Julia Sets with Polyhedral Symmetries

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Julia sets with polyhedral symmetries are constructed from equivariant rational mappings of the Riemann sphere.

0. Introduction

Let $\bar{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ denote the Riemann sphere. Let us identify the Riemann sphere with the unit sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ by the stereographic projection $z = (x_1 + ix_2)/(1 - x_3)$. Regular polyhedral groups act on the unit sphere and they induce groups of Möbius transformations of the Riemann sphere.

Let Γ be a group of such Möbius transformations. Rational function $f : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ is said to be Γ-*equivariant* if for all $\gamma \in \Gamma$,

$$\gamma \circ f \ = \ f \circ \gamma$$

holds. If f is Γ-equivariant, then its Julia set J_f is invariant for all $\gamma \in \Gamma$, *i.e.*,

$$\gamma(J_f) \ = \ J_f.$$

In this note, we consider Γ-equivariant rational functions with super-attractive cycles. In the simplest case, the obtained function has a Julia set which is locally homeomorphic to the Sierpinskii's gasket. Other functions have generalized Sierpinskii's gaskets as their Julia sets. The construction of Γ-equivariant mappings is classical and some of them were known to Klein[**?**]. P.Doyle and C.McMullen[**?**] has produced a picture of one of these functions.

1. Invariant Functions and Equivariant Mappings

P.Doyle and C.McMullen[**?**] gave a method to produce all rational maps with given symmetries. In this section we reproduce their method. Let $E = \mathbf{C}^2$ be a 2-dimensional complex vector space. A point $z \in \bar{\mathbf{C}}$ corresponds to a complex line $l = \{(x, y) \in E \mid z = x/y\}$. Let E^* denote the vector space of linear functionals of E. E^* is isomorphic to E and we use coordinates (ξ, η) on E^* such that if $v \ = \ (\xi, \eta)$ and $p \ = \ (x, y)$ then

$$v(p) \ = \ \xi x + \eta y.$$

Line $l = \{(x, y) \in E \mid z = x/y\}$ corresponds to a line in the dual space E^* given by

$$l^* \ = \ \{(\xi, \eta) \in E^* \mid z = -\frac{\eta}{\xi}\}.$$

Let $f : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ be a rational function of degree d defined by

$$f(z) = \frac{P(z)}{Q(z)},$$

where $P(z)$ and $Q(z)$ are polynomials without common factor. It defines, up to a scale factor, a homogeneous polynomial map $X : E \to E$ by

$$X(x,y) = (\tilde{P}(x,y), \tilde{Q}(x,y)),$$

where

$$\tilde{P}(x,y) = y^d P(\frac{x}{y}) \qquad \text{and} \qquad \tilde{Q}(x,y) = y^d Q(\frac{x}{y}).$$

As E is a vector space, X can also be regarded as a homogeneous vectorfield on E. Let $\Gamma \subset \mathrm{Aut}(\bar{\mathbf{C}})$ be a finite group of Möbius transformations. Let $\tilde{\Gamma} \subset SL(E)$ denote its pre-image in the group of linear transformations of determinant 1.

A vector field X on E is said to be *invariant* if there exists a character $\chi : \tilde{\Gamma} \to U(1)$, a group homomorphism of $\tilde{\Gamma}$ into the group of complex numbers with modulus 1, such that

$$X \circ \tilde{\gamma}^{-1} = \chi(\tilde{\gamma}) X$$

for all $\tilde{\gamma} \in \tilde{\Gamma}$. If the character χ is trivial, then X is said to be *absolutely* $\tilde{\Gamma}$-*invariant.*

