Coexisting attractive cycles in Hénon dynamics

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Abstract

The Hénon map can have three attracting cycles for certain values of parameters. A rigorous proof is given for some parameter values having attracting cycles of periods 1, 3, and 4. The proof is given by elementary calculations. Saddle-node locus curves in the parameter space for periodic points of periods 3 and 4 are computed. Formula for periodic points of period up to five are obtained.

0. Introduction

Classical Hénon map is given by the following formula.

\[
\begin{align*}
    x_{n+1} &= 1 + y_n - ax_n^2 \\
    y_{n+1} &= bx_n
\end{align*}
\]

In this note, the Hénon map \( H_{b,c} : (x, y) \mapsto (X, Y) \) is defined by the following formula.

\[
\begin{align*}
    X &= x^2 + c + by \\
    Y &= x
\end{align*}
\]

These maps are conjugate by affine change of coordinates with parameter correspondence \( c = -a \). In this note, we try to compute periodic points of period up to 5 explicitly. Parameter value is detected for which saddle node bifurcation of period three periodic point and saddle node bifurcation of period four occur simultaneously while one fixed point remains attracting. Further calculation reveals the existence of attracting cycles of periods 1, 3 and 4 for some nearby parameters.

A variable will be called a **cycle variable**, if a cycle can be computed from the variable, and the value can be computed from a cycle. An equation of cycle variable and parameters will be called a **cycle equation**.
The following picture shows an example of a part of unstable manifold of a saddle point. Observe that there are three basins of attractions.

1. Attracting fixed point

Fixed points are of the form \((x_0, x_0)\) with \(x_0\) given as a root of quadratic equation

\[
x_0^2 - (1 - b)x_0 + c = 0,
\]

\(i.e.,\)

\[
x_0 = \frac{1}{2}(1 - b \pm \sqrt{(1 - b)^2 - 4c}),
\]

and

\[
c = -(x_0 - \frac{1}{2}(1 - b))^2 + \frac{1}{4}(1 - b)^2.
\]

The value \(x_0\) can be regarded as a cycle variable of 1-cycles. And the cycle equation of 1-cycle is

\[
(CE1) \quad x_0^2 - (1 - b)x_0 + c = 0.
\]

The Jacobian matrix of \(H_{b,c}\) is given by

\[
DH_{b,c,(x,y)} = \begin{pmatrix} 2x & b \\ 1 & 0 \end{pmatrix}.
\]

And

\[
\det DH_{b,c,(x,y)} = -b, \quad \text{tr } DH_{b,c,(x,y)} = 2x.
\]
The eigenvalues of the fixed points are given by
\[ \lambda = x_0 \pm \sqrt{x_0^2 + b}. \]

For given \( b \) and eigenvalue \( \lambda \), the fixed point and parameter \( c \) are given by
\[ x_0 = \frac{1}{2} \left( \lambda - \frac{b}{\lambda} \right), \]
\[ c = -\frac{1}{4\lambda^2}(\lambda - 1)^2(\lambda + b)^2 + \frac{1}{4}(1 - b)^2. \]

For fixed \( b \), parameter \( c \) can be regarded as a rational function of \( \lambda \). This rational function has a pole at \( \lambda = 0 \), and non-degenerate critical points at \( \lambda = 1, -b \), and \( \pm \sqrt{-b} \), since
\[ \frac{\partial c}{\partial \lambda} = -\frac{1}{2\lambda^3}(\lambda - 1)(\lambda + b)(\lambda^2 + b). \]

The critical value \( c = \frac{1}{4}(1 - b)^2 \) is the locus of saddle-node bifurcation.

By setting \( \tau = \text{tr } DH_{b,c} \), we have
\[ \tau = \lambda - \frac{b}{\lambda} = 2x_0, \quad c = -\frac{1}{4}(\tau - (1 - b))^2 + \frac{1}{4}(1 - b)^2. \]

For fixed \( b \) with \( |b| < 1 \), the parameter region of \( c \), having an attracting fixed point, is the image of annulus \( \{ \lambda \in \mathbb{C} \mid |b| < |\lambda| < 1 \} \) by the rational map. This annulus is mapped onto an ellipse in \( \tau \) space, followed by a quadratic map with a critical point at \( \tau = 1 - b \) on the boundary of the ellipse, sending the ellipse onto an “elliptic cardioid” in \( c \) space with a cusp point at critical value \( \frac{1}{4}(1 - b)^2 \).

Let \( \mathbb{D}_2 \) denote the quotient space of polydisc \( \mathbb{D}^2 \) by equivalence \( (\lambda_1, \lambda_2) \sim (\lambda_2, \lambda_1) \). Canonical coordinate of \( \mathbb{D}_2 \) is \((\delta, \tau) = (\lambda_1\lambda_2, \lambda_1 + \lambda_2) \). The annulus \( \{ (\lambda_1, \lambda_2) \in \mathbb{D}^2 \mid \lambda_1\lambda_2 = -b \} \) is isomorphic to the annulus \( \{ \lambda \in \mathbb{D} \mid |b| < |\lambda| < 1 \} \) and mapped onto a vertical ellipse in \( \mathbb{D}_2 \). Let \( \Omega_1 \) denote the set of parameters \( (b, c) \in \mathbb{C}^2 \) for which \( H_{b,c} \) has an attracting fixed point. Coordinate \( x_0 \) represents the fixed point \((x_0, x_0)\). And let \( \widetilde{\Omega}_1 \) denote the set of points \( ((b, c), x_0) \in \Omega_1 \times \mathbb{C} \) with \((x_0, x_0)\) being the attracting fixed point, i.e., \( x_0^2 - (1 - b)x_0 + c = 0 \). We call this equation the cycle equation of period 1.

**Proposition 1.** \( \widetilde{\Omega}_1 \) is isomorphic to \( \mathbb{D}_2 \).
PROOF. For point \(((b, c), x_0) \in \tilde{\Omega}_1\), the eigenvalues \(\lambda_1, \lambda_2\) of the attracting fixed point are of modulus smaller than 1. Hence, \((\lambda_1, \lambda_2) \in D^2\). The product \(\delta = \lambda_1 \lambda_2 = -b\) and the sum \(\tau = \lambda_1 + \lambda_2 = 2x_0\) define a holomorphic mapping from \(\tilde{\Omega}_1\) to \(D^2\). The inverse holomorphic mapping is given by \(b = -\lambda_1 \lambda_2 = -\delta, \quad x_0 = \frac{1}{2} (\lambda_1 + \lambda_2) = \frac{1}{2} \tau,\) and \(c = (1 - b)x_0 - x_0^2\).

The saddle-node locus, where the fixed point has an eigenvalue \(\lambda = 1\), is a quadratic curve \(\{(b, c) \in \mathbb{C}^2 \mid c = \frac{1}{4} (1 - b)^2\}\). These parameters are located at the cusp points of “elliptic cardioid” \(\partial\Omega_1 \cap \{b = \text{const.}\}\).

2. Saddle fixed point and unstable manifold

When a fixed point \(P\) is a saddle, there exists an unstable manifold \(W^u(P)\). Computations of saddle fixed point and eigenvalues are exactly same as in the case of attracting fixed points. The coordinate \(x_0\) in the previous section can be used as the coordinate of the space of cycle of period 1. Let \(\mathbb{C}_2\) denote the quotient space of \(\mathbb{C}^2\) by equivalence \((\lambda_1, \lambda_2) \sim (\lambda_2, \lambda_1)\). This space is used as the space of eigenvalues and is isomorphic to \(\mathbb{C}^2\) by mapping \([[(\lambda_1, \lambda_2)] \mapsto (\delta, \tau)],\) defined by

\[\begin{align*}
\delta &= \lambda_1 \lambda_2, \\
\tau &= \lambda_1 + \lambda_2.
\end{align*}\]

Let \(\tilde{\Gamma}_1 = \{((b, c), x_0) \in \mathbb{C}^2 \times \mathbb{C} \mid x_0^2 - (1 - b)x_0 + c = 0\}\) denote the space of cycles of period 1.

