Coexisting attractive cycles in Hénon dynamics

Shigehiro Ushiki
Graduate School of Human and Environmental Studies
Kyoto University

Abstract

The Hénon map can have three attracting cycles for certain values of parameters. A rigorous proof is given for some parameter values having attracting cycles of periods 1,3 and 4. The proof is given by elementary calculations. Saddle-node locus curves in the parameter space for periodic points of periods 3 and 4 are computed. Formula for periodic points of period upto five are obtained.

0. Introduction

Classical Hénon map is given by the following formula.

$$\begin{cases} x_{n+1} &= 1 + y_n - a x_n^2 \\ y_{n+1} &= b x_n \end{cases}$$

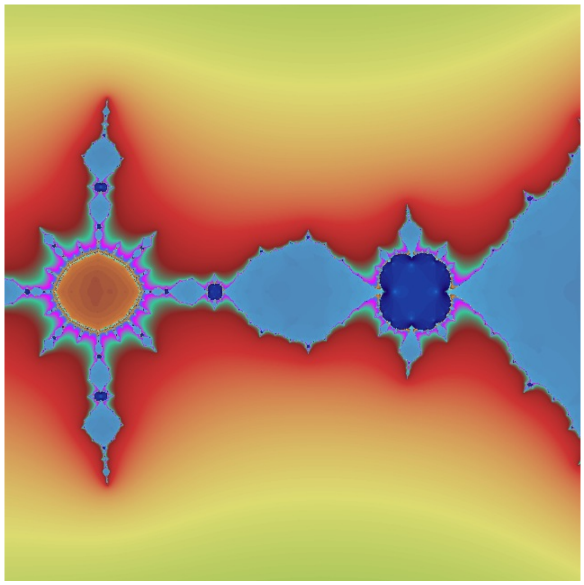
In this note, the Hénon map $H_{b,c} : (x, y) \mapsto (X, Y)$ is defined by the following formula.

$$\begin{cases} X &= x^2 + c + by \\ Y &= x \end{cases}$$

These maps are conjugate by affine change of coordinates with parameter correspondence $c = -a$. In this note, we try to compute periodic points of period up to 5 explicitly. Parameter value is detected for which saddle node bifurcation of period three periodic point and saddle node bifurcation of period four occur simultaneously while one fixed point remains attracting. Further calculation reveals the existence of attracting cycles of periods 1,3 and 4 for some nearby parameters.

A variable will be called a **cycle variable**, if a cycle can be computed from the variable, and the value can be computed from a cycle. An equation of cycle variable and parameters will be called a **cycle equation**.

The following picture shows an example of a part of unstable manifold of a saddle point. Observe that there are three basins of attractions.



1. Attracting fixed point

Fixed points are of the form (x_0, x_0) with x_0 given as a root of quadratic equation

$$x_0^2 - (1-b)x_0 + c = 0,$$

i.e.,

$$x_0 = \frac{1}{2}(1 - b \pm \sqrt{(1-b)^2 - 4c}),$$

and

$$c = -(x_0 - \frac{1}{2}(1-b))^2 + \frac{1}{4}(1-b)^2.$$

The value x_0 can be regarded as a **cycle variable** of 1-cycles. And the **cycle equation** of 1-cycle is

$$\text{(CE1)} \qquad x_0^2 - (1-b)x_0 + c = 0.$$

The Jacobian matrix of $H_{b,c}$ is given by

$$DH_{b,c,(x,y)} = \begin{pmatrix} 2x & b \\ 1 & 0 \end{pmatrix}.$$

And

$$\det DH_{b,c,(x,y)} = -b, \qquad \operatorname{tr} DH_{b,c,(x,y)} = 2x.$$

The eigenvalues of the fixed points are given by

$$\lambda = x_0 \pm \sqrt{x_0^2 + b}.$$

For given b and eigenvalue λ, the fixed point and parameter c are given by

$$x_0 = \frac{1}{2}\left(\lambda - \frac{b}{\lambda}\right),$$

$$c = -\frac{1}{4\lambda^2}(\lambda-1)^2(\lambda+b)^2 + \frac{1}{4}(1-b)^2.$$

For fixed b, parameter c can be regarded as a rational function of λ. This rational function has a pole at $\lambda = 0$, and non-degenerate critical points at $\lambda = 1, \ -b$, and $\pm\sqrt{-b}$, since

$$\frac{\partial c}{\partial \lambda} = -\frac{1}{2\lambda^3}(\lambda-1)(\lambda+b)(\lambda^2+b).$$

The critical value $c = \frac{1}{4}(1-b)^2$ is the locus of saddle-node bifurcation.

By setting $\tau = \operatorname{tr} DH_{b,c}$, we have

$$\tau = \lambda - \frac{b}{\lambda} = 2x_0, \quad c = -\frac{1}{4}(\tau - (1-b))^2 + \frac{1}{4}(1-b)^2.$$

For fixed b with $|b| < 1$, the parameter region of c, having an attracting fixed point, is the image of annulus $\{\lambda \in \mathbb{C} \mid |b| < |\lambda| < 1\}$ by the rational map. This annulus is mapped onto an ellipse in τ space, followed by a quadratic map with a critical point at $\tau = 1-b$ on the boundary of the ellipse, sending the ellipse onto an "elliptic cardioid" in c space with a cusp point at critical value $\frac{1}{4}(1-b)^2$.

Let $\mathbb{D}_2$ denote the quotient space of polydisc $\mathbb{D}^2$ by equivalence $(\lambda_1, \lambda_2) \sim (\lambda_2, \lambda_1)$. Canonical coordinate of $\mathbb{D}_2$ is $(\delta, \tau) = (\lambda_1\lambda_2, \lambda_1+\lambda_2)$. The annulus $\{(\lambda_1, \lambda_2) \in \mathbb{D}^2 \mid \lambda_1\lambda_2 = -b\}$ is isomorphic to the annulus $\{\lambda \in \mathbb{D} \mid |b| < \lambda < 1\}$ and mapped onto a vertical ellipse in $\mathbb{D}_2$. Let Ω_1 denote the set of parameters $(b,c) \in \mathbb{C}^2$ for which $H_{b,c}$ has an attracting fixed point. Coordinate x_0 represents the fixed point (x_0, x_0). And let $\tilde{\Omega}_1$ denote the set of points $((b,c), x_0) \in \Omega_1 \times \mathbb{C}$ with (x_0, x_0) being the attracting fixed point, *i.e.*, $x_0^2 - (1-b)x_0 + c = 0$. We call this equation the cycle equation of period 1.

PROPOSITION 1. $\tilde{\Omega}_1$ is isomorphic to $\mathbb{D}_2$.

PROOF. For point $((b, c), x_0) \in \tilde{\Omega}_1$, the eigenvalues λ_1, λ_2 of the attracting fixed point are of modulus smaller than 1. Hence, $(\lambda_1, \lambda_2) \in \mathbb{D}^2$. The product $\delta = \lambda_1\lambda_2 = -b$ and the sum $\tau = \lambda_1 + \lambda_2 = 2x_0$ define a holomorphic mapping from $\tilde{\Omega}_1$ to $\mathbb{D}_2$. The inverse holomorphic mapping is given by $b = -\lambda_1\lambda_2 = -\delta$, $x_0 = \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{1}{2}\tau$, and $c = (1-b)x_0 - x_0^2$.

The saddle-node locus, where the fixed point has an eigenvalue $\lambda = 1$, is a quadratic curve $\{(b, c) \in \mathbb{C}^2 \mid c = \frac{1}{4}(1-b)^2\}$. These parameters are located at the cusp points of "elliptic cardioid" $\partial\Omega_1 \cap \{b = \text{ const.}\}$.

2. Saddle fixed point and unstable manifold

When a fixed point P is a saddle, there exists an unstable manifold $W^u(P)$. Computations of saddle fixed point and eigenvalues are exactly same as in the case of attracting fixed points. The coordinate x_0 in the previous section can be used as the coordinate of the space of cycle of period 1. Let $\mathbb{C}_2$ denote the quotient space of $\mathbb{C}^2$ by equivalence $(\lambda_1, \lambda_2) \sim (\lambda_2, \lambda_1)$. This space is used as the space of eigenvalues and is isomorphic to $\mathbb{C}^2$ by mapping $[(\lambda_1, \lambda_2)] \mapsto (\delta, \tau)$, defined by

$$\delta = \lambda_1\lambda_2, \quad \tau = \lambda_1 + \lambda_2.$$

Let $\tilde{\Gamma}_1 = \{((b, c), x_0) \in \mathbb{C}^2 \times \mathbb{C} \mid x_0^2 - (1-b)x_0 + c = 0\}$ denote the space of cycles of period 1.

PROPOSITION 2. $\tilde{\Gamma}_1$ is isomorphic to $\mathbb{C}_2$.

PROOF. For point $((b, c), x) \in \tilde{\Gamma}_1$, the determinant and the trace of the Jacobian matrix $DH_{b,c}$ at fixed point (x_0, x_0) are given by

$$\delta = -b, \quad \tau = 2x_0.$$

And the inverse map is given by

$$b = -\delta, \quad c = \frac{\tau}{2}((1 + \delta) - \frac{\tau}{2}), \quad x_0 = \frac{1}{2}\tau.$$

"How do you compute unstable manifolds numerically?" is a frequently asked question. There are several methods. The author uses a classical method found by Poincaré in the 19th century, and generalized to higher dimensional saddle points by the author. Here, let us explain this method using the power series expansion for the simplest case.