Proposition 1.1. Homogeneous vector field X on E is $\tilde{\Gamma}$-invariant if and only if its corresponding rational function $f : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ satisfies

$$f \circ \gamma = f \quad \text{for all } \gamma \in \Gamma.$$

By an isomorphism between E and E^* :

$$E \ni (\xi, \eta) \longleftrightarrow -\eta dx + \xi dy \in E^*,$$

the vectorfield $X = (\tilde{P}(x,y), \tilde{Q}(x,y))$ corresponds to a 1-form

$$\theta = -\tilde{Q}(x,y)dx + \tilde{P}(x,y)dy.$$

Hence, rational function $f(z)$ corresponds to a homogeneous 1-form $\theta : E \to E^*$ up to a scale factor. To the identity map $z \mapsto z$ of the Riemann sphere corresponds the "identity" vectorfield $E \to E$

$$(x,y) \longmapsto (x,y),$$

more precisely,

$$(x,y) \longmapsto x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

And the 1-form $\lambda : E \to E^*$ corresponding to this vector field is given by

$$\lambda = -ydx + xdy.$$

A vector field X on E is said to be $\tilde{\Gamma}$-*equivariant* if there exists a character

$$\chi : \tilde{\Gamma} \to U(1)$$

such that

$$\tilde\gamma\circ X\circ\tilde\gamma^{-1} \;=\; \chi(\tilde\gamma)X$$

holds for all $\tilde\gamma\in\tilde\Gamma$. If the character χ is trivial, X is said to be absolutely $\tilde\Gamma$-equivariant.

Proposition 1.2. Vector field X is $\tilde\Gamma$-equivariant if and only if its crresponding rational function $f:\bar{\mathbf{C}}\to\bar{\mathbf{C}}$ is Γ-equivariant,*i.e.*,

$$\gamma\circ f \;=\; f\circ\gamma \quad \text{for all} \quad \gamma \;\in\; \Gamma.$$

A homogeneous 1-form θ is said to be $\tilde\Gamma$-*equivariant* if there exists a character $\chi:\tilde\Gamma\to U(1)$ such that

$$\theta\circ\tilde\gamma^{-1} \;=\; \chi(\tilde\gamma)\tilde\gamma^*\theta \quad \text{for all } \tilde\gamma \;\in\; \tilde\Gamma,$$

where $\tilde\gamma^*\theta$ is a 1-form $E\to E^*$ defined by

$$(\tilde\gamma^*\theta(x,y))(\xi,\eta) \;=\; \theta(x,y)(\tilde\gamma(\xi,\eta)).$$

Homogeneous 1-form θ is said to be *absolutely* $\tilde\Gamma$-*equivariant* if the character $\chi:\tilde\Gamma\to U(1)$ is trivial.

Proposition 1.3. Vectorfield $X=E\to E$ is $\tilde\Gamma$-equivariant if and only if its corresponding 1-form $\theta:E\to E^*$ is equivariant.

Proof. Let $\tilde\gamma\in SL(E)$. Then

$$\tilde\gamma\begin{pmatrix}\tilde P(\tilde\gamma^{-1}(x,y))\\ \tilde Q(\tilde\gamma^{-1}(x,y))\end{pmatrix} \;=\; \chi(\tilde\gamma)\begin{pmatrix}\tilde P(x,y)\\ \tilde Q(x,y)\end{pmatrix}$$

implies

$$(-\tilde Q(\tilde\gamma^{-1}(x,y)),\tilde P(\tilde\gamma^{-1}(x,y))) \;=\; \chi(\tilde\gamma)(-\tilde Q(x,y),\tilde P(x,y))\tilde\gamma.$$

Note that 1-form $\lambda=-ydx+xdy$ is absolutely $SL(E)$-equivariant. Let $\omega=dx\wedge dy$ be the volume form. Then $d\lambda=2\omega$. Obviously, ω is also absolutely $SL(E)$-equivariant.

Theorem 1.4. (P.Doyle and C.McMullen)
A homogeneous 1-form θ is $\tilde\Gamma$-equivariant if and only if

$$\theta \;=\; f(x,y)\lambda+dg(x,y)$$

where f and g are $\tilde\Gamma$-invariant homogeneous polynomials with the same character $\chi:\tilde\Gamma\to U(1)$ and $\deg(g)=\deg(f)+2$.