PROPOSITION 2. \(\tilde{\Gamma}_1\) is isomorphic to \(\mathbb{C}_2\).

PROOF. For point \(((b, c), x) \in \tilde{\Gamma}_1\), the determinant and the trace of the Jacobian matrix \(DH_{b,c}\) at fixed point \((x_0, x_0)\) are given by

\[\begin{align*}
\delta &= -b, \\
\tau &= 2x_0.
\end{align*}\]

And the inverse map is given by

\[\begin{align*}
\delta &= -\delta, \\
\tau &= \frac{\tau}{2} ((1 + \delta) - \frac{\tau}{2}), \\
x_0 &= \frac{1}{2} \tau.
\end{align*}\]

“How do you compute unstable manifolds numerically?” is a frequently asked question. There are several methods. The author uses a classical method found by Poincaré in the 19th century, and generalized to higher
dimensional saddle points by the author. Here, let us explain this method using the power series expansion for the simplest case.

Let $P = (x_0, x_0)$ be a saddle fixed point of Hénon map $H_{b,c}$, and let $\lambda$ be an eigenvalue of the Jacobian matrix $DH_{b,c}(P)$ of modulus greater than 1. Let $E^u$ denote the eigenspace of $\lambda$. By Poincaré, there exists a transcendental entire map $\Phi : \mathbb{C} \to \mathbb{C}^2$, satisfying $\Phi(0) = P$, $\text{Image}(D\Phi_0) = E^u$, and the following function equation

$$
(*) \quad \Phi(\lambda z) = H_{b,c} \circ \Phi(z).
$$

His proof is quite simple. Compute the Taylor coefficients of $\Phi$, and show the convergence of the power series obtained from the Taylor coefficients. Let

$$
\Phi(z) = (x_0 + \varphi(z), x_0 + \psi(z)), \quad \varphi(z) = \sum_{k=1}^{\infty} \varphi_k z^k.
$$

By function equation $(*)$, we have the followings.

$$
x_0 + \varphi(\lambda z) = (x_0 + \varphi(z))^2 + c + b(x_0 + \psi(z)),
$$

$$
x_0 + \psi(\lambda z) = x_0 + \varphi(z).
$$

As $P$ is a fixed point of $H_{b,c}$, we can eliminate $\psi$ to obtain

$$
\varphi(\lambda z) = 2x_0 \varphi(z) + (\varphi(z))^2 + b\varphi\left(\frac{z}{\lambda}\right).
$$

Coefficients of the first order terms with respect to $z$ must satisfy

$$
\lambda \varphi_1 = 2x_0 \varphi_1 + \frac{b}{\lambda} \varphi_1,
$$

which is automatically satisfied for any $\varphi_1$, since $\lambda$ is an eigenvalue of the saddle point. Note that we can choose $\varphi_1$ to define a coordinate in the unstable manifold. For $k > 1$, by comparing the $k$-th order terms, we have

$$
\lambda^k \varphi_k = 2x_0 \varphi_k + \sum_{l=1}^{k-1} \varphi_l \varphi_{k-l} + \frac{b}{\lambda^k} \varphi_k
$$

so that $\varphi_k$ can be computed inductively by

$$
\varphi_k = \frac{\sum_{l=1}^{k-1} \varphi_l \varphi_{k-l}}{\lambda^k - 2x_0 - \frac{b}{\lambda^k}}.
$$
Note that in our case $2x_0 = \lambda - \frac{b}{\lambda}$, we assume $|b| < 1$, $|\lambda| > 1$. The denominators

$$\lambda^k - 2x_0 - \frac{b}{\lambda^k} = \lambda(\lambda^{k-1} - 1)(1 + \frac{b}{\lambda^{k+1}})$$

do not vanish and the power series

$$\varphi(z) = x_0 + \sum_{k=1}^{\infty} \varphi_k z^k$$

converges. Finally set $\psi(z) = \varphi\left(\frac{z}{\lambda}\right)$ to obtain $\Phi(z)$.

3. Periodic points of period two

In this section, we consider attracting cycles of period two. Although this case is easy to compute, we try to apply the discrete Fourier expansion method. There is only one cycle of period two for each parameter $(b,c)$. The following construction of the space of cycles may appear to be redundant.

Let $\{(x_n, x_{n-1})\}$ be the periodic orbit of period 2, and set

$$x_n = \alpha_0 + (-1)^n \alpha_1.$$  

The difference equation for $H_{b,c}$:

$$x_{n+1} = x_n^2 + c + bx_{n-1}$$

yields the following system of equations for Fourier coefficients.

(EF2)  \[ \begin{cases} 
\alpha_0 &= \alpha_0^2 + \alpha_1^2 + c + b\alpha_0 \\
-\alpha_1 &= 2\alpha_0\alpha_1 - b\alpha_1 
\end{cases} \]

By excluding the fixed point case, i.e., by assuming $\alpha_1 \neq 0$, we have

$$\alpha_0 = -\frac{1}{2} (1 - b), \quad \alpha_1 = -\frac{3}{4} (1 - b)^2 - c.$$  

The cycle of period two consists of two points $(\alpha_0 + \alpha_1, \alpha_0 - \alpha_1)$ and $(\alpha_0 - \alpha_1, \alpha_0 + \alpha_1)$. The value $\alpha_0$ represents the 2-cycle. Thus, $\alpha_0$ can be regarded as a **cycle variable** of period 2, and the **cycle equation** is

(CE2) \[ 2\alpha_0 + 1 - b = 0. \]
Let \( \tilde{\Gamma}_2 = \{(b, c), \alpha_0) \in \mathbb{C}^2 \times \mathbb{C} | \alpha_0 = -\frac{1}{2}(1 - b) \} \) denote the space of 2-cycles.

**Proposition 3.** \( \tilde{\Gamma}_2 \) is a branched double-covering space over \( \mathbb{C}_2 \), with critical locus \( \{b = 0\} \subseteq \tilde{\Gamma}_2 \) and branching locus \( \{\delta = 0\} \subseteq \mathbb{C}_2 \).

**Proof.** The determinant and the trace gives the branched covering map

\[
\delta = (-b)^2, \quad \tau = \text{tr}DH_{b,c}^2 = 4(\alpha_0^2 - \alpha_1^2) + 2b = 4(1 - b)^2 + 4c + 2b.
\]

and the “inverse” map is given by

\[
b = \pm \sqrt{\delta}, \quad c = \frac{1}{4}\tau - \frac{1}{2}b - (1 - b)^2, \quad \alpha_0 = -\frac{1}{2}(1 - b).
\]

Let \( \Omega_2 \) denote the set of parameters \((b, c) \in \mathbb{C}^2\) for which \( H_{b,c} \) has an attracting cycle of period two. And let \( \tilde{\Omega}_2 \) denote the set of points \(((b, c), \alpha_0) \in \tilde{\Gamma}_2\) with corresponding cycle being attracting.

**Proposition 4.** \( \tilde{\Omega}_2 \) is a branched double-covering space over \( D_2 \), with critical locus \( \{b = 0\} \subseteq \tilde{\Omega}_2 \) and branching locus \( \{\delta = 0\} \subseteq D_2 \).