Let $P = (x_0, x_0)$ be a saddle fixed point of Hénon map $H_{b,c}$, and let λ be an eigenvalue of the Jacobian matrix $DH_{b,c}(P)$ of modulus greater than 1. Let E^u denote the eigenspace of λ. By Poincaré, there exists a transcendental entire map $\Phi : \mathbb{C} \to \mathbb{C}^2$, satisfying $\Phi(0) = P$, Image$(D\Phi_0) = E^u$, and the following function equation

$$(*) \qquad \Phi(\lambda z) = H_{b,c} \circ \Phi(z).$$

His proof is quite simple. Compute the Taylor coefficients of Φ, and show the convergence of the power series obtained from the Taylor coefficients. Let

$$\Phi(z) = (x_0 + \varphi(z), x_0 + \psi(z)), \quad \varphi(z) = \sum_{k=1}^{\infty} \varphi_k z^k.$$

By function equation $(*)$, we have the followings.

$$x_0 + \varphi(\lambda z) = (x_0 + \varphi(z))^2 + c + b(x_0 + \psi(z)),$$

$$x_0 + \psi(\lambda z) = x_0 + \varphi(z).$$

As P is a fixed point of $H_{b,c}$, we can eliminate ψ to obtain

$$\varphi(\lambda z) = 2x_0\varphi(z) + (\varphi(z))^2 + b\varphi(\frac{z}{\lambda}).$$

Coefficients of the first order terms with respect to z must satisfy

$$\lambda\varphi_1 = 2x_0\varphi_1 + \frac{b}{\lambda}\varphi_1,$$

which is automatically satisfied for any φ_1, since λ is an eigenvalue of the saddle point. Note that we can choose φ_1 to define a coordinate in the unstable manifold. For $k > 1$, by comparing the k-th order terms, we have

$$\lambda^k\varphi_k = 2x_0\varphi_k + \sum_{l=1}^{k-1} \varphi_l\varphi_{k-l} + \frac{b}{\lambda^k}\varphi_k$$

so that φ_k can be computed inductively by

$$\varphi_k = \frac{\sum_{l=1}^{k-1} \varphi_l\varphi_{k-l}}{\lambda^k - 2x_0 - \frac{b}{\lambda^k}}.$$

Note that in our case $2x_0 = \lambda - \frac{b}{\lambda}$, we assume $|b| < 1$, $|\lambda| > 1$. The denominators

$$\lambda^k - 2x_0 - \frac{b}{\lambda^k} = \lambda(\lambda^{k-1} - 1)(1 + \frac{b}{\lambda^{k+1}})$$

do not vanish and the power series

$$\varphi(z) = x_0 + \sum_{k=1}^{\infty} \varphi_k z^k$$

converges. Finally set $\psi(z) = \varphi(\frac{z}{\lambda})$ to obtain $\Phi(z)$.

3. Periodic points of period two

In this section, we consider attracting cycles of period two. Although this case is easy to compute, we try to apply the discrete Fourier expansion method. There is only one cycle of period two for each parameter (b, c). The following construction of the space of cycles may appear to be redundant.

Let $\{(x_n, x_{n-1})\}$ be the periodic orbit of period 2, and set

$$x_n = \alpha_0 + (-1)^n \alpha_1.$$

The difference equation for $H_{b,c}$:

$$x_{n+1} = x_n^2 + c + bx_{n-1}$$

yields the following system of equations for Fourier coefficients.

$$\text{(EF2)} \qquad \begin{cases} \alpha_0 &= \alpha_0^2 + \alpha_1^2 + c + b\alpha_0 \\ -\alpha_1 &= 2\alpha_0\alpha_1 - b\alpha_1 \end{cases}$$

By excluding the fixed point case, *i.e.*, by assuming $\alpha_1 \neq 0$, we have

$$\alpha_0 = -\frac{1}{2}(1-b), \quad \alpha_1^2 = -\frac{3}{4}(1-b)^2 - c.$$

The cycle of period two consists of two points $(\alpha_0+\alpha_1, \alpha_0-\alpha_1)$ and $(\alpha_0-\alpha_1, \alpha_0+\alpha_1)$. The value α_0 represents the 2-cycle. Thus, α_0 can be regarded as a **cycle variable** of period 2, and the **cycle equation** is

$$\text{(CE2)} \qquad 2\alpha_0 + 1 - b = 0.$$

Let $\tilde{\Gamma}_2 = \{((b,c),\alpha_0) \in \mathbb{C}^2 \times \mathbb{C} \mid \alpha_0 = -\frac{1}{2}(1-b)\}$ denote the space of 2-cycles.

PROPOSITION 3. $\tilde{\Gamma}_2$ is a branched double-covering space over $\mathbb{C}_2$, with critical locus $\{b=0\} \subset \tilde{\Gamma}_2$ and branching locus $\{\delta = 0\} \subset \mathbb{C}_2$.

PROOF. The determinant and the trace gives the branched covering map

$$\delta = (-b)^2, \quad \tau = \mathrm{tr} DH_{b,c}^2 = 4(\alpha_0^2 - \alpha_1^2) + 2b = 4(1-b)^2 + 4c + 2b.$$

and the"inverse" map is given by

$$b = \pm\sqrt{\delta}, \quad c = \frac{1}{4}\tau - \frac{1}{2}b - (1-b)^2, \quad \alpha_0 = -\frac{1}{2}(1-b).$$

Let Ω_2 denote the set of parameters $(b,c) \in \mathbb{C}^2$ for which $H_{b,c}$ has an attracting cycle of period two. And let $\tilde{\Omega}_2$ denote the set of points $((b,c),\alpha_0) \in \tilde{\Gamma}_2$ with corresponding cycle being attracting.

PROPOSITION 4. $\tilde{\Omega}_2$ is a branched double-covering space over $\mathbb{D}_2$, with critical locus $\{b=0\} \subset \tilde{\Omega}_2$ and branching locus $\{\delta = 0\} \subset \mathbb{D}_2$.

PROOF. Same as the preceding proposition. Each slice of Ω_2 by $\{b = \text{ const.}\}$ is an ellipse.

4. Unstable manifold of periodic saddle

Unstable manifolds of periodic saddles can be expanded into power series. Similarly as in the case of fixed saddle, the coefficients of the power series can be computed inductively.

Let $P_0, \cdots, P_{p-1}$ be a cycle of period p, and let λ denote the eigenvalue of the Jacobian matrix $DH_{b,c}^p$ at the periodic points. For $i = 0, \cdots, p-1$, let

$$P_i = (x_i, y_i), \quad \Phi_i : \mathbb{C} \to \mathbb{C}^2, \quad \Phi_i(z) = (x_i + \varphi_i(z), y_i + \psi_i(z))$$

and assume

$$\Phi_{i+1}(z) = H_{b,c} \circ \Phi_i(z), \qquad \text{for} \quad i = 0, \cdots, p-2,$$

and

$$\Phi_0(\lambda z) = H_{b,c} \circ \Phi_{p-1}(z).$$

These function equations yield the following system of function equations.

$$\begin{cases} x_{i+1}+\varphi_{i+1}(z) &= (x_i+\varphi_i(z))^2+c+b(y_i+\psi_i(z)) \\ y_{i+1}+\psi_{i+1}(z) &= x_i+\varphi_i(z) \end{cases} \qquad (i=0,\cdots,p-2),$$

and

$$\begin{cases} x_0+\varphi_0(\lambda z) &= (x_{p-1}+\varphi_{p-1}(z))^2+c+b(y_{p-1}+\psi_{p-1}(z)) \\ y_0+\psi_0(\lambda z) &= x_{p-1}+\varphi_{p-1}(z) \end{cases}.$$

As x_i and y_i are coordinates of periodic points, we have

$$\begin{cases} \varphi_{i+1}(z) &= 2x_i\varphi_i(z)+(\varphi_i(z))^2+b\psi_i(z) \\ \psi_{i+1}(z) &= \varphi_i(z) \end{cases} \qquad (i=0,\cdots,p-2),$$

and

$$\begin{cases} \varphi_0(\lambda z) &= 2x_{p-1}\varphi_{p-1}(z)+(\varphi_{p-1}(z))^2+b\psi_{p-1}(z) \\ \psi_0(\lambda z) &= \varphi_{p-1}(z) \end{cases}.$$

Expand functions $\varphi_i(z)$ and $\psi_i(z)$ to power series

$$\varphi_i(z)=\sum_{k=1}^{\infty}\varphi_{i,k}z^k, \qquad \psi_i(z)=\sum_{k=1}^{\infty}\psi_{i,k}z^k.$$

Let DH_i denote the Jacobian matrix $DH_{b,c}$ at P_i. The linear terms must satisfy

$$\begin{pmatrix} \varphi_{i+1,1} \\ \psi_{i+1,1} \end{pmatrix} = DH_i \begin{pmatrix} \varphi_{i,1} \\ \psi_{i,1} \end{pmatrix}, \qquad \text{for} \quad i=0,\cdots,p-2,$$

and

$$\begin{pmatrix} \lambda\varphi_{0,1} \\ \lambda\psi_{0,1} \end{pmatrix} = DH_{p-1} \begin{pmatrix} \varphi_{p-1,1} \\ \psi_{p-1,1} \end{pmatrix}.$$

We can choose an eigenvector $\begin{pmatrix} \varphi_{0,1} \\ \psi_{0,1} \end{pmatrix}$ of $DH_{p-1}\cdots DH_0$ to get values of the linear terms. Higher order coefficients are computed inductively as follows. Let

$$\varphi_{i,k}^{[2]}=\sum_{l=1}^{k-1}\varphi_{i,l}\varphi_{i,k-l}$$

denote the term of degree k of power series $(\varphi_i(z))^2$. Note that this term is determined by terms of orders smaller than k.