Proof. Suppose θ is $\tilde\Gamma$-equivariant. Then $d\theta=h(x,y)\omega$ is a $\tilde\Gamma$-equivariant 2-form. Since ω is absolutely equivariant, $h(x,y)$ is a homogeneous polynomial which is $\tilde\Gamma$-invariant with the same character as θ. Let $f(x,y)=h(x,y)/(\deg(h)+2)$. Then $d(f\lambda)=h(x,y)\omega$ holds and $\theta-f(x,y)\lambda$ is a closed 1-form. Integrating this closed form along lines from the origin, we get a unique homogeneous primitive function $g(x,y)$. By uniqueness, $g(x,y)$ is $\tilde\Gamma$-invariant with the same character as θ. The converse is clear. The condition on degrees assures that the sum is homogeneous.

2. Rational Functions with Tetrahedral Symmetry

In this section, we consider the tetrahedral group acting on the Riemann sphere. We identify the unit sphere $S^2 \subset \mathbf{R}^3$ and the Riemann sphere $\bar{\mathbf{C}}$ by the stereographic projection. The Riemann sphere and the complex projective space $\mathbf{P}E$ are identified by $z = x/y$.

Consider a regular tetrahedron with their vetices at

$$V_4 \;=\; \{0, \sqrt{2}, \sqrt{2}\omega, \sqrt{2}\omega^2\},$$

where $\omega = (-1+\sqrt{3}i)/2$ is a cubic root of 1. Homogeneous function

$$h_4(x,y) \;=\; x^4 - 2\sqrt{2}xy^3$$

vanishes at these vertices. The centers of the faces of the tetrahedron

$$V_4' \;=\; \{\infty, -\frac{1}{\sqrt{2}}, -\frac{\omega^2}{\sqrt{2}}, -\frac{\omega}{\sqrt{2}}\}$$

defines

$$h_4'(x,y) \;=\; 2\sqrt{2}x^3y + y^4.$$

The midpoints of the edges of the tetrahedron are given by

$$V_6 \;=\; \{\frac{-1 \pm \sqrt{3}}{\sqrt{2}}\omega^k \;\mid\; k = 0, 1, 2\}$$

and they define

$$h_6(x,y) \;=\; x^6 + 5\sqrt{2}x^3y^3 - y^6.$$

Let

$$h_8(x,y) \;=\; h_4h_4' \;=\; 2\sqrt{2}x^7y - 7x^4y^4 - 2\sqrt{2}xy^7.$$

The tetrahedral group is isomorphic to the alternative group A_4. Let $\tilde{A}_4$ denote its pre-image in $SL(E)$. Define $\tilde{g}_1, \tilde{g}_2 \in \tilde{A}_4$ by

$$\tilde{g}_1 = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad \tilde{g}_2 = \frac{i}{\sqrt{3}}\begin{pmatrix} -1 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}.$$

These elements and $-I : (x,y) \mapsto (-x,-y)$ generate the group $\tilde{A}_4$.

Homogeneous function h_4 is $\tilde{A}_4$-invariant with character satisfying

$$\chi(\tilde{g}_1) \;=\; \omega$$

and h_4' has character satisfying

$$\chi(\tilde{g}_1) \;=\; \omega^2.$$

Homogeneous functions h_6, h_8, h_4^3 and $(h_4')^3$ are absolutely $\tilde{A}_4$-invariant. They have a relation

$$h_6^2 \;=\; h_4^3 + (h_4')^3.$$

Let

$$h_{12} \;=\; h_4^3 - (h_4')^3 \;=\; x^{12} - 22\sqrt{2}x^9y^3 - 22\sqrt{2}x^3y^9 - y^{12}.$$

Let R^{A_4} denote the ring of absolutely $\tilde{A}_4$-invariant polynomials. This ring is generated by h_6, h_8 and h_{12}. There is a relation between these generators :

$$h_6^4 \;=\; h_{12}^2 + 4h_8^3.$$

Since all the orbits of the A_4-action on the Riemann sphere consist of even number of points, R^{A_4} is a graded ring and can be decomposed as

$$R^{A_4} \;=\; \bigoplus_{k=0}^{\infty} R_{2k}^{A_4},$$

where $R_{2k}^{A_4}$ denotes the vector space of absolutely $\tilde{A}_4$-invariant homogeneous polynomials of degree $2k$.