**Proof.** Same as the preceding proposition. Each slice of \( \Omega_2 \) by \( \{b = \text{const.}\} \) is an ellipse.

### 4. Unstable manifold of periodic saddle

Unstable manifolds of periodic saddles can be expanded into power series. Similarly as in the case of fixed saddle, the coefficients of the power series can be computed inductively.

Let \( P_0, \ldots, P_{p-1} \) be a cycle of period \( p \), and let \( \lambda \) denote the eigenvalue of the Jacobian matrix \( DH_{b,c}^p \) at the periodic points. For \( i = 0, \ldots, p - 1 \), let

\[
P_i = (x_i, y_i), \quad \Phi_i : \mathbb{C} \to \mathbb{C}^2, \quad \Phi_i(z) = (x_i + \varphi_i(z), y_i + \psi_i(z))
\]

and assume

\[
\Phi_{i+1}(z) = H_{b,c} \circ \Phi_i(z), \quad \text{for } i = 0, \ldots, p - 2,
\]

and

\[
\Phi_0(\lambda z) = H_{b,c} \circ \Phi_{p-1}(z).
\]
These function equations yield the following system of function equations.

\[
\begin{align*}
  x_{i+1} + \varphi_{i+1}(z) &= (x_i + \varphi_i(z))^2 + c + b(y_i + \psi_i(z)) & (i = 0, \ldots, p - 2), \\
  y_{i+1} + \psi_{i+1}(z) &= x_i + \varphi_i(z)
\end{align*}
\]

and

\[
\begin{align*}
  x_0 + \varphi_0(\lambda z) &= (x_{p-1} + \varphi_{p-1}(z))^2 + c + b(y_{p-1} + \psi_{p-1}(z)) \\
  y_0 + \psi_0(\lambda z) &= x_{p-1} + \varphi_{p-1}(z)
\end{align*}
\]

As \( x_i \) and \( y_i \) are coordinates of periodic points, we have

\[
\begin{align*}
  \varphi_{i+1}(z) &= 2x_i \varphi_i(z) + (\varphi_i(z))^2 + b\psi_i(z) & (i = 0, \ldots, p - 2), \\
  \psi_{i+1}(z) &= \varphi_i(z)
\end{align*}
\]

and

\[
\begin{align*}
  \varphi_0(\lambda z) &= 2x_{p-1} \varphi_{p-1}(z) + (\varphi_{p-1}(z))^2 + b\psi_{p-1}(z) \\
  \psi_0(\lambda z) &= \varphi_{p-1}(z)
\end{align*}
\]

Expand functions \( \varphi_i(z) \) and \( \psi_i(z) \) to power series

\[
\varphi_i(z) = \sum_{k=1}^{\infty} \varphi_{i,k} z^k, \quad \psi_i(z) = \sum_{k=1}^{\infty} \psi_{i,k} z^k.
\]

Let \( DH_i \) denote the Jacobian matrix \( DH_{b,c} \) at \( P_i \). The linear terms must satisfy

\[
\begin{pmatrix}
  \varphi_{i+1,1} \\
  \psi_{i+1,1}
\end{pmatrix} = DH_i
\begin{pmatrix}
  \varphi_{i,1} \\
  \psi_{i,1}
\end{pmatrix}, \quad \text{for} \quad i = 0, \ldots, p - 2,
\]

and

\[
\begin{pmatrix}
  \lambda \varphi_{0,1} \\
  \lambda \psi_{0,1}
\end{pmatrix} = DH_{p-1} \begin{pmatrix}
  \varphi_{p-1,1} \\
  \psi_{p-1,1}
\end{pmatrix}.
\]

We can choose an eigenvector \( \begin{pmatrix} \varphi_{0,1} \\ \psi_{0,1} \end{pmatrix} \) of \( DH_{p-1} \cdots DH_0 \) to get values of the linear terms. Higher order coefficients are computed inductively as follows. Let

\[
\varphi_{i,k}^{[2]} = \sum_{l=1}^{k-1} \varphi_{i,l} \varphi_{i,k-l}
\]

denote the term of degree \( k \) of power series \((\varphi_i(z))^2\). Note that this term is determined by terms of orders smaller than \( k \).
The coefficients of degree $k$ greater than 1 satisfy
\[
\begin{pmatrix}
\varphi_{i+1,k} \\
\psi_{i+1,k}
\end{pmatrix} = DH_i \begin{pmatrix}
\varphi_{i,k} \\
\psi_{i,k}
\end{pmatrix} + \begin{pmatrix}
\varphi_{i+1,k}^{[2]} \\
0
\end{pmatrix}, \quad \text{for} \quad i = 0, \ldots, p - 2,
\]
and
\[
\lambda^k \begin{pmatrix}
\varphi_{0,k} \\
\psi_{0,k}
\end{pmatrix} = DH_{p-1} \begin{pmatrix}
\varphi_{p-1,k} \\
\psi_{p-1,k}
\end{pmatrix} + \begin{pmatrix}
\varphi_{p-1,k}^{[2]} \\
0
\end{pmatrix}.
\]

These equations can be solved as follows
\[
\begin{pmatrix}
\varphi_{0,k} \\
\psi_{0,k}
\end{pmatrix} = (\lambda^k E - DH_{p-1} \cdots DH_0)^{-1} \left( \sum_{j=0}^{p-1} DH_{p-1} \cdots DH_{j+1} \begin{pmatrix}
\varphi_{j,k}^{[2]} \\
0
\end{pmatrix} \right).
\]

These formulas recursively determine all the coefficients of the convergent power series $\Phi_0(z), \ldots, \Phi_{p-1}$.

5. Cycles of period three

In this section, we compute the periodic points of period three using the discrete Fourier expansion. Let $\omega = \frac{1}{2}(-1 + \sqrt{3}i)$ denote a cubic root of unity. Periodic sequence $\{x_n\}$ of period three can be expressed as
\[
x_n = u_0 + \omega^n u_1 + \omega^{2n} u_2.
\]

To exclude fixed point cases, we assume $(u_1, u_2) \neq (0, 0)$.

Sequence $P_n = (x_n, x_{n-1})$ is an orbit of $H_{b,c}$ if and only if
\[
x_{n+1} - bx_{n-1} = (x_n)^2 + c
\]
holds. We obtain the following system of equations for Fourier coefficients.

\[
\text{(EF3)} \quad \begin{cases}
(1 - b)u_0 &= u_0^2 + 2u_1u_2 + c, \\
(\omega - b\omega^2)u_1 &= u_1^2 + 2u_0u_1, \\
(\omega^2 - \omega b)u_2 &= u_1^2 + 2u_0u_2.
\end{cases}
\]

As we assumed $u_1u_2 \neq 0$, the second and the third equations give
\[
u_2^2u_1^{-1} = \omega - b\omega^2 - 2u_0, \quad u_1^2u_2^{-1} = \omega^2 - b\omega - 2u_0.
\]

The product and the sum of these yield
\[
u_1u_2 = 4u_0^2 + 2(1 - b)u_0 + b^2 + b + 1,
\]
\[
\frac{u_1^3 + u_2^3}{u_1 u_2} = -(1 - b) - 4u_0.
\]
Note that \(u_1\) and \(u_2\) are determined from \(b\) and \(u_0\). The cycle does not depend on the choice of \(u_1\) and \(u_2\) as solutions. Hence the value \(u_0\) represents the cycle. We call \(u_0\) the **cycle variable** of 3-cycle. The condition for this cycle to be a cycle of \(H_{b,c}\) reduces to equation
\[
c = -9u_0^2 - 3(1 - b)u_0 - 2(b^2 + b + 1),
\]
or
\[
(CE3) \quad 9u_0^2 + 3(1 - b)u_0 + 2(b^2 + b + 1) + c = 0,
\]
which we call the **cycle equation** of period 3.

