Parabolic bifurcations of area-preserving Hénon maps

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Abstract

Real one parameter family of volume preserving complex Hénon maps is studied. Cycle of neutral periodic points bifurcates from a parabolic fixed point. Cases of periods 3 and 4 are computed directly. In the area preserving real Hénon maps, pair of a cycle of saddle type and a cycle of center type appears from a parabolic fixed point. Neutral periodic cycles are observed as so-called "islands" between KAM circles around a neutral fixed points. In this note, the appearance of pair of periodic orbits of center type and saddle type is proved for period 5 cases.

1. Area-preserving complex Hénon map

In this section, we consider a complex one-parameter family $H_\alpha : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$H_\alpha(x, y) = (y, y^2 + \alpha - x),$$

parametrized by a complex parameter α. The determinant of the Jacobian matrix is always equal to 1. If α is real, then H_α maps the real axis $\mathbb{R}^2$ into itself and defines an area-preserving real diffeomorphism.

The fixed point $P_* = (y_*, y_*)$ of our Hénon map is given by quadratic equation $y_*^2 - 2y_* + \alpha = 0$. The Jacobian matrix at the fixed point is as follows.

$$DH_\alpha|_{P_*} = \begin{pmatrix} 0 & 1 \\ -1 & 2y_* \end{pmatrix}, \qquad \text{trace } DH_\alpha = 2y_*, \quad \det DH_\alpha = 1.$$

2. Parabolic bifurcation of order 3

Let us consider the case where $\omega = \frac{-1+\sqrt{3}i}{2}$ and $\bar{\omega} = \frac{-1-\sqrt{3}i}{2}$ are the eigenvalues of DH_α at the fixed point P_*. Then,

$$y_* = \frac{1}{2}(\omega+\bar{\omega}) = -\frac{1}{2} \quad \text{and} \quad \alpha_* = 2y_* - y_*^2 = -\frac{5}{4}.$$

Compute periodic points of period 3 as follows. Let $y_n = u_0 + \omega^n u_1 + \bar{\omega}^n u_2$, and suppose $y_{n+1} = y_n^2 + \alpha - y_{n-1}$ holds. We have a system of equations for cycles of period 3.

$$(F) \qquad \begin{cases} 2u_0 &= u_0^2 + 2u_1u_2 + \alpha \\ (\omega+\bar{\omega})u_1 &= 2u_0u_1 + u_2^2 \\ (\bar{\omega}+\omega)u_2 &= 2u_0u_2 + u_1^2 \end{cases}$$

When $\alpha = \alpha_*$, then we have a solution $u_0 = y_*, u_1 = u_2 = 0$. We fix constants $\alpha_* = -\frac{4}{5}$, $y_* = -\frac{1}{2}$ and set $u_0 = u_0(\varepsilon) = y_* - \frac{\varepsilon}{2}$. The second and third equations of (F) are rewritten as follows.

$$\begin{cases} \varepsilon u_1 &= u_2^2 \\ \varepsilon u_2 &= u_1^2 \end{cases}$$

We obtain $u_1 = \varepsilon\omega^k, u_2 = \varepsilon\bar{\omega}^k$, $(k = 0, 1, 2)$. The choice of k corresponds to the choice of initial point in the periodic orbit. We choose $k = 0$ and obtain the solution

$$u_0 = y_* - \frac{\varepsilon}{2}, \qquad u_1 = \varepsilon, \qquad u_2 = \varepsilon.$$

It follows that

$$\alpha = \alpha_* - \frac{3}{2}\varepsilon - \frac{9}{4}\varepsilon^2 = -\frac{9}{4}(\varepsilon + \frac{1}{3})^2 - 1,$$

$$y_0 = -\frac{1}{2} + \frac{3}{2}\varepsilon, \qquad y_1 = -\frac{1}{2} - \frac{3}{2}\varepsilon, \qquad y_2 = -\frac{1}{2} - \frac{3}{2}\varepsilon.$$

The trace of the Jacobian matrix of the 3-cycle is given by the following.

$$\tau(\varepsilon) = \mathrm{trace}(DH_{P_2}DH_{P_1}DH_{P_0}) = 8y_2y_1y_0 - 2(y_2+y_1+y_0) = 2 + 9\varepsilon^2 + 27\varepsilon^3.$$

And

$$\tau(0) = 2, \quad \frac{d\tau}{d\varepsilon} = 9\varepsilon(9\varepsilon+2), \quad \tau(-\frac{1}{3}) = 2, \quad \tau(\frac{2}{3}) = -2.$$