Proposition 2.1.

$$\dim R_{2k}^{A_4} \;=\; \frac{1}{12}\{2k+1+3(-1)^k+8(1-(k \bmod 3))\}.$$

Proof. This formula can be obtained by directly computing

$$\dim R_{2k}^{A_4} \;=\; \frac{1}{\mid A_4 \mid}\sum_{g\in A_4} \mathrm{trace}\rho_{2k}(g),$$

where $\rho_{2k} : A_4 \to GL(R_{2k})$ is the representation of the tetrahedral group on the linear space of homogeneous polynomials of degree $2k$.

As a corollary, we have :

Proposition 2.2.

$$R_0^{A_4} = \mathbf{C}, \quad R_2^{A_4} = 0, \quad R_4^{A_4} = 0,$$
$$R_6^{A_4} = \mathbf{C}h_6 = R_0^{A_4}h_6, \quad R_8^{A_4} = \mathbf{C}h_8 = R_0^{A_4}h_8,$$
$$R_{10}^{A_4} = 0, \quad R_{12} = \mathbf{C}h_{12} + \mathbf{C}h_6^2 = R_0^{A_4}h_{12} + R_6^{A_4}h_6$$
$$R_{14}^{A_4} = \mathbf{C}h_6h_8 = R_8^{A_4}h_6 = R_6^{A_4}h_8, \quad R_{16}^{A_4} = \mathbf{C}h_8^2 = R_8^{A_4}h_8.$$
$$R_{18}^{A_4} = \mathbf{C}h_6h_{12} + \mathbf{C}h_6^3 = R_{12}^{A_4}h_6,$$
$$R_{20}^{A_4} = \mathbf{C}h_8h_{12} + \mathbf{C}h_8h_6^2 = R_{12}^{A_4}h_8, \quad R_{22}^{A_4} = \mathbf{C}h_6h_8^2 = R_{16}^{A_4}h_6 = R_{14}^{A_4}h_8,$$
$$R_{24}^{A_4} = \mathbf{C}h_{12}^2 + \mathbf{C}h_{12}h_6^2 + \mathbf{C}h_6^4.$$

In general, $R_{2k+12}^{A_4} = R_{2k}^{A_4}h_{12} + \mathbf{C}h_6^{\alpha}h_8^{\beta}$ holds for some α and β with $6\alpha + 8\beta = 2k+12$, $h_6^{\alpha}h_8^{\beta} \notin R_{2k}^{A_4}h_{12}$.

Equivariant mappings are constructed by Theorem 1.4. If $f(x,y)$ and $g(x,y)$ are $\tilde{\Gamma}$-invariant homogeneous polynomials with the same character and if $\deg(g) = \deg(f) + 2$, then

$$\theta \;=\; f(x,y)\lambda + dg(x,y)$$

defines a $\tilde{\Gamma}$-equivariant 1-form. We denote the Γ-equivariant rational mapping corresponding to θ by

$$\psi(f,g) \;:\; \bar{\mathbf{C}} \;\to\; \bar{\mathbf{C}}.$$

$\psi(f,g)(z)$ is obtained by setting $z = x/y$ in

$$\frac{xf(x,y)+\frac{\partial g}{\partial y}(x,y)}{yf(x,y)-\frac{\partial g}{\partial x}(x,y)}.$$

Note that the numerator and the denominator may have a common factor and the degree of the obtained mapping $\psi(f,g)$ may be smaller than $\deg(g)-1$.

A_4-equivariant maps are computed as follows. $\tilde{A}_4$-invariant homogeneous functions with character $\chi(\tilde{g}_1)=\omega$ are the functions $h_4, (h_4')^2$ and absolutely $\tilde{A}_4$-invariant homegeneous functions multiplied by one of these two functions. For $\tilde{A}_4$-invariant homogeneous functions with character $\chi(\tilde{g}_1)=\omega^2$, we have absolutely $\tilde{A}_4$-invariant homogeneous functions multiplied by h_4' or h_4^2. Hence, for these functions we obtain, for example,

$$\psi(0,h_4^k) \;=\; \frac{3z}{\sqrt{2}z^3-1}, \quad k=1,2,\ldots$$

and

$$\psi(0,(h_4')^k) \;=\; -\frac{z^3+\sqrt{2}}{3z^2}, \quad k=1,2,\ldots$$

From absolutely $\tilde{A}_4$-invariant homogeneous functions, we obtain, for example, the following A_4-equivariant mappings :