For each parameter \((b, c)\), there are two cycles of period three. If the above equation has two distinct solutions for \(u_0\), they correspond to the two 3-cycles. For each root \(u_0\) of the cycle equation, \(u_1\) and \(u_2\) are obtained from the above equations. The choice of the cubic root corresponds to the choice of the starting point of the cycle. Saddle-node cycle must correspond to the double root case. The saddle-node parameters satisfy the discriminant equation
\[
9(1 - b)^2 - 36(c + 2(b^2 + b + 1)) = 0,
\]
or,
\[
(SN3_L) \quad c = -\frac{1}{4}(7b^2 + 10b + 7),
\]
which we call the **saddle-node equation** of period 3. The curve \(\{(b, c) \in \mathbb{C}^2 \mid c = -\frac{1}{4}(7b^2 + 10b + 7)\}\) in the parameter space is called the **saddle-node locus** of period 3. The value of cycle variable for the saddle-node cycle is given by the double root
\[
(SN3_C) \quad u_0 = -\frac{1}{6}(1 - b)
\]
of the cycle equation.

Note that this curve is invariant under change of parameters \((b, c) \mapsto (b^{-1}, cb^{-2})\), since the condition for cycle to have eigenvalue 1 is invariant under the involution \(H_{b,c} \mapsto H_{b,c}^{-1} \sim H_{b^{-1},cb^{-2}}\).

The cycle variable \(u_0\) of the saddle-node is the double root of the cycle equation and is given by \(u_0 = \frac{1}{6}(b - 1)\).
The trace of the Jacobian matrix $D(H^3_{b,c})$ along the cycle is given by
\[ \tau = \text{tr} \ D(H^3_{b,c}) = 8x_0x_1x_2 + 2b(x_0 + x_1 + x_2). \]

As
\[ x_0 + x_1 + x_2 = 3u_0, \]
and
\[ x_0x_1x_2 = u_0^3 + u_1^3 + u_2^3 - 3u_0u_1u_2 \]
\[ = -27u_0^3 - 18(1 - b)u_0^2 - 3(3b^2 + b + 3)u_0 - (1 - b^3), \]
we have a formula for trace of 3-cycle
\[(TF3) \quad \tau = -8 \left(27u_0^3 + 18(1 - b)u_0^2 + 9(b^2 + \frac{b}{4} + 1)u_0 + 1 - b^3 \right). \]

Let $\tilde{\Gamma}_3 = \{((b, c), u_0) \in \mathbb{C}^2 \times \mathbb{C} \mid c = -9u_0^2 - 3(1 - b)u_0 - 2(b^2 + b + 1)\}$ denote the cycle space of period 3. The trace $\tau : \tilde{\Gamma}_3 \to \mathbb{C}$, is called the trace function.

**PROPOSITION 5.** Holomorphic map $\chi : \tilde{\Gamma}_3 \to \mathbb{C}_2$, with $\chi((b, c), u_0) = (\delta, \tau)$, defined by
\[ \delta = \text{det} \ D(H^3_{b,c}) = -b^3, \quad \tau = \text{tr} \ D(H^3_{b,c}) \]
is regular near the saddle-node locus $\{c = -\frac{1}{4}(7b^2 + 10b + 7)\}$, except at $(b, c) = -\frac{1}{14}(13 \pm 3\sqrt{3}i)(1, \frac{3}{4})$ and $\{b = 0\}$.

**PROOF.** Partial derivative $\frac{\partial \tau}{\partial u_0}$ at saddle-node cycle $((b, -\frac{1}{4}(7b^2 + 10b + 7)), -\frac{1}{6}(1 - b)) \in \tilde{\Gamma}_3$ is given by
\[ \frac{\partial \tau}{\partial u_0} \bigg|_{u_0 = -\frac{1-b}{6}} = -6(7b^2 + 13b + 7). \]
Hence, $\chi$ is regular if $b \neq 0$ and $7b^2 + 13b + 7 \neq 0$. Note that $|b| = 1$ if $7b^2 + 13b + 7 = 0$.

**PROPOSITION 6.** Cycle of period 3 has eigenvalue 1 on the saddle-node locus $\{c = -\frac{1}{4}(7b^2 + 10b + 7)\}$ with $u_0 = -\frac{1}{6}(1 - b)$. On the parabolic bifurcation locus $\{u_0 = \frac{1}{2}(\omega - b\omega^2)\} \cup \{u_0 = \frac{1}{2}(\omega^2 - b\omega)\}$ of period tripling, the cycle degenerates to a parabolic fixed point. Apart from these cases, 3-cycle does not have eigenvalue 1.
Proof. When an eigenvalue $\lambda$ of the cycle is 1, the other eigenvalue is $-b^3$. The trace $\tau$ computed above can be decomposed as

$$\tau - (1 - b^3) = -216(u_0 + \frac{1}{6}(1 - b))(u_0^2 + \frac{1}{2}(1 - b)u_0 + \frac{1}{4}(b^2 + b + 1)).$$

The first factor gives the saddle-node locus. The second factor can be solved as

$$u_0 = \frac{1}{2}(\omega - b\omega^2) \text{ or } u_0 = \frac{1}{2}(\omega^2 - b\omega).$$

In these cases, we have $u_1 = u_2 = 0$. Hence the cycle degenerates to a fixed point. The parameter $c$ is given by

$$c = \frac{1}{4}(2\omega^2 - \omega)b^2 + b + \frac{1}{4}(2\omega - \omega^2)$$

or

$$c = \frac{1}{4}(2\omega - \omega^2)b^2 + b + \frac{1}{4}(2\omega^2 - \omega).$$

These parameters correspond to the bifurcation locus.

Let $\tilde{\Omega}_3 = \chi^{-1}(D_2) \subset \tilde{\Gamma}_3$ denote the set of attracting cycles of period 3. And let $\Omega_3 \subset \mathbb{C}^2$ denote its projected image in the parameter space.

Proposition 7. For each parameter $b$ with $|b| < 1$, locus of attracting cycles $\Omega_3 \cap (\{b\} \times \mathbb{C})$ has a cusp point at $c = -\frac{1}{4}(7b^2 + 10b + 7)$.

Proof. We fix the parameter $b$. Then the trace $\tau$ can be regarded as a holomorphic function of the cycle variable $u_0$. To the saddle-node parameter value $(b, c)$ corresponds a saddle-node cycle $((b, c), \frac{1}{6}(1 - b)) \in \tilde{\Gamma}_3$. As computed in proposition 5, $\frac{\partial \tau}{\partial u_0} \neq 0$ there. At the saddle-node cycle, one of the eigenvalues takes value 1, so that $\tau = 1 - b^3$. There is a neighborhood of $\lambda = 1$, such that the local inverse map $\lambda \mapsto u_0$ is holomorphic and regular. Projection from the cycle space to the parameter space, for fixed $b$ has a quadratic singularity at the saddle-node cycle point. Hence, the locus of attracting cycles must include the image of a part of the unit disk in the $\lambda$ space by a holomorphic map with quadratic singularity at $\lambda = 1$. More precisely, the “cardioid-like” set may intersect with itself. In this case, instead of the usual saddle-node bifurcation, we may see “collision” of two coexisting attracting cycles.

Proposition 8. Saddle-node locus of 3-cycle and period doubling locus of fixed point intersect at two points $(b, c) = (-2 \pm \sqrt{3}, 9(1 \mp \frac{1}{2}\sqrt{3})).$
**Proof.** Saddle-node locus of period three cycle is given by \( c = -\frac{1}{4}(7b^2 + 10b + 7) \) and period doubling locus of period 1 to 2 is given by \( c = -\frac{3}{4}(1 - b)^2 \). By eliminating \( c \) from these equations, we obtain

\[
b^2 + 4b + 1 = 0,
\]

from which we get

\[
b = -2 \pm \sqrt{3}, \quad c = 9 \mp \frac{9}{2}\sqrt{3}.
\]

Note that \( b + \frac{1}{b} = -4 \). This fact will be used in section 9.