The coefficients of degree k greater than 1 satisfy

$$\begin{pmatrix} \varphi_{i+1,k} \\ \psi_{i+1,k} \end{pmatrix} = DH_i \begin{pmatrix} \varphi_{i,k} \\ \psi_{i,k} \end{pmatrix} + \begin{pmatrix} \varphi^{[2]}_{i,k} \\ 0 \end{pmatrix}, \qquad \text{for} \quad i = 0, \cdots, p-2,$$

and

$$\lambda^k \begin{pmatrix} \varphi_{0,k} \\ \psi_{0,k} \end{pmatrix} = DH_{p-1} \begin{pmatrix} \varphi_{p-1,k} \\ \psi_{p-1,k} \end{pmatrix} + \begin{pmatrix} \varphi^{[2]}_{p-1,k} \\ 0 \end{pmatrix}.$$

These equations can be solved as follows

$$\begin{pmatrix} \varphi_{0,k} \\ \psi_{0,k} \end{pmatrix} = (\lambda^k E - DH_{p-1} \cdots DH_0)^{-1} \left(\sum_{j=0}^{p-1} DH_{p-1} \cdots DH_{j+1} \begin{pmatrix} \varphi^{[2]}_{j,k} \\ 0 \end{pmatrix} \right).$$

These formulas recursively determine all the coefficients of the convergent power series $\Phi_0(z), \cdots, \Phi_{p-1}$.

5. Cycles of period three

In this section, we compute the periodic points of period three using the discrete Fourier expansion. Let $\omega = \frac{1}{2}(-1+\sqrt{3}i)$ denote a cubic root of unity. Periodic sequence $\{x_n\}$ of period three can be expressed as

$$x_n = u_0 + \omega^n u_1 + \omega^{2n} u_2.$$

To exclude fixed point cases, we assume $(u_1, u_2) \neq (0, 0)$.

Sequence $P_n = (x_n, x_{n-1})$ is an orbit of $H_{b,c}$ if and only if

$$x_{n+1} - bx_{n-1} = (x_n)^2 + c$$

holds. We obtain the following system of equations for Fourier coefficients.

$$\text{(EF3)} \qquad \begin{cases} (1-b)u_0 &= u_0^2 + 2u_1u_2 + c, \\ (\omega - b\omega^2)u_1 &= u_2^2 + 2u_0u_1, \\ (\omega^2 - \omega b)u_2 &= u_1^2 + 2u_0u_2. \end{cases}$$

As we assumed $u_1u_2 \neq 0$, the second and the third equations give

$$u_2^2 u_1^{-1} = \omega - b\omega^2 - 2u_0, \quad u_1^2 u_2^{-1} = \omega^2 - b\omega - 2u_0.$$

The product and the sum of these yield

$$u_1 u_2 = 4u_0^2 + 2(1-b)u_0 + b^2 + b + 1,$$

$$\frac{u_1^3+u_2^3}{u_1u_2} = -(1-b) - 4u_0.$$

Note that u_1 and u_2 are determined from b and u_0. The cycle does not depend on the choice of u_1 and u_2 as solutions. Hence the value u_0 represents the cycle. We call u_0 the **cycle variable** of 3-cycle. The condition for this cycle to be a cycle of $H_{b,c}$ reduces to equation

$$c = -9u_0^2 - 3(1-b)u_0 - 2(b^2+b+1),$$

or

$$\text{(CE3)} \qquad 9u_0^2 + 3(1-b)u_0 + 2(b^2+b+1) + c = 0,$$

which we call the **cycle equation** of period 3.

For each parameter (b,c), there are two cycles of period three. If the above equation has two distinct solutions for u_0, they correspond to the two 3-cycles. For each root u_0 of the cycle equation, u_1 and u_2 are obtained from the above equations. The choice of the cubic root corresponds to the choice of the starting point of the cycle. Saddle-node cycle must correspond to the double root case. The saddle-node parameters satisfy the discriminant equation

$$9(1-b)^2 - 36(c+2(b^2+b+1)) = 0,$$

or,

$$\text{(SN3}_L\text{)} \qquad c = -\frac{1}{4}(7b^2+10b+7),$$

which we call the **saddle-node equation** of period 3. The curve $\{(b,c) \in \mathbb{C}^2 \mid c = -\frac{1}{4}(7b^2+10b+7)\}$ in the parameter space is called the **saddle-node locus** of period 3. The value of cycle variable for the saddle-node cycle is given by the double root

$$\text{(SN3}_C\text{)} \qquad u_0 = -\frac{1}{6}(1-b)$$

of the cycle equation.

Note that this curve is invariant under change of parameters $(b,c) \mapsto (b^{-1}, cb^{-2})$, since the condition for cycle to have eigenvalue 1 is invariant under the involution $H_{b,c} \mapsto H_{b,c}^{-1} \sim H_{b^{-1},cb^{-2}}$.

The cycle variable u_0 of the saddle-node is the double root of the cycle equation and is given by $u_0 = \frac{1}{6}(b-1)$.

The trace of the Jacobian matrix $D(H^3_{b,c})$ along the cycle is given by

$$\tau = \text{tr } D(H^3_{b,c}) = 8x_0x_1x_2 + 2b(x_0 + x_1 + x_2).$$

As

$$x_0 + x_1 + x_2 = 3u_0,$$

and

$$\begin{aligned} x_0x_1x_2 &= u_0^3 + u_1^3 + u_2^3 - 3u_0u_1u_2 \\ &= -27u_0^3 - 18(1-b)u_0^2 - 3(3b^2 + b + 3)u_0 - (1 - b^3), \end{aligned}$$

we have a formula for trace of 3-cycle

$$\text{(TF3)} \qquad \tau = -8\left(27u_0^3 + 18(1-b)u_0^2 + 9(b^2 + \frac{b}{4} + 1)u_0 + 1 - b^3\right).$$

Let $\tilde{\Gamma}_3 = \{((b, c), u_0) \in \mathbb{C}^2 \times \mathbb{C} \mid c = -9u_0^2 - 3(1 - b)u_0 - 2(b^2 + b + 1)\}$ denote the **cycle space** of period 3. The trace $\tau : \tilde{\Gamma}_3 \to \mathbb{C}$, is called the **trace function**.

PROPOSITION 5. Holomorphic map $\chi : \tilde{\Gamma}_3 \to \mathbb{C}_2$, with $\chi((b, c), u_0) = (\delta, \tau)$, defined by

$$\delta = \det D(H^3_{b,c}) = -b^3, \qquad \tau = \text{tr } D(H^3_{b,c})$$

is regular near the saddle-node locus $\{c = -\frac{1}{4}(7b^2 + 10b + 7)\}$, except at $(b, c) = -\frac{1}{14}(13 \pm 3\sqrt{3}i)(1, \frac{3}{4})$ and $\{b = 0\}$.

PROOF. Partial derivative $\frac{\partial\tau}{\partial u_0}$ at saddle-node cycle $((b, -\frac{1}{4}(7b^2+10b+7)), -\frac{1}{6}(1 - b)) \in \tilde{\Gamma}_3$ is given by

$$\left.\frac{\partial \tau}{\partial u_0}\right|_{u_0 = -\frac{(1-b)}{6}} = -6(7b^2 + 13b + 7).$$

Hence, χ is regular if $b \neq 0$ and $7b^2 + 13b + 7 \neq 0$. Note that $|b| = 1$ if $7b^2 + 13b + 7 = 0$.

PROPOSITION 6. Cycle of period 3 has eigenvalue 1 on the sadlde-node locus $\{c = -\frac{1}{4}(7b^2 + 10b + 7)\}$ with $u_0 = -\frac{1}{6}(1 - b)$. On the parabolic bifurcation locus $\{u_0 = \frac{1}{2}(\omega - b\omega^2)\} \cup \{u_0 = \frac{1}{2}(\omega^2 - b\omega)\}$ of period tripling, the cycle degenerates to a parabolic fixed point. Apart from these cases, 3-cycle does not have eigenvalue 1.

PROOF. When an eigenvalue λ of the cycle is 1, the other eigenvalue is $-b^3$. The trace τ computed above can be decomposed as

$$\tau - (1 - b^3) = -216(u_0 + \frac{1}{6}(1-b))(u_0^2 + \frac{1}{2}(1-b)u_0 + \frac{1}{4}(b^2 + b + 1)).$$

The first factor gives the saddle-node locus. The second factor can be solved as

$$u_0 = \frac{1}{2}(\omega - b\omega^2) \quad \text{or} \quad u_0 = \frac{1}{2}(\omega^2 - b\omega).$$

In these cases, we have $u_1 = u_2 = 0$. Hence the cycle degenerates to a fixed point. The parameter c is given by

$$c = \frac{1}{4}(2\omega^2 - \omega)b^2 + b + \frac{1}{4}(2\omega - \omega^2)$$

or

$$c = \frac{1}{4}(2\omega - \omega^2)b^2 + b + \frac{1}{4}(2\omega^2 - \omega).$$

These parameters correspond to the trifurcation locus.

Let $\tilde{\Omega}_3 = \chi^{-1}(\mathbb{D}_2) \subset \tilde{\Gamma}_3$ denote the set of attracting cycles of period 3. And let $\Omega_3 \subset \mathbb{C}^2$ denote its projected image in the parameter space.

PROPOSITION 7. For each parameter b with $|b| < 1$, locus of attracting cycles $\Omega_3 \cap (\{b\} \times \mathbb{C})$ has a cusp point at $c = -\frac{1}{4}(7b^2 + 10b + 7)$.

PROOF. We fix the parameter b. Then the trace τ can be regarded as a holomorphic function of the cycle variable u_0. To the saddle-node parameter value (b, c) corresponds a saddle-node cycle $((b, c), -\frac{1}{6}(1-b)) \in \tilde{\Gamma}_3$. As computed in proposition 5, $\frac{\partial\tau}{\partial u_0} \neq 0$ there. At the saddle-node cycle, one of the eigenvalues takes value 1, so that $\tau = 1 - b^3$. There is a neighborhood of $\lambda = 1$, such that the local inverse map $\lambda \mapsto u_0$ is holomorphic and regular. Projection from the cycle space to the parameter space, for fixed b has a quadratic singularity at the saddle-node cycle point. Hence, the locus of attracting cycles must include the image of a part of the unit disk in the λ space by a holomorphic map with quadratic singularity at $\lambda = 1$. More precisely, the "cardioid-like" set may intersect with itself. In this case, instead of the usual saddle-node bifurcation, we may see "collision" of two coexisting attracting cycles.