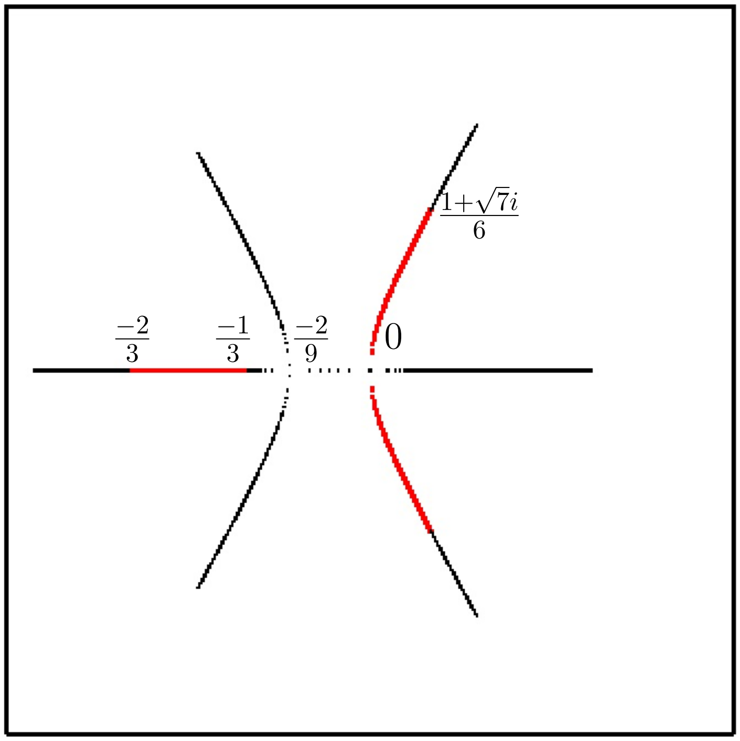

Fig.1 $\{\varepsilon \mid \tau(\varepsilon) \in [-2,2]\}$ is drawn in red.

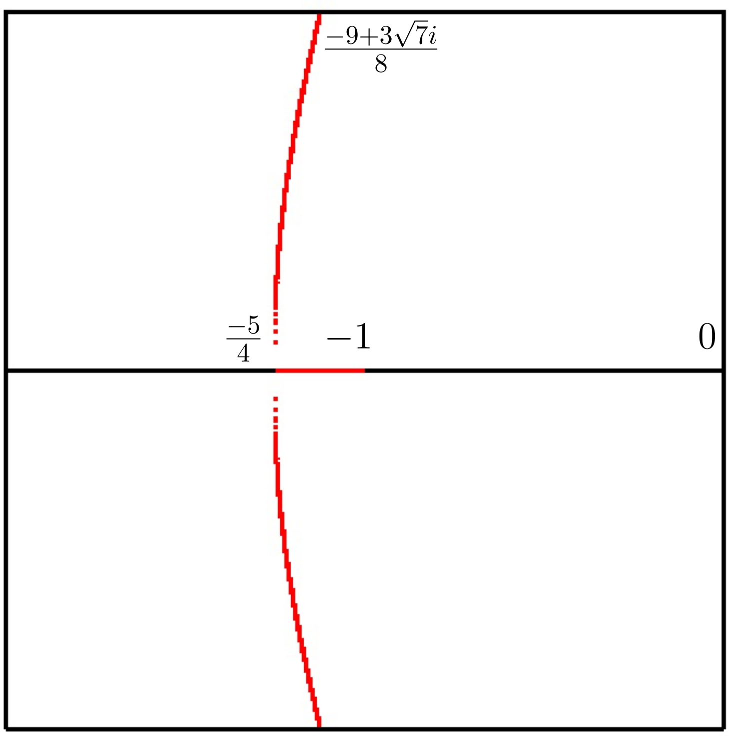

Fig.2 $\{\alpha(\varepsilon) \mid \tau(\varepsilon) \in [-2,2]\}$ is drawn in red.