6. Cycles of period four

In this section, we compute the periodic points of period four. Periodic sequence \( \{x_n\} \) of period four can be expressed as

\[
x_n = v_0 + i^n v_1 + (-1)^n v_2 + (-i)^n v_3.
\]

As in the previous section, sequence \( \{x_n x_{n-1}\} \) is an orbit of \( H_{b,c} \) if and only if

\[
x_{n+1} - bx_{n-1} = (x_n)^2 + c
\]

holds. We obtain the following system of equations for Fourier coefficients.

\[
\begin{cases}
(1 - b)v_0 = v_0^2 + v_2^2 + 2v_1v_3 + c \\
i(1 + b)v_1 = 2(v_0v_1 + v_2v_3) \\
-(1 - b)v_2 = v_1^2 + v_3^2 + 2v_0v_2 \\
-i(1 + b)v_3 = 2(v_1v_2 + v_0v_3)
\end{cases}
\]

By setting \( b_1 = 1 - b \) and \( b_2 = i(1 + b)/2 \), we get the following system of equations.

\[
\begin{cases}
b_1v_0 = v_0^2 + v_2^2 + 2v_1v_3 + c \\
b_2v_1 = v_0v_1 + v_2v_3 \\
-b_1v_2 = v_1^2 + v_3^2 + 2v_0v_2 \\
-b_2v_3 = v_1v_2 + v_0v_3
\end{cases}
\]

Let us eliminate \( v_1, v_2, v_3 \) to obtain an equation of \( v_0 \), which represents the cycle. Second and fourth line of (*) are equivalent to

\[
\begin{pmatrix}
v_0 & v_2 \\
-v_2 & -v_0
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_3
\end{pmatrix}
= b_2
\begin{pmatrix}
v_1 \\
v_3
\end{pmatrix}.
\]
To exclude cycles of period 2 or 1, we assume \((v_1, v_3) \neq (0, 0)\). We get
\[
v_2^2 = v_0^2 - b_2^2. \tag{**}
\]
Making sums and differences of equations, system of equations (\(\ast\)) is equivalent to the following.
\[
\begin{align*}
  b_1(v_0 - v_2) &= (v_0 + v_2)^2 + (v_1 + v_3)^2 + c \\
  b_1(v_0 + v_2) &= (v_0 - v_2)^2 - (v_1 - v_3)^2 + c \\
  b_2(v_1 - v_3) &= (v_0 + v_2)(v_1 + v_3) \\
  b_2(v_1 + v_3) &= (v_0 - v_2)(v_1 - v_3)
\end{align*}
\]
By setting
\[
s_0 = v_0 + v_2, \quad s_1 = v_1 + v_3, \quad t_0 = v_0 - v_2, \quad t_1 = v_1 - v_3,
\]
it is rewritten as follows.
\[
\begin{align*}
  b_1 t_0 &= s_0^2 + s_1^2 + c \\
  b_1 s_0 &= t_0^2 - t_1^2 + c \\
  b_2 t_1 &= s_0 s_1 \\
  b_2 s_1 &= t_0 t_1
\end{align*}
\]
Let us assume \(b_2 \neq 0, \text{i.e., } b \neq -1\). To suppress cycles of periods smaller than 4, we can assume \(s_1 t_1 \neq 0\). Then, we have \(s_0 t_0 = b_2^2 \neq 0\).
By using these, our system of equations can be rewritten as follows.
\[
\begin{align*}
  t_0 &= \frac{b_2^2}{s_0} \\
  t_1 &= \frac{s_0 s_1}{b_2} \\
  s_1^2 &= b_1 \frac{b_2^2}{s_0} - s_0^2 - c \\
  b_1 s_0 &= \frac{b_2^4}{s_0^2} - \frac{s_0^2}{b_2^2} \left( b_1 \frac{b_2^2}{s_0} - s_0^2 - c \right) + c
\end{align*}
\]
Note that \(\ast \ast \ast\) is equivalent to \((\ast)\), under the condition \(b_2 s_1 t_1 \neq 0\). And note that the last equation, divided by \(s_0 b_2\), gives
\[
\left( -\frac{b_2}{s_0} \right)^3 + \left( -\frac{s_0}{b_2} \right)^3 - \frac{2b_1}{b_2} + \frac{1}{b_2^4} \left( \frac{s_0}{b_2} + \frac{b_2}{s_0} \right) c = 0.
\]
Now, as \( b_2^2 = v_0^2 - v_2^2 = s_0 t_0 \), we have

\[
\frac{s_0}{b_2} + \frac{b_2}{s_0} = \frac{s_0^2 + b_2^2}{b_2 s_0} = \frac{s_0 + t_0}{b_2} = \frac{2v_0}{b_2}.
\]

When we need to solve back from \( v_0 \) to \( s_0 \), use quadratic equation

\[
s_0^2 - 2v_0 s_0 + b_2^2 = 0.
\]

We obtain the cycle equation of period four, by eliminating \( s_0 \),

\[
v_0^3 + \frac{1}{4}(c - 3b_2^2)v_0 - \frac{1}{4}b_1 b_2^2 = 0.
\]

Or, in terms of parameters \((b, c)\),

\[
(CE4) \quad v_0^3 + \frac{1}{4}(c + \frac{3}{4}(b + 1)^2)v_0 - \frac{1}{16}(b^2 - 1)(b + 1) = 0.
\]

If \( v_0 \) is a solution of this equation, by (**) and (***) , other Fourier coefficients are obtained and a cycle of period four is derived. Hence, each solution \( v_0 \) represents a 4-cycle, and is considered as a cycle variable.

Let

\[
\bar{\Gamma}_4 = \{(b, c), v_0) \in \mathbb{C}^2 \times \mathbb{C} \mid v_0^3 + \frac{1}{4}(c + \frac{3}{4}(b + 1)^2)v_0 - \frac{1}{16}(b^2 - 1)(b + 1) = 0\}
\]

denote the space of 4-cycles. It is a co-dimension one hyper-surface with a singularity at \((-1, 0), 0\). Let \( \bar{\Omega}_4 \) denote the subset of \( \bar{\Gamma}_4 \) of attracting cycles, and let \( \Omega_4 \) denote its projection into the parameter space.

Let \( \tau = \text{tr} \, DH_{b,c}^4 \) denote the trace of the derivative along the cycle,

\[
\tau = 16x_3 x_2 x_1 x_0 + 4b(x_3 x_2 + x_2 x_1 + x_1 x_0 + x_0 x_3) + 2b^2
\]

\[
= 16(t_0^2 + t_1^2)(s_0^2 - s_1^2) + 16bs_0 t_0 + 2b^2.
\]

Here, we used, for example,

\[
x_3 x_1 = (v_0 - iv_1 - v_2 + iv_3)(v_0 + iv_1 - v_2 - iv_3) = (v_0 - v_2)^2 + (v_1 - v_3)^2 = t_0^2 + t_1^2,
\]

\[
x_2 x_0 = (v_0 - v_1 + v_2 - v_3)(v_0 + v_1 + v_2 + v_3) = (v_0 + v_2)^2 - (v_1 + v_3)^2 = s_0^2 - s_1^2.
\]

For fixed \( b \), trace \( \tau \) can be expressed as a rational function of cycle variable \( v_0 \). To see this, note that \( c \) can be expressed as

\[
c = -4v_0^2 + \frac{1}{4v_0}(b^2 - 1)(b + 1) - \frac{3}{4}(b + 1)^2.
\]
and, by eliminating $s_0, s_1, t_0, t_1$, we have
\begin{equation}
\tau = 16(-16b_1v_0^3 + 8cv_0^2 + 2b_1(6b_2^2 - 2c)v_0 + 4b_2^4 + b_1^2b_2^2 + c^2 - 4b_2^2c) + 16bb_2^2 + 2b^2.
\end{equation}
This trace function will be decomposed in section 8.