$$\begin{aligned}
\psi(0,h_6^k) &= -\frac{5z^3-\sqrt{2}}{z^2(\sqrt{2}z^3+5)}, \quad k=1,2,\ldots\\
\psi(0,h_8^k) &= -\frac{z(z^6-7\sqrt{2}z^3-7)}{7z^6-7\sqrt{2}z^3-1}, \quad k=1,2,\ldots\\
\psi(0,h_{12}^k) &= \frac{11z^9+33z^3+\sqrt{2}}{z^2(\sqrt{2}z^9-33z^6-11)}, \quad k=1,2,\ldots
\end{aligned}$$

If a rational function $\varphi:\bar{\mathbf{C}}\to\bar{\mathbf{C}}$ is A_4-equivariant, then orbits of the A_4-action on the Riemann sphere are mapped onto some A_4-orbits. The vertices V_4 of the tetrahedron are mapped either onto themselves or onto the centers of the faces V_4'. The midpoints of the edges of the tetrahedron V_6 are mapped onto midpoints of edges, since they are the only points whose orbits consist of six points. The intersection of the tetrahedron and its dual tetrahedron is a regular octahedron. Homogeneous equation $h_{12}=0$ defines the set of midpoints of the edges of this octahedron

$$V_{12} \;=\; \{\pm i, \pm\frac{\sqrt{3}\pm i}{2}, (\sqrt{2}\pm\sqrt{3})\omega^k\},$$

where $k=0,1,2$. Centers of the faces of the octahedron

$$V_8 \;=\; \{0,\infty,\sqrt{2}\omega^k,-\frac{\omega^k}{\sqrt{2}}\}$$

are defined by $h_8=0$, and the vertices are given by $h_6=0$. The homogeneous equation $h_8=0$ defines also the vetices of the cube which is the dual of the regular octahedron. Similarly, $h_6=0$ defines the centers of the faces of this regular cube.

By looking at the equivariance, we see that the mapping φ restricted to the set of vertices or the set of the centers of the faces is either the identity map or the antipodal map $z \mapsto -1/\bar{z}$. If φ has critical points at these special points whose orbits consist of numbers smaller than the order of the polyhedral group, they are : super-attractive fixed points, super-attractive cycles of period two, or mapped into the antipodal unstable fixed points. The last of these cases occur when φ is A_4-equivariant with critical points at V_4 or V_4', mapping these points into their antipodal point and these antipodal points are unstable fixed points. In this case,

$$\tilde{\varphi} = -\frac{1}{\varphi(z)}$$

gives a rational function which has super-attractive fixed points on the real line and super-attractive cycles of period two. The restriction of $\tilde{\varphi}$ to these critical points is the conjugation $z \mapsto \bar{z}$. $\tilde{\varphi}$ is *not* A_4-equivariant ($-1/\bar{\varphi}(z)$ is A_4-equivariant but not complex analytic).

Now, let us consider A_4-equivariant maps which have super-attractive basins. We can verify the following facts.

A_4-equivariant function	critical points	critical type	restriction to the critical points
$\psi(0, h_4)$	V_4'	$h_4' = 0$	antipodal map
$\psi(0, h_4')$	V_4	$h_4 = 0$	antipodal map
$\psi(0, h_6)$	V_8	$h_8 = 0$	antipodal map
$\psi(0, h_8)$	V_6	$h_6^2 = 0$	antipodal map
$\psi(0, h_{12})$	$V_6 \cup V_8$	$h_6^2 h_8 = 0$	antipodal map
$\psi(-14\sqrt{2}h_6, h_8)$	V_4	$h_4^3 = 0$	identity map
$\psi(14\sqrt{2}h_6, h_8)$	V_4'	$(h_4')^3 = 0$	identity map
$\psi(2\sqrt{14}h_6, h_8)$	V_{12}	$h_{12} = 0$	neither antipodal nor identity

$$\begin{aligned} \psi(2\sqrt{2}h_6, h_8) &= \psi(0, h_4') \\ \psi(-2\sqrt{2}h_6, h_8) &= \psi(0, h_4) \end{aligned}$$

