for example,

$$\psi(0, h_6), \psi(0, h_8), \psi(0, h_{12}), \text{etc.}.$$ 

As $h_6$ and $h_{12}$ have the same character and $h_8$ is absolutely $\tilde{S}_4$-invariant, $h_{12}$ and $h_6 h_8$ have the same character. Therefore, one-parameter family

$$\psi(p h_{12}, h_6 h_8), \quad p \in \mathbb{C}$$

gives $S_4$-invariant rational mappings. In this family, we find the following rational functions.

$$\psi\left(-\frac{13\sqrt{2}}{24} h_{12}, h_6 h_8\right) = \frac{z(5z^{12} + 130\sqrt{2}z^9 - 936z^6 - 182\sqrt{2}z^3 + 91)}{91z^{12} + 182\sqrt{2}z^9 - 936z^6 - 130\sqrt{2}z^3 + 5}$$

has critical points at $V_{12}$ (given by $h_{12}^2 = 0$) and these critical points are mapped into their antipodal points.

$$\psi\left(26\sqrt{2}h_{12}, h_6 h_8\right) = \frac{z^4(4z^9 - 78\sqrt{2}z^6 - 39z^3 - 91\sqrt{2})}{-91\sqrt{2}z^9 + 39z^6 - 78\sqrt{2}z^3 - 4}$$
has fixed critical points at $V_8$ (given by $h_8^3 = 0$).

$$\psi\left(-\frac{39}{\sqrt{2}}h_{12}, h_6h_8\right) = \frac{z(-5z^{12} + 130\sqrt{2}z^9 - 78z^6 + 104\sqrt{2}z^3 + 13)}{-13z^{12} + 104\sqrt{2}z^9 + 78z^6 + 130\sqrt{2}z^3 + 5}$$

has fixed critical points at $V_6$ ($h_6^4 = 0$). In this family, $\psi(-2\sqrt{2}h_{12}, h_6h_8) = \psi(0, h_6)$ and $\psi\left(\frac{3}{\sqrt{2}}h_{12}, h_6h_8\right) = \psi(0, h_8)$ are found.

### 4. Rational mappings with dodecahedral symmetry

In this section, we calculate some rational mappings with the symmetry of a regular dodecahedron. Homogeneous function

$$f_{12} = x^{11}y + 11x^6y^6 - xy^{12}$$

defines the 12 centers of the faces of a regular dodecahedron

$$W_{12} = \{0, \infty, \sqrt{5\sqrt{5} - 11}2\Omega^k, -\sqrt{5\sqrt{5} + 11}2\Omega^k\}, \quad k = 0, 1, 2, 3, 4,$$

where $\Omega = \frac{\sqrt{5} - 1}{4} + \frac{\sqrt{5 + \sqrt{5}}}{8}i$ is a quintic root of 1. The 20 vertices of the dodecahedron

$$W_{20} = \{\sqrt{57 + 25\sqrt{5} + 5\sqrt{255} + 114\sqrt{5}\Omega^k}, \sqrt{57 - 25\sqrt{5} + 5\sqrt{255} - 114\sqrt{5}\Omega^k}, \sqrt{-57 + 25\sqrt{5} + 5\sqrt{255} + 114\sqrt{5}\Omega^k}, -\sqrt{-57 - 25\sqrt{5} + 5\sqrt{255} - 114\sqrt{5}\Omega^k}\}$$

are given by

$$H_{20} = x^{20} - 228x^{15}y^5 + 494x^{10}y^{10} + 228x^5y^{15} + y^{20}$$

The midpoints of the edges

$$W_{30} = \{i\Omega^k, -i\Omega^k, \sqrt{\frac{\pm15\sqrt{650} - 290\sqrt{5} + 125\sqrt{5} - 261\Omega^k}{2}}, \sqrt{\frac{\pm15\sqrt{650} + 290\sqrt{5} - 125\sqrt{5} - 261\Omega^k}{2}}\}$$

are given by

$$T_{30} = x^{30} + y^{30} + 522(x^{25}y^5 - x^5y^{25}) - 10005(x^{20}y^{10} + x^{10}y^{20}).$$

Since the group of the symmetries of the regular dodecahedron is isomorphic to the alternative group $A_5$, and $A_5$ is a simple group, $A_5$-invariant homogeneous polynomials are always absolutely $A_5$-invariant.

As described in P. Doyle and C. McMullen[?],
\[
\psi(0, f_{12}) = -\frac{z(11z^{10} + 66z^5 - 11)}{11z^{10} + 66z^5 - 1},
\]
\[
\psi(0, H_{20}) = \frac{57z^{15} - 247z^{10} - 171z^5 - 1}{z^4(z^{15} - 171z^{10} + 247z^5 + 57)},
\]
\[
\psi(0, T_{30}) = -\frac{87z^{25} - 3335z^{20} - 6670z^{10} - 435z^5 + 1}{z^4(z^{25} + 435z^{20} - 18570z^{15} - 3335z^5 - 87)}
\]
are the only \(A_5\)-equivariant rational functions with degrees smaller than 31.

There is a family of \(A_5\)-equivariant rational functions of degree 31 since \(f_{12}H_{20}\) is of degree 32 and \(T_{30}\) is of degree 30. One parameter family

\[
\psi(pT_{30}, f_{12}H_{20}), \quad p \in \mathbb{C}
\]
gives \(A_5\)-equivariant rational mappings. Among these mappings, we find, for example,

\[
\psi\left(-\frac{31}{45}T_{30}, f_{12}H_{20}\right) = \frac{z(19z^{30} - 10602z^{25} - 326895z^{20} + 1060200z^{15} + 398505z^{10} - 67518z^5 - 341)}{341z^{30} - 67518z^{25} - 398505z^{20} + 1060200z^{15} + 326895z^{10} - 10602z^5 - 19}
\]
which has degenerate critical points at \(W_{30}\) (given by \(T_{30}^2 = 0\)) and these critical points are mapped into their antipodal points. In the same family we find

\[
\psi(-31T_{30}, f_{12}H_{20}) = \frac{z^6(z^{25} + 465z^{20} - 10385z^{15} + 2945z^{10} - 8370z^5 - 682)}{682z^{25} - 8370z^{20} - 2945z^{15} - 10385z^{10} - 465z^5 + 1}
\]
which has critical points at \(W_{12}\) (\(f^5_{12} = 0\)) and these degenerate critical points are fixed points. Another one found in this family is

\[
\psi\left(\frac{155}{3}T_{30}, f_{12}H_{20}\right) = \frac{z(19z^{30} + 10602z^{25} - 185535z^{20} - 35340z^{15} - 209715z^{10} - 7998z^5 + 31)}{31z^{30} + 7998z^{25} - 209715z^{20} + 35340z^{15} - 185535z^{10} - 10602z^5 + 19}
\]
which has degenerate critical points at \(W_{20}\) (\(H^3_{20} = 0\)) and these critical points are fixed points.
References