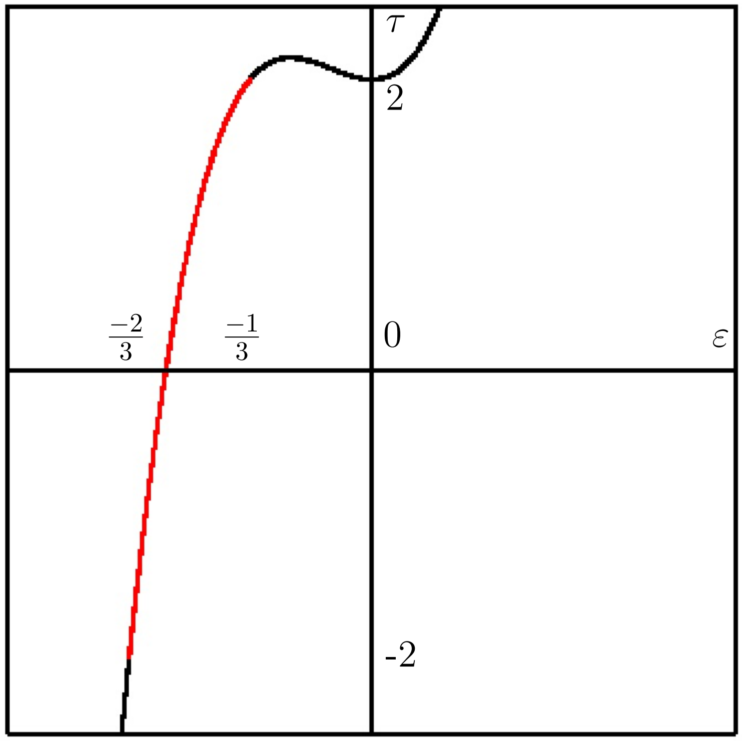

Fig.3. graph of $\tau(\varepsilon)$ for $-1 \leq \varepsilon \leq 1$.

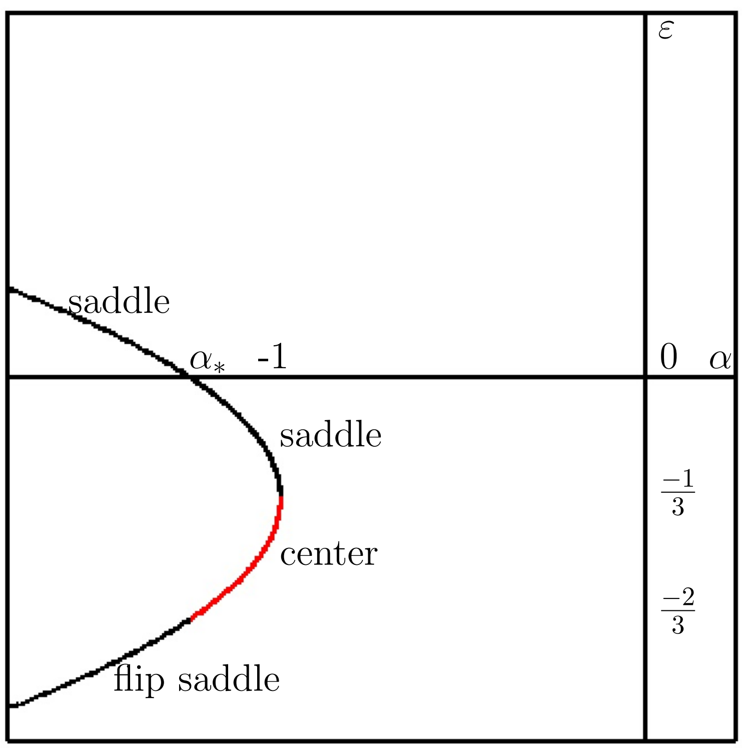

Fig.4. Bifurcation diagram of cycles of period 3.

As for the real cycles for real parameter α, we pick up the cycles corresponding to real values of ε, as shown in Fig.3. The bifurcation diagram of 3-cycles is plotted in Fig.4. Cycle of saddle type with negative eigenvalues are plotted as flip saddle.

3. Parabolic bifurcation of order 4

Let us consider the case where $\pm i$ are the eigenvalues of DH_α at the fixed point P_*. Then $y_* = 0$ and $\alpha_* = 0$.