Formulas of these functions are as follows.

$$\psi(2\sqrt{14}h_6, h_8) = \frac{z((1+\sqrt{7})z^6 + (5\sqrt{14} - 7\sqrt{2})z^3 - 7 - \sqrt{7})}{(\sqrt{7} - 7)z^6 + (5\sqrt{14} + 7\sqrt{2})z^3 + 1 - \sqrt{7}}$$

$$\psi(-14\sqrt{2}h_6, h_8) = \frac{3z^4(z^3 + 7\sqrt{2})}{7z^6 + 14\sqrt{2}z^3 - 4}$$

$$\psi(14\sqrt{2}h_6, h_8) = -\frac{z(\sqrt{2}z^6 + 14\sqrt{2}z^3 - 7)}{3(1 - 7\sqrt{2}z^3)}$$

Function $\psi(0, h_4)$ is conjugate to $\psi(0, h_4')$ and function $\psi(-14\sqrt{2}h_6, h_8)$ is conjugate to $\psi(14\sqrt{2}h_6, h_8)$.

3. Rational functions with octahedral symmetry

Six points V_6 of the Riemann sphere defines a regular octahedron. Points of V_6 are its vertices. The centers of the faces correspond to the points of $V_8 = V_4 \cup V_4'$ defined by $h_8 = 0$. The midpoints of the edges V_{12} are given by $h_{12} = 0$. The polyhedral group acting on the regular octahedron is isomorphic to the symmetric group S_4. Let $\tilde{S}_4 \subset SL(E)$ denote its pre-image.

Homogeneous functions h_6 and h_{12} are $\tilde{S}_4$-invariant with character

$$\chi(\tilde{g}_3) \;=\; -1,$$

where

$$\tilde{g}_3 \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$\tilde{g}_3$ is the generator of $\tilde{S}_4/\tilde{A}_4 \simeq \mathbf{Z}_2$. Homogeneous functions h_4 and h_4' are *not* $\tilde{S}_4$-invariant. Homogeneous function h_8 is absolutely $\tilde{S}_4$-invariant.

The ring of absolutely $\tilde{S}_4$-invariant polynomials have only terms of even degree. The octahedral group S_4 can be considered to act on the polynomials with even degree terms. We denote this ring by R^{S_4} and decompose it into homogeneous parts :

$$R^{S_4} \;=\; \bigoplus_{k=0}^{\infty} R_{2k}^{S_4}.$$

R^{S_4} is generated by h_8, h_6^2, and $h_6 h_{12}$. They have a relation

$$h_6^4 \;=\; h_{12}^2 + 4h_8^3.$$

Proposition 3.1.

$$\dim R_{2k}^{S_4} \;=\; \frac{1}{24}(2k + 1 + 9(-1)^k + 8(1 - (k \bmod 3)) + 6(-1)^{[k/2]})$$

Proposition 3.2. $\tilde{S}_4$-invariant functions with character $\chi(\tilde{g}_3) = -1$ form a vector space

$$R^{S_4} h_6 + R^{S_4} h_{12}.$$

Since $A_4 \subset S_4$, $\tilde{S}_4$-invariant homogeneous polynomials are also $\tilde{A}_4$-invariant. And S_4-equivariant mappings are also A_4-equivariant mappings. Among S_4-equivari-ant mappings, we have, for example,

$$\psi(0, h_6), \psi(0, h_8), \psi(0, h_{12}), etc.$$

As h_6 and h_{12} have the same character and h_8 is absolutely $\tilde{S}_4$-invariant, h_{12} and $h_6 h_8$ have the same character. Therefore, one-parameter family

$$\psi(p h_{12}, h_6 h_8), \qquad p \in \mathbf{C}$$

gives S_4-invariant rational mappings. In this family, we find the following rational functions.

$$\psi(-\frac{13\sqrt{2}}{24} h_{12}, h_6 h_8) \;=\; -\frac{z(5z^{12} + 130\sqrt{2}z^9 - 936z^6 - 182\sqrt{2}z^3 + 91)}{91z^{12} + 182\sqrt{2}z^9 - 936z^6 - 130\sqrt{2}z^3 + 5}$$

has critical points at V_{12} (given by $h_{12}^2 = 0$) and these critical points are mapped into their antipodal points.