PROPOSITION 8. Saddle-node locus of 3-cycle and period doubling locus of fixed point intersect at two points $(b, c) = (-2 \pm \sqrt{3}, 9(1 \mp \frac{1}{2}\sqrt{3}))$.

PROOF. Saddle-node locus of period three cycle is given by $c = -\frac{1}{4}(7b^2 + 10b + 7)$ and period doubling locus of period 1 to 2 is given by $c = -\frac{3}{4}(1-b)^2$. By eliminating c from these equations, we obtain

$$b^2 + 4b + 1 = 0,$$

from which we get

$$b = -2 \pm \sqrt{3}, \qquad c = 9 \mp \frac{9}{2}\sqrt{3}.$$

Note that $b + \frac{1}{b} = -4$. This fact will be used in section 9.

6. Cycles of period four

In this section, we compute the periodic points of period four. Periodic sequence $\{x_n\}$ of period four can be expressed as

$$x_n = v_0 + i^n v_1 + (-1)^n v_2 + (-i)^n v_3.$$

As in the previous section, sequence $\{(x_n x_{n-1})\}$ is an orbit of $H_{b,c}$ if and only if

$$x_{n+1} - bx_{n-1} = (x_n)^2 + c$$

holds. We obtain the following system of equations for Fourier coefficients.

$$\text{(EF4)} \qquad \left\{ \begin{array}{rcc} (1-b)v_0 & = & v_0^2 + v_2^2 + 2v_1v_3 + c \\ i(1+b)v_1 & = & 2(v_0v_1 + v_2v_3) \\ -(1-b)v_2 & = & v_1^2 + v_3^2 + 2v_0v_2 \\ -i(1+b)v_3 & = & 2(v_1v_2 + v_0v_3) \end{array} \right.$$

By setting $b_1 = 1 - b$ and $b_2 = i(1+b)/2$, we get the following system of equations.

$$(*) \left\{ \begin{array}{rcc} b_1v_0 & = & v_0^2 + v_2^2 + 2v_1v_3 + c \\ b_2v_1 & = & v_0v_1 + v_2v_3 \\ -b_1v_2 & = & v_1^2 + v_3^2 + 2v_0v_2 \\ -b_2v_3 & = & v_1v_2 + v_0v_3 \end{array} \right.$$

Let us eliminate v_1, v_2, v_3 to obtain an equation of v_0, which represents the cycle. Second and fourth line of $(*)$ are equivalent to

$$\begin{pmatrix} v_0 & v_2 \\ -v_2 & -v_0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_3 \end{pmatrix} = b_2 \begin{pmatrix} v_1 \\ v_3 \end{pmatrix}.$$

To exclude cycles of period 2 or 1, we assume $(v_1, v_3) \neq (0,0)$. We get

$$v_2^2 = v_0^2 - b_2^2. \qquad (**)$$

Making sums and differences of equations, system of equations $(*)$ is equivalent to the following.

$$\begin{cases} b_1(v_0 - v_2) &= (v_0 + v_2)^2 + (v_1 + v_3)^2 + c \\ b_1(v_0 + v_2) &= (v_0 - v_2)^2 - (v_1 - v_3)^2 + c \\ b_2(v_1 - v_3) &= (v_0 + v_2)(v_1 + v_3) \\ b_2(v_1 + v_3) &= (v_0 - v_2)(v_1 - v_3) \end{cases}$$

By setting

$$s_0 = v_0 + v_2, \quad s_1 = v_1 + v_3, \quad t_0 = v_0 - v_2, \quad t_1 = v_1 - v_3,$$

it is rewritten as follows.

$$\begin{cases} b_1 t_0 &= s_0^2 + s_1^2 + c \\ b_1 s_0 &= t_0^2 - t_1^2 + c \\ b_2 t_1 &= s_0 s_1 \\ b_2 s_1 &= t_0 t_1 \end{cases}$$

Let us assume $b_2 \neq 0$,*i.e.*,$b \neq -1$. To suppress cycles of periods smaller than 4, we can assume $s_1 t_1 \neq 0$. Then, we have $s_0 t_0 = b_2^2 \neq 0$.
By using these, our system of equations can be rewritten as follows.

$$(***) \begin{cases} t_0 &= \dfrac{b_2^2}{s_0} \\ t_1 &= \dfrac{s_0 s_1}{b_2} \\ s_1^2 &= b_1 \dfrac{b_2^2}{s_0} - s_0^2 - c \\ b_1 s_0 &= \dfrac{b_2^4}{s_0^2} - \dfrac{s_0^2}{b_2^2}(b_1 \dfrac{b_2^2}{s_0} - s_0^2 - c) + c \end{cases}$$

Note that $(***)$ is equivalent to $(*)$, under the condition $b_2 s_1 t_1 \neq 0$. And note that the last equation, divided by $s_0 b_2$, gives

$$(\frac{b_2}{s_0})^3 + (\frac{s_0}{b_2})^3 - \frac{2b_1}{b_2} + \frac{1}{b_2^2}(\frac{s_0}{b_2} + \frac{b_2}{s_0})c = 0.$$

Now, as $b_2^2 = v_0^2 - v_2^2 = s_0 t_0$, we have

$$\frac{s_0}{b_2} + \frac{b_2}{s_0} = \frac{s_0^2 + b_2^2}{b_2 s_0} = \frac{s_0 + t_0}{b_2} = \frac{2v_0}{b_2}.$$

When we need to solve back from v_0 to s_0, use quadratic equation

$$s_0^2 - 2v_0 s_0 + b_2^2 = 0.$$

We obtain the **cycle equation** of period four, by eliminating s_0,

$$v_0^3 + \frac{1}{4}(c - 3b_2^2)v_0 - \frac{1}{4}b_1 b_2^2 = 0.$$

Or, in terms of parameters (b, c),

$$\text{(CE4)} \qquad v_0^3 + \frac{1}{4}\left(c + \frac{3}{4}(b+1)^2\right)v_0 - \frac{1}{16}(b^2 - 1)(b+1) = 0.$$

If v_0 is a solution of this equation, by $(**)$ and $(***)$, other Fourier coefficients are obtained and a cycle of period four is derived. Hence, each solution v_0 represents a 4-cycle, and is considered as a **cycle variable**.

Let

$$\tilde{\Gamma}_4 = \{((b,c), v_0) \in \mathbb{C}^2 \times \mathbb{C} \mid v_0^3 + \frac{1}{4}(c + \frac{3}{4}(b+1)^2)v_0 - \frac{1}{16}(b^2-1)(b+1) = 0\}$$

denote the **space of 4-cycles**. It is a co-dimension one hyper-surface with a singularity at $((-1,0),0)$. Let $\tilde{\Omega}_4$ denote the subset of $\tilde{\Gamma}_4$ of attracting cycles, and let Ω_4 denote its projection into the parameter space.

Let $\tau = \operatorname{tr} DH_{b,c}^4$ denote the trace of the derivative along the cycle,

$$\begin{aligned}\tau &= 16x_3x_2x_1x_0 + 4b(x_3x_2 + x_2x_1 + x_1x_0 + x_0x_3) + 2b^2 \\ &= 16(t_0^2 + t_1^2)(s_0^2 - s_1^2) + 16bs_0t_0 + 2b^2.\end{aligned}$$

Here, we used, for example,

$$x_3x_1 = (v_0 - iv_1 - v_2 + iv_3)(v_0 + iv_1 - v_2 - iv_3) = (v_0 - v_2)^2 + (v_1 - v_3)^2 = t_0^2 + t_1^2,$$

$$x_2x_0 = (v_0 - v_1 + v_2 - v_3)(v_0 + v_1 + v_2 + v_3) = (v_0 + v_2)^2 - (v_1 + v_3)^2 = s_0^2 - s_1^2.$$

For fixed b, trace τ can be expressed as a rational function of cycle variable v_0. To see this, note that c can be expressed as

$$c = -4v_0^2 + \frac{1}{4v_0}(b^2 - 1)(b+1) - \frac{3}{4}(b+1)^2,$$

and, by eliminating s_0, s_1, t_0, t_1, we have

$$\tau = 16(-16b_1v_0^3 + 8cv_0^2 + 2b_1(6b_2^2 - 2c)v_0 + 4b_2^4 + b_1^2b_2^2 + c^2 - 4b_2^2c) + 16bb_2^2 + 2b^2. \tag{TF4}$$

This **trace function** will be decomposed in section 8.

7. Saddle-node locus of 4-cycle

PROPOSITION 9. Saddle-node locus of 4-cycle is given by

$$c = -\frac{3}{4}(b+1)(b+1+\sqrt[3]{4(b-1)^2(b+1)}),$$

and the cycle variable of the saddle-node is given by

$$v_0 = \frac{1}{4}\sqrt[3]{2(b+1)^2(1-b)},$$

where choice of cubic root is specified by

$$c = -\frac{3}{4}(b+1)^2 - 12v_0^2.$$

PROOF. The cycle equation

$$v_0^3 + \frac{1}{4}(c - 3b_2^2)v_0 - \frac{1}{4}b_1b_2^2 = 0$$

of 4-cycle has a double root at saddle-node locus. Hence, the discriminant of the cubic equation must vanish there. We have the **saddle-node equation**,

$$4(\frac{1}{4}(c-3b_2^2))^3 + 27(-\frac{1}{4}b_1b_2^2)^2 = 0,$$

or,

$$(c - 3b_2^2)^3 = -27b_1^2b_2^4. \tag{SN4$_L$}$$

Hence, we have a formula for the **saddle-node locus**,

$$\begin{aligned} c &= 3b_2^2 - 3\sqrt[3]{b_1^2b_2^4} \\ &= 3\left(\frac{(b+1)i}{2}\right)^2 - 3\sqrt[3]{(1-b)^2\left(\frac{(b+1)i}{2}\right)^4} \end{aligned}$$

$$= -\frac{3}{4}(b+1)(b+1+\sqrt[3]{4(b-1)^2(b+1)}).$$

The cycle variable of the saddle-node v_0 is the double root of the cycle equation. From the derived function

$$3v_0^2 + \frac{1}{4}(c - 3b_2^2) = 0$$

of the cycle equation, we have

$$c = 3b_2^2 - 12v_0^2.$$

Use this formula to eliminate c in the cycle equation to get

$$\text{(SN4}_C\text{)} \qquad v_0^3 = -\frac{1}{8}\ b_1 b_2^2.$$

Note that for fixed b, there are three saddle-node loci of c, which form the three vertices of a regular triangle.