Recall the equation of 4-periodic point. $(y_{n+4} = y_n)$

$$y_{n+1} = y_n^2 + \alpha - y_{n-1}, \qquad n = 0, \cdots, 3.$$

Discrete Fourier expansion

$$y_n = u_0 + i^n u_1 + (-1)^n u_2 + (-i)^n u_3$$

gives rise to the following system of equations.

$$\begin{aligned}
(F_0) \quad 2u_0 &= u_0^2 + u_2^2 + 2u_1u_3 + \alpha, \\
(F_1) \quad 0 &= 2u_0u_1 + 2u_2u_3, \\
(F_2) \quad -2u_2 &= 2u_0u_2 + u_1^2 + u_3^2, \\
(F_3) \quad 0 &= 2u_0u_3 + 2u_1u_2.
\end{aligned}$$

From (F_1) and (F_3), we have

$$\begin{pmatrix} u_0 & u_2 \\ u_2 & u_0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_3 \end{pmatrix} = \mathbf{0}.$$

To have a non-trivial 4-cycle, it is necessary to have $u_0^2 = u_2^2$.

Case I $\quad u_0 = u_2 = 0$.
In this case, from (F_2) and (F_0), we have two sub-cases

$$u_3 = iu_1, \quad \alpha = 2iu_1^2, \quad y_0 = y_1 = (1+i)u_1, \quad y_2 = y_3 = -(1+i)u_1,$$

and

$$u_3 = -iu_1, \quad \alpha = -2iu_1^2, \quad y_0 = y_1 = (1-i)u_1, \quad y_2 = y_3 = -(1-i)u_1.$$

They give the same 4-cycle. And the trace of the 4-cycle is given by $\tau = 2 - 64u_1^4$.

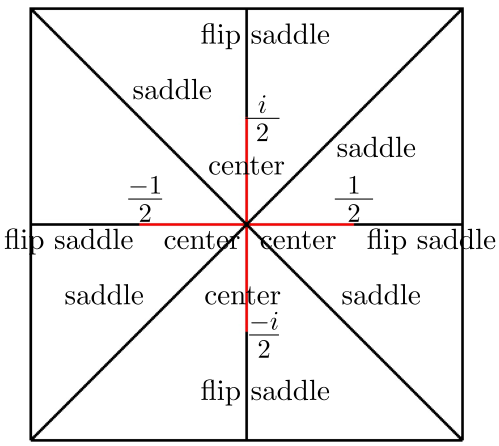

Fig.5. $\{u_1 \mid \tau(u_1) \in [-2, 2]\}$ is drawn in red in u_1 space.

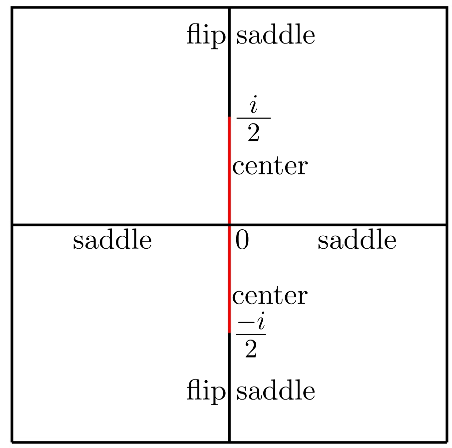

Fig.6. α space for CASE I.

In this case, real cycles for real α are all saddles.

CASE II $\quad u_2 = -u_0$ and $u_3 = u_1$.
From (F_2), $u_1 = \pm\sqrt{u_0^2 + u_0}$.

$$y_0 = 2u_1, \quad y_1 = 2u_0, \quad y_2 = -2u_1, \quad y_3 = 2u_0.$$

CASE III $\quad u_0 = u_2$ and $u_3 = -u_1$.
From (F_2), $u_1 = \pm\sqrt{-u_0^2 - u_0}$.

$$y_0 = 2u_0, \quad y_1 = 2iu_1, \quad y_2 = 2u_0, \quad y_3 = -2iu_1.$$

This gives the same orbit as in CASE II.

In these cases the trace of the 4-cycle is given by $\tau = 2 - 256u_0^3(1 + u_0)$. And $\alpha = -4u_0^2$. For real u_0, the trace τ is real and plotted in Fig.7.
Location of the u_0 values with $\tau(u_0) \in [-2, 2]$ is plotted in Fig.8.
And the corresponding values of α are plotted in Fig.9 and Fig.10.
In Figs 9 and 10, segment $[\frac{-i}{2}, \frac{i}{2}]$ of CASE I is plotted,too.