7. Saddle-node locus of 4-cycle

**Proposition 9.** Saddle-node locus of 4-cycle is given by
\begin{equation}
c = -\frac{3}{4} \left( b + 1 \right) \left( b + 1 + \frac{3}{4} \sqrt{4(b-1)^2(b+1)} \right),
\end{equation}
and the cycle variable of the saddle-node is given by
\begin{equation}
v_0 = \frac{1}{4} \frac{3}{2} \sqrt{2(b+1)^2(1-b)},
\end{equation}
where choice of cubic root is specified by
\begin{equation}
c = -\frac{3}{4} \left( b + 1 \right)^2 - 12v_0^2.
\end{equation}

**Proof.** The cycle equation
\begin{equation}
v_0^3 + \frac{1}{4} (c - 3b_2^2)v_0 - \frac{1}{4} b_1b_2^2 = 0
\end{equation}
of 4-cycle has a double root at saddle-node locus. Hence, the discriminant of the cubic equation must vanish there. We have the saddle-node equation,
\begin{equation}
4 \left( \frac{1}{4} (c - 3b_2^2) \right)^3 + 27 \left( -\frac{1}{4} b_1b_2^2 \right)^2 = 0,
\end{equation}
or,
\begin{equation}
(SN_{4L}) \quad (c - 3b_2^2)^3 = -27b_1^2b_2^4.
\end{equation}
Hence, we have a formula for the saddle-node locus,
\begin{align*}
c &= 3b_2^2 - 3 \frac{3}{2} \sqrt{b_1^2b_2^4} \\
&= 3 \left( \frac{(b+1)i}{2} \right)^2 - 3 \sqrt{2^2} \left( \frac{(b+1)i}{2} \right)^4
\end{align*}
The cycle variable of the saddle-node $v_0$ is the double root of the cycle equation. From the derived function

$$3v_0^2 + \frac{1}{4}(c - 3b_2^2) = 0$$

of the cycle equation, we have

$$c = 3b_2^2 - 12v_0^2.$$ 

Use this formula to eliminate $c$ in the cycle equation to get

$$(SN4C) \quad v_0^3 = -\frac{1}{8}b_1b_2^2.$$

Note that for fixed $b$, there are three saddle-node loci of $c$, which form the three vertices of a regular triangle.

8. Regularity of trace

In section 6, we computed the trace function $\tau$ of 4-cycles. As the determinant of Jacobian matrix $DH_{b,c}^4$ is $b^4$, the value of the trace for saddle-node cycle is given by $\tau = 1 + b^4$. Eigenvalue of cycle can be 1 for cycles bifurcating from period doubling from 2-cycle, or period quadrupling from fixed point (satellite bifurcation).

First, let us verify that $\tau = 1 + b^4$ for saddle-node cycles. Use saddle-node condition $c = 3b_2^2 - 12v_0^2$ to eliminate $c$ from the trace to get

$$\tau = 16((8v_0^3 + b_1b_2^2)(6v_0 + b_1) + b_2^4) + 16b_2^2 + 2b^2.$$ 

Then use $v_0^3 = -\frac{1}{8}b_1b_2^2$ and $b_2^2 = -\frac{1}{4}(b + 1)^2$ to get $\tau = 1 + b^4$ at saddle-node cycles.

So $\tau - (1 + b^4)$, as a function of $v_0$ should be factorized. Set $c = \Delta - 12v_0^2 + 3b_2^2$ and eliminate $c$ in $\tau$, to decompose it as

$$\tau - (1 + b^4) = 16\Delta(-10v_0^2 - b_1v_0 + \Delta + 2b_2^2).$$

Then set back $\Delta = c - 3b_2^2 + 12v_0^2$ and see

$$\tau - (1 + b^4) = 16(c - 3b_2^2 + 12v_0^2)(c + 2v_0^2 - b_1v_0 - b_2^2).$$
The last factor of the above gives rise to equation
\[ c = -2v_0^2 + b_1v_0 + b_2^2. \]
This equation, combined with the cycle equation
\[ v_0^3 + \frac{1}{4}(c - 3b_2^2)v_0 - \frac{1}{4}b_1b_2^2 = 0, \]
gives the following equation, by eliminating \( c \).
\[ (v_0^2 - b_2^2)(v_0 + \frac{1}{2}b_1) = 0. \]
Solutions \( v_0 = \pm b_2 \), with \( c = \pm b_1b_2 - b_2^2 \), corresponds to the locus of period quadrupling bifurcation from a fixed point. For, from (\( * \)), \( v_2 = 0 \) follows and and from (\( ** \)), \( v_1 = v_3 = 0 \) follow. And solution \( v_0 = -\frac{1}{2}b_1 \), with \( c = -\frac{1}{2}(b^2 + 1) \) corresponds to the locus of period-doubling from 2-cycle. In this case, as \( 2v_0 = -b_1 \), from the third equation of (\( * \)) of section 6, we have \( v_1^2 + v_3^2 = 0 \). And, from
\[ \frac{s_0}{b_2} + \frac{b_2}{s_0} = \frac{2v_0}{b_2} = -\frac{b_1}{b_2}, \]
\[ s_0 = -\frac{1}{2}b_1 \pm \frac{1}{2}\sqrt{b_1^2 - b_2^2}, \]
we have
\[ v_2^2 = \frac{1}{4}b_1^2 - b_2^2 = \frac{1}{2}(b^2 + 1). \]
Use these and the first equation of (\( * \)) to obtain \( v_1v_3 = 0 \), which shows that our cycle degenerates to a cycle of period two.

PROPOSITION 10. Holomorphic map \( \chi : \tilde{\Gamma}_4 \to \mathbb{C}_2 \), with \( \chi(b, c, v_0) = (\delta, \tau) \), defined by
\[ \delta = \det D(H_{b,c}^4) = b^4, \quad \tau = \text{tr} \ D(H_{b,c}^4) \]
is regular on saddle-node locus \( \{c = -\frac{3}{4}(b+1)(b+1+\frac{3}{4}(b-1)^2(b+1))\} \), for \( 0 < |b| < 1 \), saddle-node being specified in proposition 9.

PROOF. Trace \( \tau \) is a function of \( b \) and \( v_0 \) computed at the end of section 6. And \( c \) in the formula of \( \tau \) should be considered as a function of \( b \) and \( v_0 \). By differentiating the cycle equation with respect to \( v_0 \), we have
\[ 3v_0^2 + \frac{1}{4}(c - 3b_2^2) + \frac{1}{4}v_0 \frac{\partial c}{\partial v_0} = 0. \]
As $c = 3b_2^2 - 12v_0^2$ holds at the saddle-node cycle, we see $\frac{\partial c}{\partial v_0} = 0$ if $v_0 \neq 0$. For saddle-node cycle locus, we have

$$\frac{\partial \tau}{\partial v_0} = 16(-48b_1v_0^2 + 16cv_0 + 2b_1(6b_2^2 - 2c))$$

$$= 16(-48b_1v_0^2 + 16(3b_2^2 - 12v_0^2)v_0 + 2b_1(6b_2^2 - 2(3b_2^2 - 12v_0^2)))$$

$$= 96(-8b_1v_0^2 + 8b_2^2v_0 - 32v_0^3 + b_1(2b_2^2 - 2b_2^2 + 8v_0^2))$$

$$= 96(-32v_0^3 + 8b_2^2v_0) = 96(4b_1b_2^2 + 8b_2^2v_0) = 384b_2^2(2v_0 + b_1).$$

As $v_0$ corresponds to a saddle-node cycle, $v_0^3 = -\frac{1}{8}b_1b_2^2$, therefore the case $2v_0 + b_1 = 0$ implies $b = 1$ or $b = \frac{1}{5}(3 \pm 4i)$, and the case $b_2 = 0$ implies $b = \pm 1$. In all of these cases, $|b| = 1$. The case $v_0 = 0$ can occur only when $b = \pm 1$. Hence, we conclude that $\tau$ is regular with respect to $v_0$ along the saddle-node locus, if $|b| < 1$.