8. Regularity of trace

In section 6, we computed the trace function τ of 4-cycles. As the determinant of Jacobian matrix $DH^4_{b,c}$ is b^4, the value of the trace for saddle-node cycle is given by $\tau = 1 + b^4$. Eigenvalue of cycle can be 1 for cycles bifurcating from period doubling from 2-cycle, or period quadrupling from fixed point (satellite bifurcation).

First, let us verify that $\tau = 1+b^4$ for saddle-node cycles. Use saddle-node condition $c = 3b_2^2 - 12v_0^2$ to eliminate c from the trace to get

$$\tau = 16((8v_0^3 + b_1 b_2^2)(6v_0 + b_1) + b_2^4) + 16bb_2^2 + 2b^2.$$

Then use $v_0^3 = -\frac{1}{8}b_1 b_2^2$ and $b_2^2 = -\frac{1}{4}(b+1)^2$ to get $\tau = 1 + b^4$ at saddle-node cycles.

So $\tau - (1 + b^4)$, as a function of v_0 should be factorized. Set $c = \Delta - 12v_0^2 + 3b_2^2$ and eliminate c in τ, to decompose it as

$$\tau - (1 + b^4) = 16\Delta(-10v_0^2 - b_1 v_0 + \Delta + 2b_2^2).$$

Then set back $\Delta = c - 3b_2^2 + 12v_0^2$ and see

$$\tau - (1 + b^4) = 16(c - 3b_2^2 + 12v_0^2)(c + 2v_0^2 - b_1 v_0 - b_2^2).$$

The last factor of the above gives rise to equation

$$c = -2v_0^2 + b_1 v_0 + b_2^2.$$

This equation, combined with the cycle equation

$$v_0^3 + \frac{1}{4}(c - 3b_2^2)v_0 - \frac{1}{4}b_1 b_2^2 = 0,$$

gives the following equation, by eliminating c.

$$(v_0^2 - b_2^2)(v_0 + \frac{1}{2}b_1) = 0.$$

Solutions $v_0 = \pm b_2$, with $c = \pm b_1 b_2 - b_2^2$, corresponds to the locus of period quadrupling bifurcation from a fixed point. For, from $(*)$, $v_2 = 0$ follows and and from $(**)$, $v_1 = v_3 = 0$ follow. And solution $v_0 = -\frac{1}{2}b_1$, with $c = -\frac{1}{2}(b^2 + 1)$ corresponds to the locus of period-doubling from 2-cycle. In this case, as $2v_0 = -b_1$, from the third equation of $(*)$ of section 6, we have $v_1^2 + v_3^2 = 0$. And, from

$$\frac{s_0}{b_2} + \frac{b_2}{s_0} = \frac{2v_0}{b_2} = -\frac{b_1}{b_2},$$

$$s_0 = -\frac{1}{2}b_1 \pm \frac{1}{2}\sqrt{b_1^2 - b_2^2},$$

we have

$$v_2^2 = \frac{1}{4}b_1^2 - b_2^2 = \frac{1}{2}(b^2 + 1).$$

Use these and the first equation of $(*)$ to obtain $v_1 v_3 = 0$, which shows that our cycle degenerates to a cycle of period two.

PROPOSITION 10. Holomorphic map $\chi : \tilde{\Gamma}_4 \to \mathbb{C}_2$, with $\chi(b, c, v_0) = (\delta, \tau)$, defined by

$$\delta = \det\, D(H_{b,c}^4) = b^4, \qquad \tau = \mathrm{tr}\, D(H_{b,c}^4)$$

is regular on saddle-node locus $\{c = -\frac{3}{4}(b+1)(b+1+\sqrt[3]{4(b-1)^2(b+1)})\}$, for $0 < |b| < 1$, saddle-node being specified in proposition 9.

PROOF. Trace τ is a function of b and v_0 computed at the end of section 6. And c in the formula of τ should be considered as a function of b and v_0. By differentiating the cycle equation with respect to v_0, we have

$$3v_0^2 + \frac{1}{4}(c - 3b_2^2) + \frac{1}{4}v_0 \frac{\partial c}{\partial v_0} = 0.$$

As $c = 3b_2^2 - 12v_0^2$ holds at the saddle-node cycle, we see $\frac{\partial c}{\partial v_0} = 0$ if $v_0 \neq 0$. For saddle-node cycle locus, we have

$$
\begin{aligned}
\frac{\partial \tau}{\partial v_0} &= 16(-48b_1v_0^2 + 16cv_0 + 2b_1(6b_2^2 - 2c)) \\
&= 16(-48b_1v_0^2 + 16(3b_2^2 - 12v_0^2)v_0 + 2b_1(6b_2^2 - 2(3b_2^2 - 12v_0^2))) \\
&= 96(-8b_1v_0^2 + 8b_2^2v_0 - 32v_0^3 + b_1(2b_2^2 - 2b_2^2 + 8v_0^2)) \\
&= 96(-32v_0^3 + 8b_2^2v_0) = 96(4b_1b_2^2 + 8b_2^2v_0) = 384b_2^2(2v_0 + b_1).
\end{aligned}
$$

As v_0 corresponds to a saddle-node cycle, $v_0^3 = -\frac{1}{8}b_1b_2^2$, therefore the case $2v_0 + b_1 = 0$ implies $b = 1$ or $b = \frac{1}{5}(3 \pm 4i)$, and the case $b_2 = 0$ implies $b = -1$. In all of these cases, $|b| = 1$. The case $v_0 = 0$ can occur only when $b = \pm 1$. Hence, we conclude that τ is regular with respect to v_0 along the saddle-node locus, if $|b| < 1$.

PROPOSITION 11. For each parameter b with $|b| < 1$, locus of attracting cycles $\Omega_4 \cap (\{b\} \times \mathbb{C})$ has cusp points at saddle-node loci.

PROOF. The proof is similar to that of proposition 7. We fix the parameter b. Trace τ can be regarded as a holomorphic function of the cycle variable v_0. To the saddle-node parameter value (b, c) corresponds a saddle-node cycle $((b, c), v_0) \in \tilde{\Gamma}_4$, as specified in proposition 9. As computed in proposition 10, $\frac{\partial \tau}{\partial u_0} \neq 0$ there. At the saddle-node cycle, one of the eigenvalues takes value 1, so that $\tau = 1 + b^4$. There is a neighborhood of $\lambda = 1$, such that the inverse map $\lambda \mapsto u_0$ is holomorphic and regular. Projection from the cycle space to the parameter space, for fixed b has a quadratic singularity at the saddle-node cycle point. The cycle equation in the definition of $\tilde{\Gamma}_4$,

$$v_0^3 + \frac{1}{4}(c - 3b_2^2)v_0 - \frac{1}{4}b_1b_2^2 = 0,$$

defines c as a rational function of v_0 for fixed b. By derivations with respect to v_0, we have

$$3v_0^2 + \frac{1}{4}(c - 3b_2^2) + \frac{1}{4}v_0\frac{\partial c}{\partial v_0} = 0,$$

and

$$6v_0 + \frac{1}{2}\frac{\partial c}{\partial v_0} + \frac{1}{4}v_0\frac{\partial^2 c}{\partial v_0^2} = 0.$$

At the saddle-node cycle, saddle-node condition

$$3v_0^2 + \frac{1}{4}(c - 3b_2^2) = 0$$

is satisfied. Hence we have $\frac{\partial c}{\partial v_0} = 0$ and $\frac{\partial^2 c}{\partial v_0^2} = -24$, since $v_0 \neq 0$. Hence, the locus of attracting cycles must include the image of a part of the unit disk in the λ space by a holomorphic map with quadratic singularity at $\lambda = 1$. More precisely, the "cardioid-like" set may intersect with itself. The image of the unit circle in the λ space forms a cusp in the c space.

9. Coexistence of attractive cycles

In this section, we prove the existence of an open set of parameters (b, c), such that $H_{b,c}$ has an attractive fixed point, an attractive 3-cycle, and an attractive 4-cycle, at the same time.

PROPOSITION 12. The intersection $\Omega_1 \cap \Omega_3 \cap \Omega_4$ is not empty.