$$\psi(26\sqrt{2} h_{12}, h_6 h_8) \;=\; \frac{z^4(4z^9 - 78\sqrt{2}z^6 - 39z^3 - 91\sqrt{2})}{-91\sqrt{2}z^9 + 39z^6 - 78\sqrt{2}z^3 - 4}$$

has fixed critical points at V_8 (given by $h_8^3 = 0$).

$$\psi(-\frac{39}{\sqrt{2}}h_{12}, h_6h_8) \;=\; \frac{z(-5z^{12} + 130\sqrt{2}z^9 - 78z^6 + 104\sqrt{2}z^3 + 13)}{-13z^{12} + 104\sqrt{2}z^9 + 78z^6 + 130\sqrt{2}z^3 + 5}$$

has fixed critical points at V_6 ($h_6^4 = 0$). In this family, $\psi(-2\sqrt{2}h_{12}, h_6h_8) = \psi(0, h_6)$ and $\psi(\frac{3}{\sqrt{2}}h_{12}, h_6h_8) = \psi(0, h_8)$ are found.

4. Rational mappings with dodecahedral symmetry

In this section, we calculate some rational mappings with the symmetry of a regular dodecahedron. Homogeneous function

$$f_{12} \;=\; x^{11}y + 11x^6y^6 - xy^{12}$$

defines the 12 centers of the faces of a regular dodecahedron

$$W_{12} \;=\; \{0, \infty, \sqrt[5]{\frac{5\sqrt{5}-11}{2}}\Omega^k, -\sqrt[5]{\frac{5\sqrt{5}+11}{2}}\Omega^k\}, \quad k = 0, 1, 2, 3, 4,$$

where $\Omega = \frac{\sqrt{5}-1}{4} + \sqrt{\frac{5+\sqrt{5}}{8}}i$ is a quintic root of 1. The 20 vertices of the dodecahedron

$$W_{20} \;=\; \{\sqrt[5]{57 + 25\sqrt{5} + 5\sqrt{255 + 114\sqrt{5}}}\Omega^k, \sqrt[5]{57 - 25\sqrt{5} + 5\sqrt{255 - 114\sqrt{5}}}\Omega^k,$$
$$- \sqrt[5]{-57 + 25\sqrt{5} + 5\sqrt{255 + 114\sqrt{5}}}\Omega^k, -\sqrt[5]{-57 - 25\sqrt{5} + 5\sqrt{255 - 114\sqrt{5}}}\Omega^k\}$$

are given by

$$H_{20} \;=\; x^{20} - 228x^{15}y^5 + 494x^{10}y^{10} + 228x^5y^{15} + y^{20}$$

The midpoints of the edges

$$W_{30} \;=\; \{i\Omega^k, -i\Omega^k, \sqrt[5]{\frac{\pm 15\sqrt{650 - 290\sqrt{5}} + 125\sqrt{5} - 261}{2}}\Omega^k,$$
$$\sqrt[5]{\frac{\pm 15\sqrt{650 + 290\sqrt{5}} - 125\sqrt{5} - 261}{2}}\Omega^k\}$$

are given by

$$T_{30} \;=\; x^{30} + y^{30} + 522(x^{25}y^5 - x^5y^{25}) - 10005(x^{20}y^{10} + x^{10}y^{20}).$$

Since the group of the symmetries of the regular dodecahedron is isomorphic to the alternative group A_5, and A_5 is a simple group, $\tilde{A}_5$-invariant homogeneous polynomials are always absolutely $\tilde{A}_5$-invariant.

As described in P.Doyle and C.McMullen[**?**],

$$\psi(0,f_{12}) = -\frac{z(z^{10}+66z^5-11)}{11z^{10}+66z^5-1},$$

$$\psi(0,H_{20}) = \frac{57z^{15}-247z^{10}-171z^5-1}{z^4(z^{15}-171z^{10}+247z^5+57)},$$

$$\psi(0,T_{30}) = -\frac{87z^{25}-3335z^{20}-6670z^{10}-435z^5+1}{z^4(z^{25}+435z^{20}-6670z^{15}-3335z^5-87)}$$

are the only A_5-equivariant rational functions with degrees smaller than 31.