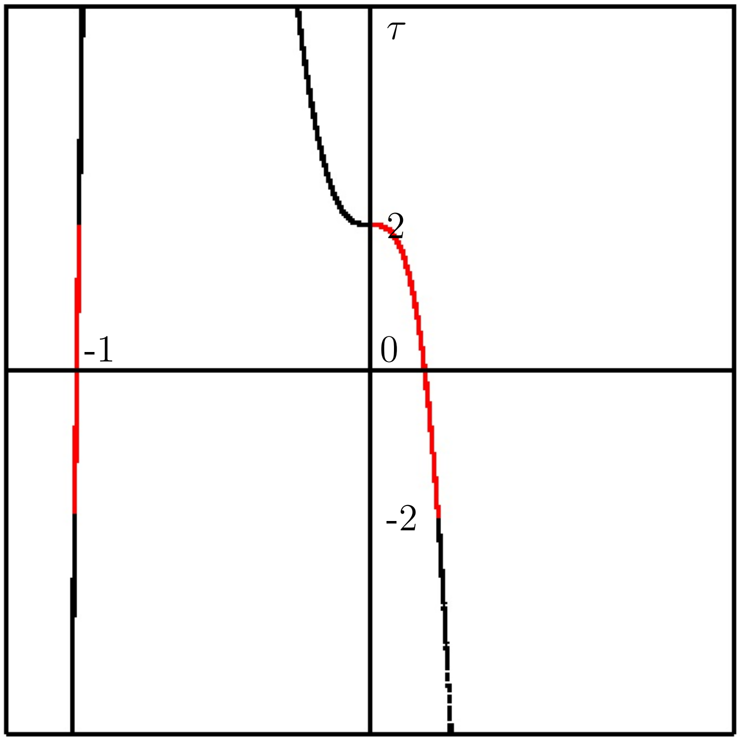

Fig.7. Graph of $\tau(u_0)$ for real u_0.

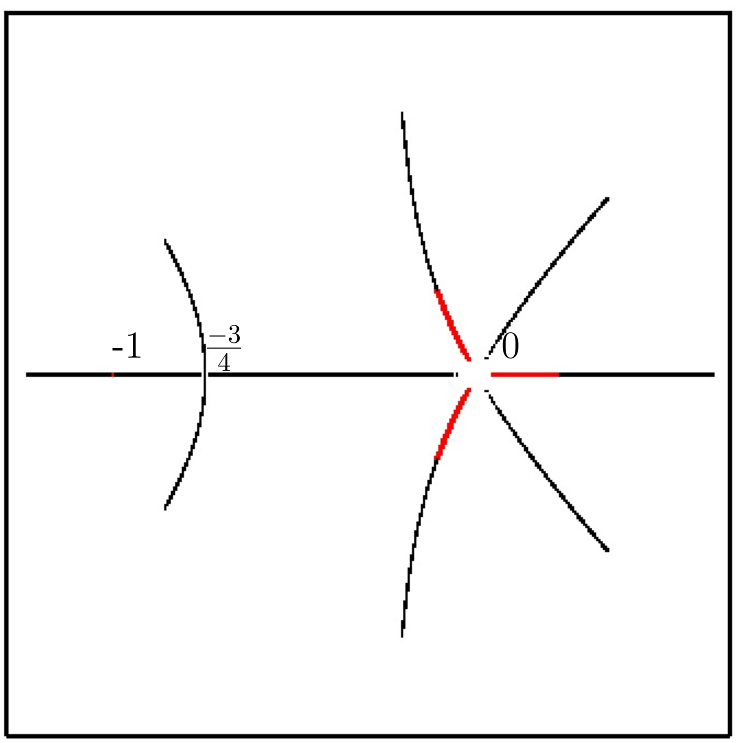

Fig.8. $\{u_0 \mid \tau(u_0) \in [-2, 2]\}$ is drawn in red.
Observe that a short interval near -1 is in red.

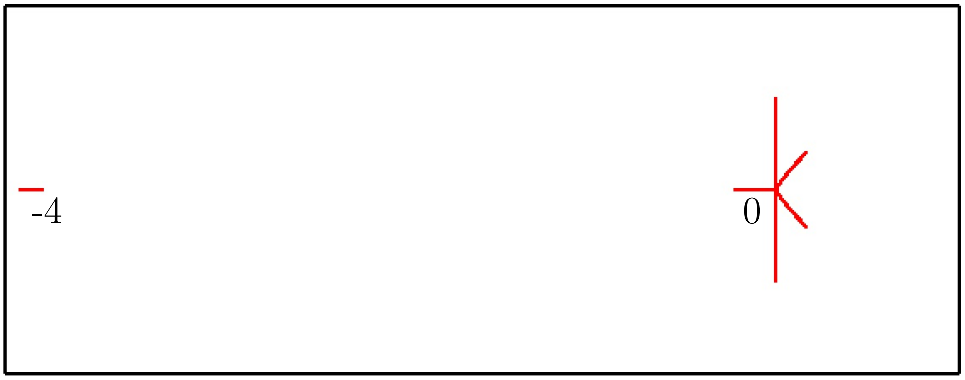

Fig.9. $\{\alpha(\varepsilon) \mid \tau(\varepsilon) \in [-2, 2]\}$ is drawn in red.