PROOF. First, let us examine the intersection of the saddle-node loci of 3-cycles and the saddle-node loci of 4-cycles. The saddle-node locus of 3-cycles is given by

$$c = -\frac{1}{4}(7b^2 + 10b + 7),$$

and the saddle-node locus of 4-cycles is given by

$$(c + \frac{3}{4}(b+1)^2)^3 + \frac{27}{16}(b-1)^2(b+1)^4 = 0$$

from proposition 9. Assume $b \neq 0$, and let $w = b + \frac{1}{b}$. Eliminate c from two equations above and obtain an equation of w,

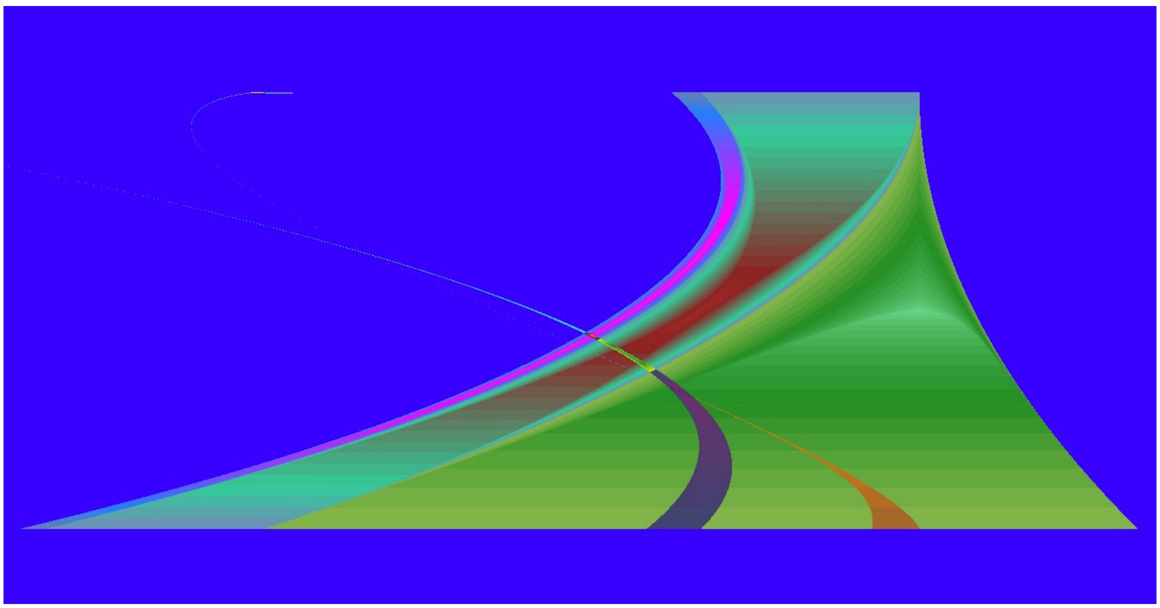
$$11w^3 + 6w^2 - 156w - 232 = 0.$$

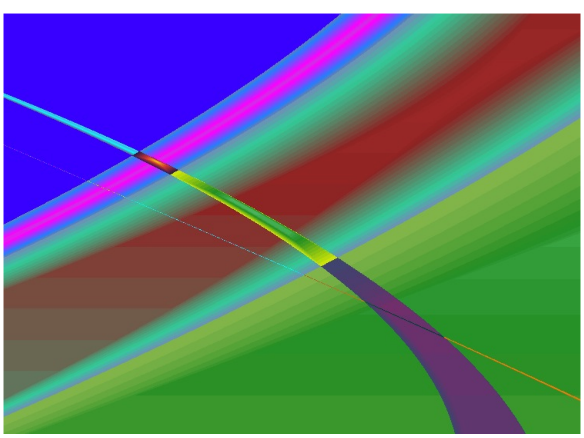
Let $f(w) = 11w^3 + 6w^2 - 156w - 232$. As $f(-4) = -216$, $f(-2) = 16$, $f(0) = -232$, and $f(2) = -432$, there are three real solutions w_1, w_2, w_3 with

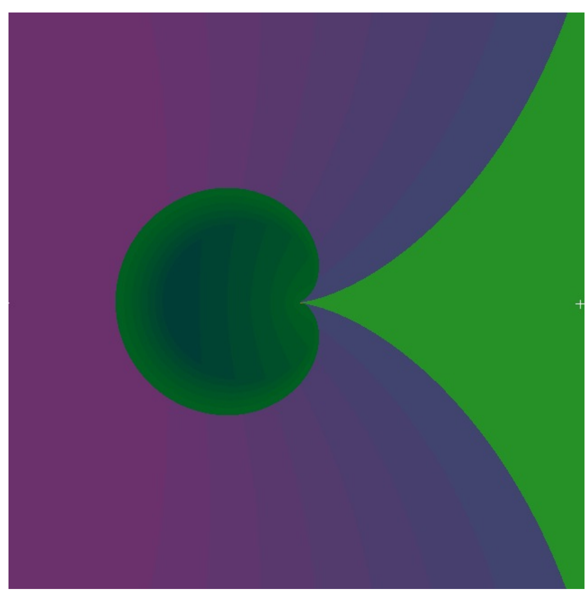
$$-4 < w_1 < -2 < w_2 < 0, \qquad 2 < w_3.$$

For each solution w_k, we have solutions b of quadratic equation $b^2 - w_k b + 1 = 0$. The six solutions give the intersection points of the saddle-node curves, specified by $c = -\frac{1}{4}(7b^2 + 10b + 7)$. For w_2, the solution b are non-real and $|b| = 1$. For w_3, the solutions are real and positive. For w_1, the solutions, say β_1, β_2, are real and negative, with $-2-\sqrt{3} < \beta_1 < -1 < \beta_2 < -2+\sqrt{3} < 0$. As noted in proposition 8, the intersection point $(\beta_2, -\frac{1}{4}(7\beta_2+10\beta+7))$ belongs to Ω_1. By proposition 7, $\Omega_3 \cap (\{\beta_2\} \times \mathbb{C})$ has a cusp point at this intersection point, and by proposition 11, $\Omega_4 \cap (\{\beta_2\} \times \mathbb{C})$ has a cusp point there, too. These slices of cardioid-like shape intersect in an open set. This shows the non-emptiness of $\Omega_1 \cap \Omega_3 \cap \Omega_4$.

Following picture shows the real section of the parameter space.

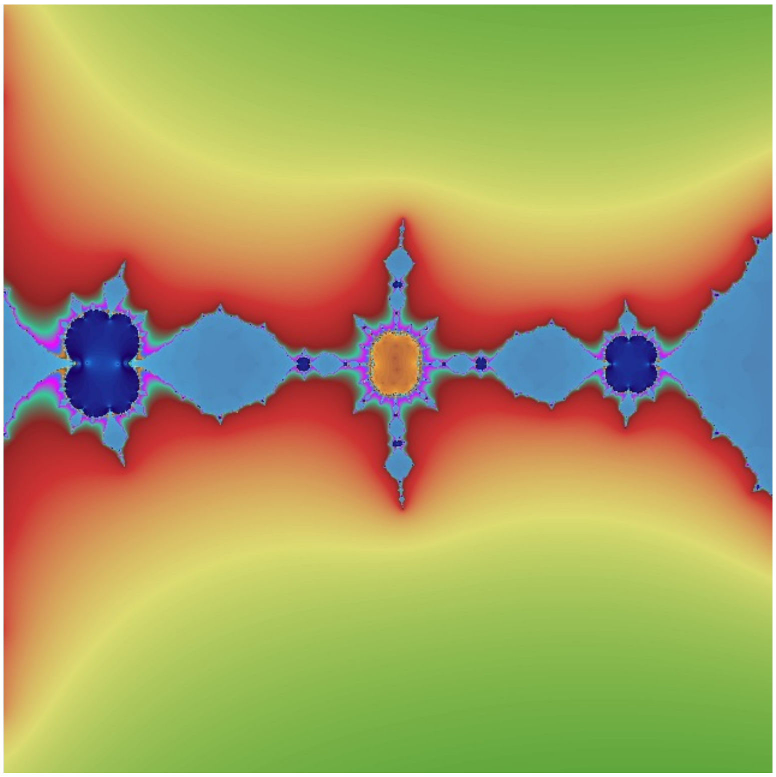






The left picture shows an enlargement of the preceding picture. Parameter region of attracting 3-cycles and attracting 4-cycles intersect in the attracting fixed point region. The right picture shows a complex slice near the intersection point of the saddle-node locus of periods 3 and 4.

The following picture shows a portion of unstable manifold of a saddle fixed point. Three different basins of attractions are observed. Parameters for this picture are $b = -0.3946, c = -1.0362$.



10. Cycles of period five

Let us compute the cycles of period five. Let $\kappa = \exp(2\pi i/5)$ denote a quintic root of unity. We begin with the discrete Fourier expansion. Let

$$x_n = \sum_{k=0}^{4} a_k \kappa^{nk}.$$

From our difference scheme

$$x_{n+1} - bx_{n-1} = x_n^2 + c,$$

we have the following system of equations for Fourier coefficients.

$$\text{(EF5)} \qquad \begin{cases} (1-b)a_0 &= a_0^2 + 2a_1a_4 + 2a_2a_3 + c \\ (\kappa - b\kappa^4)a_1 &= 2a_0a_1 + 2a_2a_4 + a_3^2 \\ (\kappa^2 - b\kappa^3)a_2 &= 2a_0a_2 + 2a_3a_4 + a_1^2 \\ (\kappa^3 - b\kappa^2)a_3 &= 2a_0a_3 + 2a_1a_2 + a_4^2 \\ (\kappa^4 - b\kappa)a_4 &= 2a_0a_4 + 2a_1a_3 + a_2^2 \end{cases}$$

Let $b_1 = 1-b$, $b_3 = 1+b$, and set constants

$$\lambda_1 = \kappa + \kappa^4 = -\frac{1}{2}(1-\sqrt{5}), \quad \lambda_2 = \kappa - \kappa^4 = \sqrt{\frac{1}{2}(5+\sqrt{5})}i,$$

$$\gamma_1 = \kappa^2 + \kappa^3 = -\frac{1}{2}(1+\sqrt{5}), \quad \gamma_2 = \kappa^2 - \kappa^3 = \sqrt{\frac{1}{2}(5-\sqrt{5})}i.$$

Let

$$p_1 = a_1a_2^2, \quad p_2 = a_2a_4^2, \quad p_3 = a_3a_1^2, \quad p_4 = a_4a_3^2,$$

and

$$r = a_1a_4, \quad q = a_2a_3.$$

The "center" of cycle a_0 and these variables do not depend on the choice of initial point of the periodic orbit. The system of equations is converted to the following system.