A_5-equivariant mapping	critical points	critical type	restriction to the critical points
$\psi(0,f_{12})$	W_{20}	$H_{20}=0$	antipodal map
$\psi(0,H_{20})$	W_{12}	$f_{12}^3=0$	antipodal map
$\psi(0,T_{30})$	$W_{12}\cup W_{20}$	$f_{12}^3H_{20}=0$	antipodal map

There is a family of A_5-equivariant rational functions of degree 31 since $f_{12}H_{20}$ is of degree 32 and T_{30} is of degree 30. One parameter family

$$\psi(pT_{30},f_{12}H_{20}),\quad p\in\mathbf{C}$$

gives A_5-equivariant rational mappings. Among these mappings, we find, for example,

$$\psi(-\frac{31}{45}T_{30},f_{12}H_{20}) =$$

$$-\frac{z(19z^{30}-10602z^{25}-326895z^{20}+1060200z^{15}+398505z^{10}-67518z^5-341)}{341z^{30}-67518z^{25}-398505z^{20}+1060200z^{15}+326895z^{10}-10602z^5-19}$$

which has degenerate critical points at W_{30} (given by $T_{30}^2=0$) and these critical points are mapped into their antipodal points. In the same family we find

$$\psi(-31T_{30},f_{12}H_{20}) = \frac{z^6(z^{25}+465z^{20}-10385z^{15}+2945z^{10}-8370z^5-682)}{682z^{25}-8370z^{20}-2945z^{15}-10385z^{10}-465z^5+1}$$

which has critical points at W_{12} ($f_{12}^5=0$) and these degenerate critical points are fixed points. Another one found in this family is

$$\psi(\frac{155}{3}T_{30},f_{12}H_{20}) =$$

$$\frac{z(19z^{30}+10602z^{25}-185535z^{20}-35340z^{15}-209715z^{10}-7998z^5+31)}{31z^{30}+7998z^{25}-209715z^{20}+35340z^{15}-185535z^{10}-10602z^5+19}$$

which has degenerate critical points at W_{20} ($H_{20}^3=0$) and these critical points are fixed points.

References

[1] B.Branner and J.H.Hubbard, The iteration of cubic polynomials, part I: The global topology of parameter space, Acta Math. **160**(1988)143-206.

[2] A.Douady and J.H.Hubbard, Systèmes dynamiques holomorphes I: Itération des polynômes complexes, Publ. Math. Orsay(1984).

[3] P.Doyle and C.McMullen, Solving the Quintic by Iteration, Acta Math., **163**(1989) 151-180.

[4] F.Klein, Vorlesungen über das Ikosaheder und die Auflösung der Gleichungen vom fünften Grade. B.G.Teubner, 1884.

[5] H.-O.Peitgen and P.Richter, The Beauty of Fractals, Springer Verlag, Berlin Heidelberg, 1986.

[6] H.-O.Peitgen and D.Saupe, The Science of Fractal Images, Springer Verlag, New York, 1988.

[7] S.Ushiki, Chaotic phenomena and fractal objects in numerical analysis, in Patterns and Waves — Qualitative Analysis of Nonlinear Differential Equations —, eds. T.Nishida, M.Mimura and H.Fujii, Studies in Math and its Appl. 18, Kinokuniya, Tokyo, 1986.

[8] S.Ushiki, The World of Fractals — intruduction to complex dynamical systems (in japanese), Nihon Hyoron Sha, Tokyo, 1987.

[9] S.Ushiki, A surgery on Blaschke's function, Japan J. of Appl. Math. **5**(1988)407-429.

[10] S.Ushiki, Arnold's tongues and swallows' tails in complex parameter spaces, Advanced Series on Dynamical Systems Vol.6, Stability Theory and Related Topics in Dynamical Systems, Eds. K.Shiraiwa and G.Ikegami, World Scientific, Singapore, 1989, 153-178.