**Proposition 11.** For each parameter $b$ with $|b| < 1$, locus of attracting cycles $\Omega_4 \cap (\{b\} \times \mathbb{C})$ has cusp points at saddle-node loci.

**Proof.** The proof is similar to that of proposition 7. We fix the parameter $b$. Trace $\tau$ can be regarded as a holomorphic function of the cycle variable $v_0$. To the saddle-node parameter value $(b, c)$ corresponds a saddle-node cycle $((b, c), v_0) \in \tilde{\Gamma}_4$, as specified in proposition 9. As computed in proposition 10, $\frac{\partial \tau}{\partial v_0} \neq 0$ there. At the saddle-node cycle, one of the eigenvalues takes value 1, so that $\tau = 1 + b^4$. There is a neighborhood of $\lambda = 1$, such that the inverse map $\lambda \mapsto u_0$ is holomorphic and regular. Projection from the cycle space to the parameter space, for fixed $b$ has a quadratic singularity at the saddle-node cycle point. The cycle equation in the definition of $\Gamma_4$,

$$v_0^3 + \frac{1}{4}(c - 3b_2^2)v_0 - \frac{1}{4}b_1b_2^2 = 0,$$

defines $c$ as a rational function of $v_0$ for fixed $b$. By derivations with respect to $v_0$, we have

$$3v_0^2 + \frac{1}{4}(c - 3b_2^2) + \frac{1}{4}v_0 \frac{\partial c}{\partial v_0} = 0,$$

and

$$6v_0 + \frac{1}{2} \frac{\partial c}{\partial v_0} + \frac{1}{4}v_0 \frac{\partial^2 c}{\partial v_0^2} = 0.$$
At the saddle-node cycle, saddle-node condition
\[3v_0^2 + \frac{1}{4}(c - 3b_2^2) = 0\]
is satisfied. Hence we have \(\frac{\partial c}{\partial v_0} = 0\) and \(\frac{\partial^2 c}{\partial v_0^2} = -24\), since \(v_0 \neq 0\). Hence, the locus of attracting cycles must include the image of a part of the unit disk in the \(\lambda\) space by a holomorphic map with quadratic singularity at \(\lambda = 1\). More precisely, the “cardioid-like” set may intersect with itself. The image of the unit circle in the \(\lambda\) space forms a cusp in the \(c\) space.

9. Coexistence of attractive cycles

In this section, we prove the existence of an open set of parameters \((b, c)\), such that \(H_{b,c}\) has an attractive fixed point, an attractive 3-cycle, and an attractive 4-cycle, at the same time.

**Proposition 12.** The intersection \(\Omega_1 \cap \Omega_3 \cap \Omega_4\) is not empty.

**Proof.** First, let us examine the intersection of the saddle-node loci of 3-cycles and the saddle-node loci of 4-cycles. The saddle-node locus of 3-cycles is given by
\[c = -\frac{1}{4}(7b^2 + 10b + 7),\]
and the saddle-node locus of 4-cycles is given by
\[(c + \frac{3}{4}(b + 1)^2)^3 + \frac{27}{16}(b - 1)^2(b + 1)^4 = 0\]
from proposition 9. Assume \(b \neq 0\), and let \(w = b + \frac{1}{b}\). Eliminate \(c\) from two equations above and obtain an equation of \(w\),
\[11w^3 + 6w^2 - 156w - 232 = 0.\]
Let \(f(w) = 11w^3 + 6w^2 - 156w - 232\). As \(f(-4) = -216, f(-2) = 16, f(0) = -232,\) and \(f(2) = -432\), there are three real solutions \(w_1, w_2, w_3\) with
\[-4 < w_1 < -2 < w_2 < 0, \quad 2 < w_3.\]
For each solution \(w_k\), we have solutions \(b\) of quadratic equation \(b^2 - w_k b + 1 = 0\). The six solutions give the intersection points of the saddle-node curves, specified by \(c = -\frac{1}{4}(7b^2 + 10b + 7)\). For \(w_2\), the solution \(b\) are
non-real and $|b| = 1$. For $w_3$, the solutions are real and positive. For $w_1$, the solutions, say $\beta_1, \beta_2$, are real and negative, with $-2 - \sqrt{3} < \beta_1 < -1 < \beta_2 < -2 + \sqrt{3} < 0$. As noted in proposition 8, the intersection point $(\beta_2, -\frac{1}{4}(7\beta_2 + 10\beta + 7))$ belongs to $\Omega_1$. By proposition 7, $\Omega_3 \cap (\{\beta_2\} \times \mathbb{C})$ has a cusp point at this intersection point, and by proposition 11, $\Omega_4 \cap (\{\beta_2\} \times \mathbb{C})$ has a cusp point there, too. These slices of cardioid-like shape intersect in an open set. This shows the non-emptiness of $\Omega_1 \cap \Omega_3 \cap \Omega_4$.

Following picture shows the real section of the parameter space.
attracting fixed point region. The right picture shows a complex slice near the intersection point of the saddle-node locus of periods 3 and 4.

The following picture shows a portion of unstable manifold of a saddle fixed point. Three different basins of attractions are observed. Parameters for this picture are $b = -0.3946, c = -1.0362$.

10. **Cycles of period five**

Let us compute the cycles of period five. Let $\kappa = \exp(2\pi i/5)$ denote a quintic root of unity. We begin with the discrete Fourier expansion. Let

$$x_n = \sum_{k=0}^{4} a_k \kappa^{nk}.$$

From our difference scheme

$$x_{n+1} - bx_{n-1} = x_n^2 + c,$$
we have the following system of equations for Fourier coefficients.

\[
(\text{EF5}) \quad \begin{cases}
(1 - b)a_0 &= a_0^2 + 2a_1a_4 + 2a_2a_3 + c \\
(\kappa - bk^4)a_1 &= 2a_0a_1 + 2a_2a_4 + a_3^2 \\
(\kappa^2 - bk^4)a_2 &= 2a_0a_2 + 2a_3a_4 + a_1^2 \\
(\kappa^3 - bk^2)a_3 &= 2a_0a_3 + 2a_1a_2 + a_4^2 \\
(\kappa^4 - bk)a_4 &= 2a_0a_4 + 2a_1a_3 + a_2^2
\end{cases}
\]

Let \(b_1 = 1 - b\), \(b_3 = 1 + b\), and set constants

\[
\lambda_1 = \kappa + \kappa^4 = -\frac{1}{2} (1 - \sqrt{5}), \quad \lambda_2 = \kappa - \kappa^4 = \sqrt{\frac{1}{2} (5 + \sqrt{5})} i, \\
\gamma_1 = \kappa^2 + \kappa^3 = -\frac{1}{2} (1 + \sqrt{5}), \quad \gamma_2 = \kappa^2 - \kappa^3 = \sqrt{\frac{1}{2} (5 - \sqrt{5})} i.
\]