$$\begin{cases} b_1a_0 &= a_0^2 + 2(r+q) + c \\ (\lambda_1 b_1 - 4a_0)r &= p_1 + 2p_2 + 2p_3 + p_4 \\ (\gamma_1 b_1 - 4a_0)q &= 2p_1 + p_2 + p_3 + 2p_4 \\ \lambda_2 b_3 r &= -p_1 + 2p_2 - 2p_3 + p_4 \\ \gamma_2 b_3 q &= -2p_1 - p_2 + p_3 + 2p_4 \end{cases}$$

Set $g_1 = \gamma_1 b_1 - 4a_0$, $g_2 = \gamma_2 b_3$, $l_1 = \lambda_1 b_1 - 4a_0$, and $l_2 = \lambda_2 b_3$, to simplify the system and get

$$\begin{pmatrix} l_1 r \\ g_1 q \\ l_2 r \\ g_2 q \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ -1 & 2 & -2 & 1 \\ -2 & -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}.$$

We have

$$a_1a_2^2 = p_1 = (-\frac{1}{6}l_1 - \frac{1}{10}l_2)r + (\frac{1}{3}g_1 - \frac{1}{5}g_2)q,$$

$$a_2a_4^2 = p_2 = (\frac{1}{3}l_1 + \frac{1}{5}l_2)r + (-\frac{1}{6}g_1 - \frac{1}{10}g_2)q,$$
$$a_3a_1^2 = p_3 = (\frac{1}{3}l_1 - \frac{1}{5}l_2)r + (-\frac{1}{6}g_1 + \frac{1}{10}g_2)q,$$
$$a_4a_3^2 = p_4 = (-\frac{1}{6}l_1 + \frac{1}{10}l_2)r + (\frac{1}{3}g_1 + \frac{1}{5}g_2)q.$$

If $rq = 0$ and $|b| < 1$, then from the equations above, we see that the cycle degenerates to a fixed point. In the following, we assume $rq \neq 0$.

Now, we introduce a new variable ρ by

$$\rho = \frac{r}{q}.$$

Then, as

$$rq^2 = a_1a_2^2a_3^2a_4 = p_1p_4, \quad r^2q = a_1^2a_3a_2a_4^2 = p_2p_3,$$

and the right-hand sides of these equations are homogeneous functions of r and q of degree 2, r and $r\rho$ are expressed as a quadratic function of ρ.

$$r = \frac{p_1p_4}{q^2} = \frac{1}{4}(\frac{1}{9}l_1^2 - \frac{1}{25}l_2^2)\rho^2 - (\frac{1}{9}l_1g_1 + \frac{1}{25}l_2g_2)\rho + \frac{1}{9}g_1^2 - \frac{1}{25}g_2^2,$$

$$r\rho = \frac{p_2p_3}{q^2} = (\frac{1}{9}l_1^2 - \frac{1}{25}l_2^2)\rho^2 - (\frac{1}{9}l_1g_1 - \frac{1}{25}l_2g_2)\rho + \frac{1}{4}(\frac{1}{9}g_1^2 - \frac{1}{25}g_2^2).$$

By eliminating r from these equations, we get a following cubic equation of ρ, with coefficients being functions of b and a_0.

$$e_3\rho^3 + e_2\rho^2 + e_1\rho + e_0 = 0,$$

where,

$$\begin{aligned} e_3 &= 25l_1^2 - 9l_2^2, \\ e_2 &= -100l_1(g_1 + l_1) - 36l_2(g_2 - l_2), \\ e_1 &= 100g_1(g_1 + l_1) - 36g_2(g_2 + l_2), \\ e_0 &= -25g_1^2 + 9g_2^2. \end{aligned}$$

For fixed parameter value b, this equation is of degree 3 in ρ and of degree 2 in a_0. This equation, expressed as a quadratic equation in a_0, is as follows.

$$f_2a_0^2 + f_1a_0 + f_0 = 0,$$

with

$$f_2 = 400(\rho^3 - 8\rho^2 + 8\rho - 1),$$

$$f_1 = 200b_1(-\lambda_1\rho^3 - 2(1-2\lambda_1)\rho^2 + 2(1-2\gamma_1)\rho + \gamma_1),$$

$$\begin{aligned} f_0 = {} & (25(2+\gamma_1)b_1^2 + 9(2-\gamma_1)b_3^2)\rho^3 - (100(1+\gamma_1)b_1^2 + 36(2-\lambda_1)b_3^2)\rho^2 \\ & + (100(1+\lambda_1)b_1^2 + 36(2-\gamma_1)b_3^2)\rho - 25(2+\lambda_1)b_1^2 - 9(2-\lambda_1)b_3^2. \end{aligned}$$

Unfortunately, in the case of cycles of period five, we do not have a single "cycle equation" to find a cycle variable, from given parameters (b, c). Our "cycle equation" is given by a system of equations.

$$\text{(CE5)} \qquad \begin{cases} f_2a_0^2 + f_1a_0 + f_0 & = \quad 0, \\ \qquad c & = \quad b_1a_0 - a_0^2 - 2(r+q). \end{cases}$$

One equation to determine the cycle variable, and the other to describe the relation to parameter c.

However, if we are given a parameter value of b and a value ρ, we can solve the above equation to find a_0(two solutions). Or alternately, if we are given parameter value of b and a value of the "center" a_0, we can solve the above equation to find ρ(three solutions). And r, q and c are computed by

$$r = \frac{1}{4}(\frac{1}{9}l_1^2 - \frac{1}{25}l_2^2)\rho^2 + (\frac{1}{9}l_1g_1 - \frac{1}{25}l_2g_2)\rho + \frac{1}{9}g_1^2 - \frac{1}{25}g_2^2,$$

$$q = r/\rho,$$

$$c = b_1a_0 - a_0^2 - 2(r+q).$$

Finally, with these data, we can compute a_1,a_2,a_3, and a_4 as follows. Take a quintic root of