Let

\[
p_1 = a_1a_2^2, \quad p_2 = a_2a_4^2, \quad p_3 = a_3a_1^2, \quad p_4 = a_4a_3^2,
\]

and

\[
r = a_1a_4, \quad q = a_2a_3.
\]

The “center” of cycle \(a_0\) and these variables do not depend on the choice of initial point of the periodic orbit. The system of equations is converted to the following system.

\[
\begin{cases}
  b_1a_0 &= a_0^2 + 2(r + q) + c \\
(\lambda_1b_1 - 4a_0)r &= p_1 + 2p_2 + 2p_3 + p_4 \\
(\gamma_1b_1 - 4a_0)q &= 2p_1 + p_2 + p_3 + 2p_4 \\
  \lambda_2b_3r &= -p_1 + 2p_2 - 2p_3 + p_4 \\
  \gamma_2b_3q &= -2p_1 - p_2 + p_3 + 2p_4
\end{cases}
\]

Set \(g_1 = \gamma_1b_1 - 4a_0\), \(g_2 = \gamma_2b_3\), \(l_1 = \lambda_1b_1 - 4a_0\), and \(l_2 = \lambda_2b_3\), to simplify the system and get

\[
\begin{pmatrix}
  l_1r \\
  g_1q \\
  l_2r \\
  g_2q
\end{pmatrix} =
\begin{pmatrix}
  1 & 2 & 2 & 1 \\
  2 & 1 & 1 & 2 \\
 -1 & 2 & -2 & 1 \\
-2 & -1 & 1 & 2
\end{pmatrix}
\begin{pmatrix}
  p_1 \\
  p_2 \\
  p_3 \\
  p_4
\end{pmatrix}.
\]

We have

\[
a_1a_2^2 = p_1 = \left( -\frac{1}{6} l_1 - \frac{1}{10} l_2 \right) r + \left( \frac{1}{3} g_1 + \frac{1}{5} g_2 \right) q.
\]
\[
a_2a_4^2 = p_2 = \left(\frac{1}{3}l_1 + \frac{1}{5}l_2\right)r + \left(-\frac{1}{6}g_1 - \frac{1}{10}g_2\right)q,
\]
\[
a_3a_1^2 = p_3 = \left(\frac{1}{3}l_1 - \frac{1}{5}l_2\right)r + \left(-\frac{1}{6}g_1 + \frac{1}{10}g_2\right)q,
\]
\[
a_4a_3^2 = p_4 = \left(-\frac{1}{6}l_1 + \frac{1}{10}l_2\right)r + \left(\frac{1}{3}g_1 + \frac{1}{5}g_2\right)q.
\]

If \(rq = 0\) and \(|b| < 1\), then from the equations above, we see that the cycle degenerates to a fixed point. In the following, we assume \(rq \neq 0\).

Now, we introduce a new variable \(\rho\) by
\[
\rho = \frac{r}{q}.
\]

Then, as
\[
\frac{p_1p_4}{q^2} = \frac{1}{4} \left(\frac{1}{9}l_1^2 - \frac{1}{25}l_2^2\right)\rho^2 - \left(\frac{1}{9}l_1g_1 + \frac{1}{25}l_2g_2\right)\rho + \frac{1}{9}g_1^2 - \frac{1}{25}g_2^2,
\]
\[
\frac{p_2p_3}{q^2} = \left(\frac{1}{9}l_1^2 - \frac{1}{25}l_2^2\right)\rho^2 - \left(\frac{1}{9}l_1g_1 - \frac{1}{25}l_2g_2\right)\rho + \frac{1}{4} \left(\frac{1}{9}g_1^2 - \frac{1}{25}g_2^2\right).
\]
By eliminating \(r\) from these equations, we get a following cubic equation of \(\rho\), with coefficients being functions of \(b\) and \(a_0\).
\[
e_3\rho^3 + e_2\rho^2 + e_1\rho + e_0 = 0,
\]
where,
\[
e_3 = 25l_1^2 - 9l_2^2,
\]
\[
e_2 = -100l_1(g_1 + l_1) - 36l_2(g_2 - l_2),
\]
\[
e_1 = 100g_1(g_1 + l_1) - 36g_2(g_2 + l_2),
\]
\[
e_0 = -25g_1^2 + 9g_2^2.
\]

For fixed parameter value \(b\), this equation is of degree 3 in \(\rho\) and of degree 2 in \(a_0\). This equation, expressed as a quadratic equation in \(a_0\), is as follows.
\[
f_2a_0^2 + f_1a_0 + f_0 = 0,
\]
with
\[
f_2 = 400(\rho^3 - 8\rho^2 + 8\rho - 1),
\]
\[ f_1 = 200b_1(-\lambda_1\rho^3 - 2(1 - 2\lambda_1)\rho^2 + 2(1 - 2\gamma_1)\rho + \gamma_1), \]
\[ f_0 = (25(2 + \gamma_1)b_1^2 + 9(2 - \gamma_1)b_3^2)\rho^3 - (100(1 + \gamma_1)b_1^2 + 36(2 - \lambda_1)b_3^2)\rho^2 \]
\[ + (100(1 + \lambda_1)b_1^2 + 36(2 - \gamma_1)b_3^2)\rho - 25(2 + \lambda_1)b_1^2 - 9(2 - \lambda_1)b_3^2. \]

Unfortunately, in the case of cycles of period five, we do not have a single “cycle equation” to find a cycle variable, from given parameters \((b, c)\). Our “cycle equation” is given by a system of equations:

\[
\begin{align*}
(CE5) \quad \left\{ 
\begin{array}{l}
2a_0^2 + f_1 + f_0 &= 0, \\
\rho &= b_1a_0 - a_0^2 - 2(r + q). \\
\end{array}
\right.
\]

One equation to determine the cycle variable, and the other to describe the relation to parameter \(c\).

However, if we are given a parameter value of \(b\) and a value \(\rho\), we can solve the above equation to find \(a_0\) (two solutions). Or alternately, if we are given parameter value of \(b\) and a value of the “center” \(a_0\), we can solve the above equation to find \(\rho\) (three solutions). And \(r, q\) and \(c\) are computed by

\[
r = \frac{1}{4}(\frac{1}{9}l_1^2 - \frac{1}{25}l_2^2)\rho^2 + (\frac{1}{9}l_1g_1 - \frac{1}{25}l_2g_2)\rho + \frac{1}{9}g_1^2 - \frac{1}{25}g_2^2, \\
q = r/\rho, \\
c = b_1a_0 - a_0^2 - 2(r + q). 
\]

Finally, with these data, we can compute \(a_1, a_2, a_3,\) and \(a_4\) as follows. Take a quintic root of

\[
a_1^5 = p_1p_3^2/q^2. 
\]

Then

\[
a_3 = p_3/a_1^2, \quad a_2 = q/a_3, \quad a_4 = r/a_1, 
\]

which gives the periodic points.

Some complex slices \(\{b = -0.3847 + 0.0854i\}\), and \(\{b = 0.15\}\) are shown in the following picture. Hyperbolic components, so to say, of periods up to 5 are seen. The three-petals region in the enlargement picture represents a region of attracting 5-cycles. It is a slice of the swallow’s tail region for period 5 attractor.
The following picture shows the real axis slice of the parameter space. For parameters in the swallow’s tail region, there is/are attracting 5-cycle(s).

Next picture is an enlargement of the swallow’s tail region.
Next picture shows a system with attracting cycles of periods 1, 3, and 5.

Next picture is a system with attracting cycles of periods 1, 4, and 5.
References