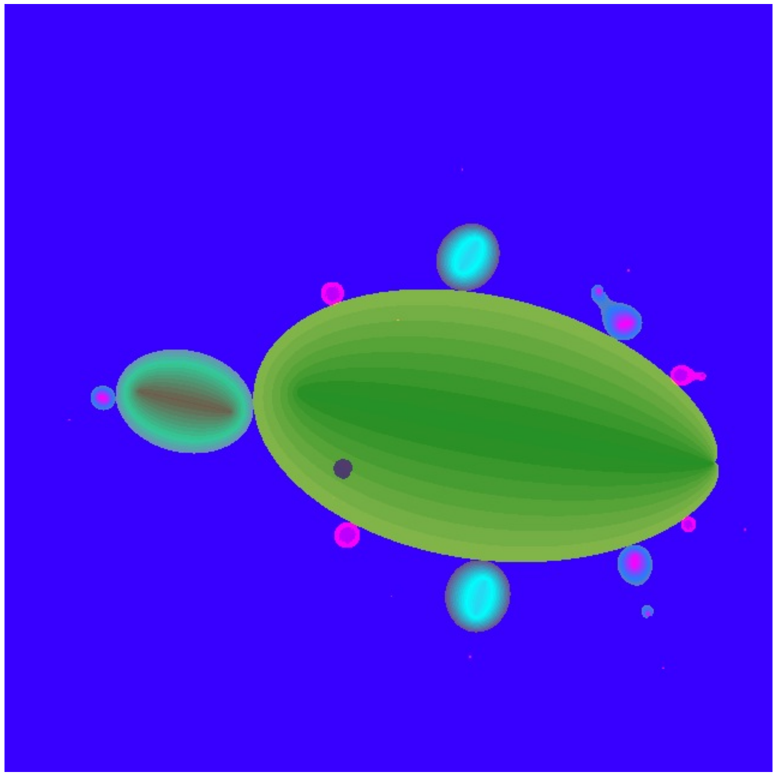
$$a_1^5 = p_1p_3^2/q^2.$$

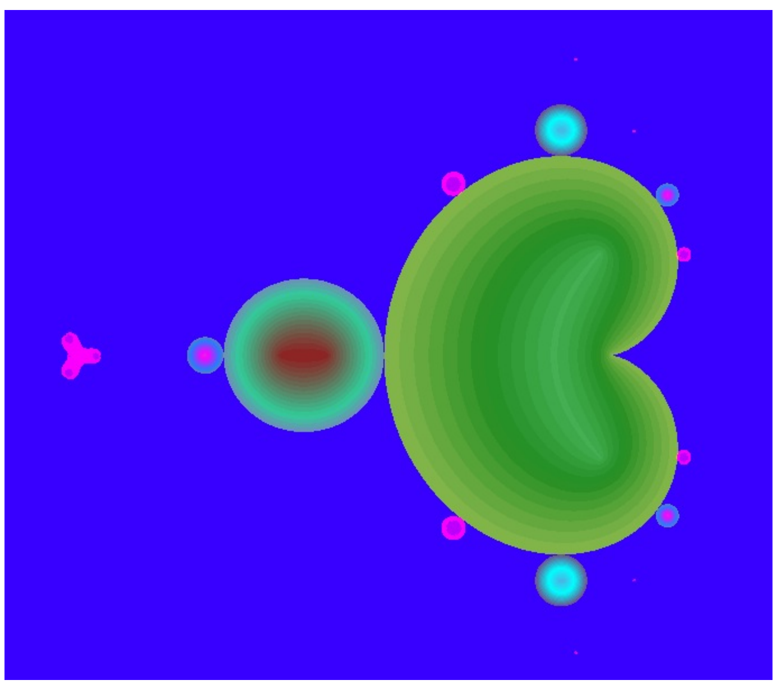
Then

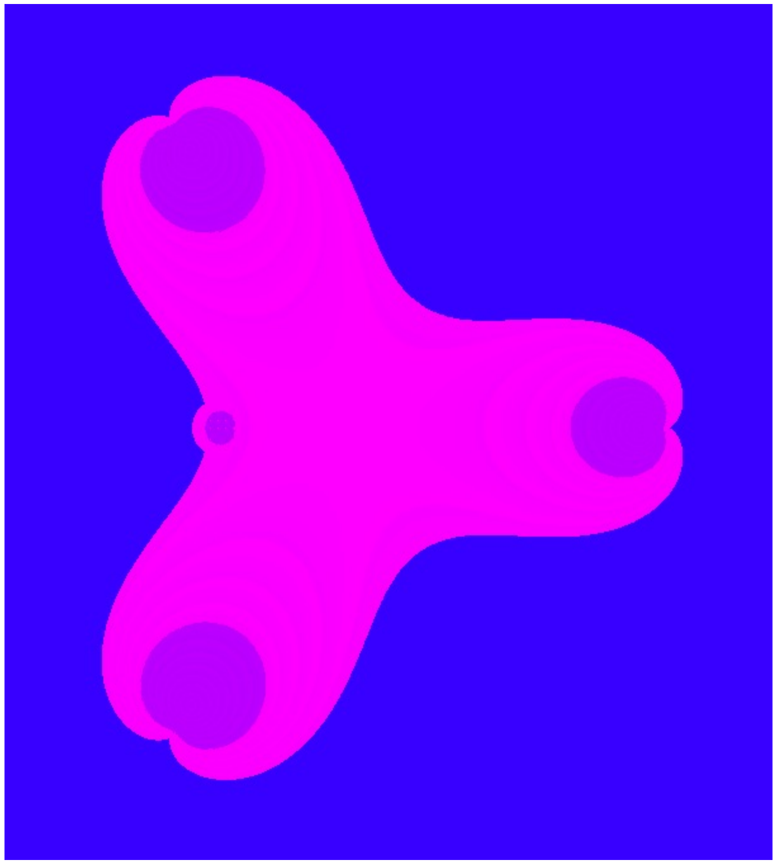
$$a_3 = p_3/a_1^2, \quad a_2 = q/a_3, \quad a_4 = r/a_1,$$

which gives the periodic points.

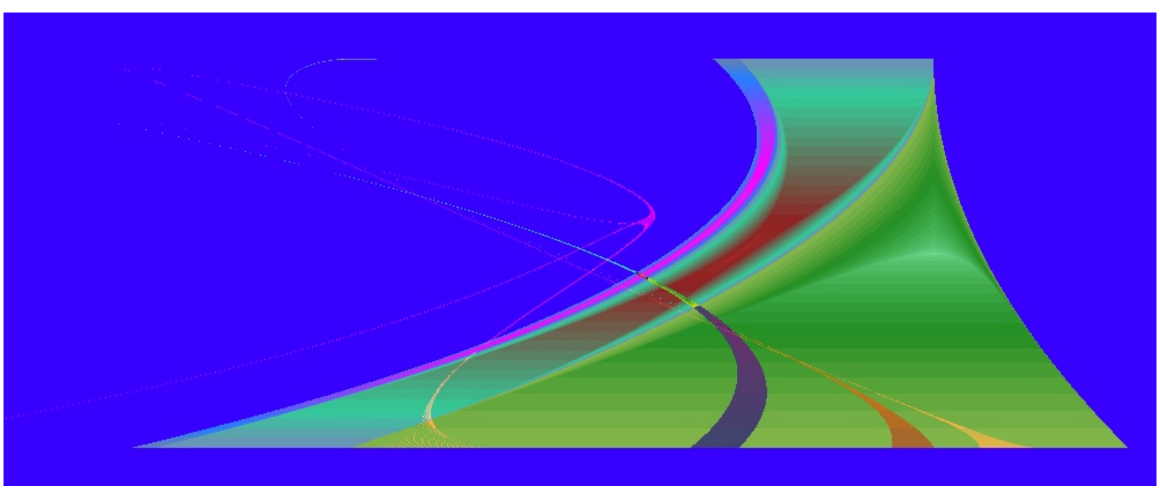
Some complex slices $\{b = -0.3847 + 0.0854i\}$, and $\{b = 0.15\}$ are shown in the following picture. Hyperbolic components, so to say, of periods up to 5 are seen. The three-petals region in the enlargement picture represents a region of attracting 5-cycles. It is a slice of the swallow's tail region for period 5 attractor.



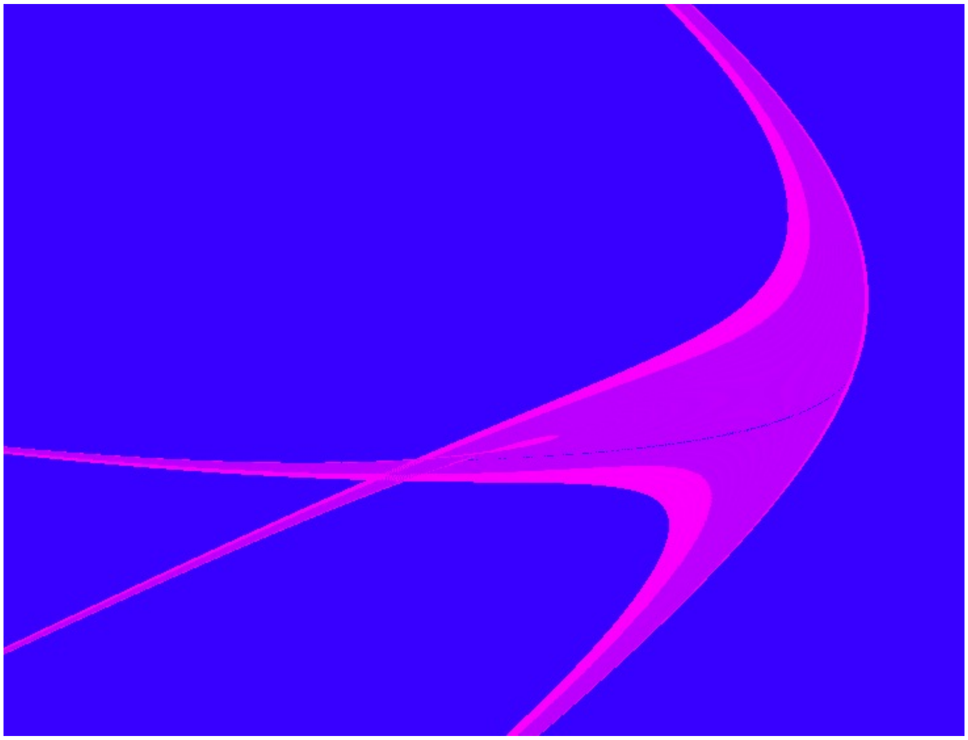




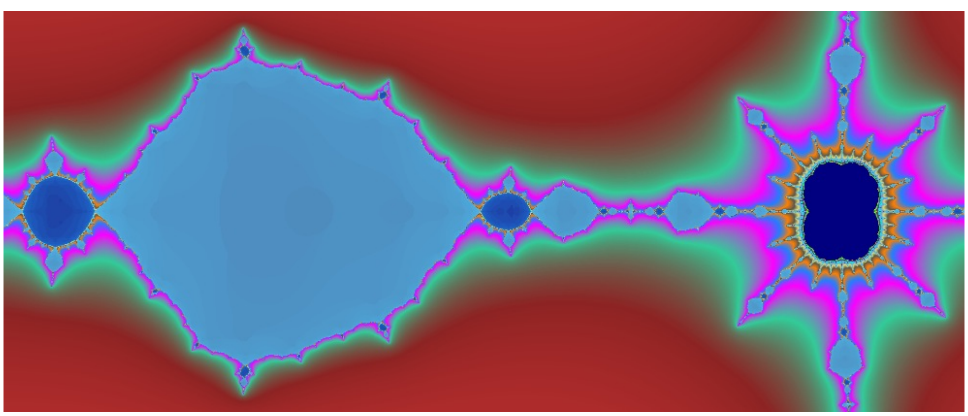
THe following picture shows the real axis slice of the parameter space. For parameters in the swallow's tail region, there is/are attracting 5-cycle(s).



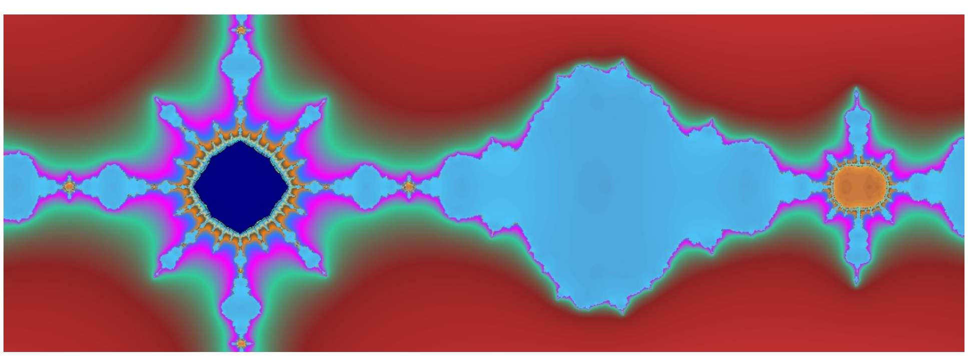
Next picture is an enlargement of the swallow's tail region.



Next picture shows a system with attracting cycles of periods 1, 3, and 5.



Next picture is a system with attracting cycles of periods 1, 4, and 5.



References

[1] H. Poincaré : Sur une classe nouvelle des transcendantes uniformes, Journal de Mathématiques, 4^e série, tôme VI, Fasc. IV(1890).
[2] M. Hénon : A two-dimensional mapping with a strange attractor, Comm. Math. Phys. 50(1976), pp69-77.
[3] S. Ushiki : Unstable manifolds of analytic dynamical systems, Journal of Mathematics of Kyoto University, Vol. 21, No. 4(1981), pp763-785.
[4] H. E. Hamouly et C. Mira : Lien entre les propriété d'un endomorphisme de dimension un et celles d'un difféomorphisme de dimension deux, C. R. Acad. Sc. Paris, t. 293(1981), Série I, pp525-528.