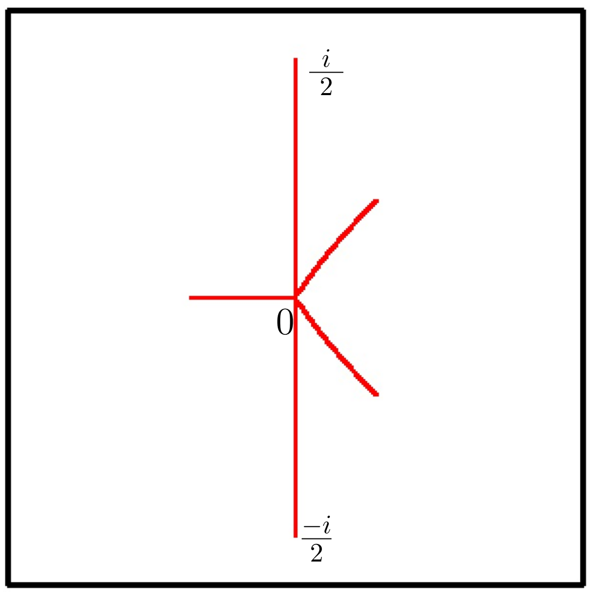

Fig.10. Enlargement of fig.6.

The bifurcation diagram for real parameter α and real 4-cycles is plotted in Fig.11.

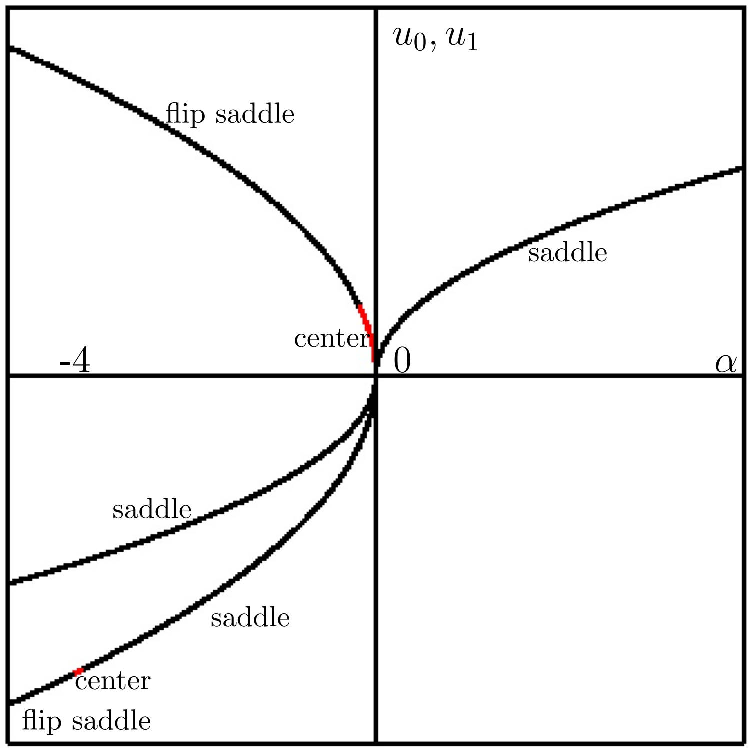

Fig.11. Bifurcation diagram of real 4-cycles for real α.

4. Parabolic bifurcation of order 5

In this section, we consider the case where the eigenvalue of the fixed point is a primary fifth root of unity. Let $\Omega = e^{\frac{2\pi i}{5}}$ or $\Omega = e^{\frac{4\pi i}{5}}$ denote a primary fifth root of unity. And suppose eigenvalues at the fixed point P_* are Ω and $\bar{\Omega}$. In the following, we treat two cases with same notations.

We set $\omega_1 = \Omega + \bar{\Omega} = \frac{\pm\sqrt{5}-1}{2}$, $\omega_2 = \Omega^2 + \bar{\Omega}^2 = \frac{\mp\sqrt{5}-1}{2}$, with $\omega_1\omega_2 < 0$. then,

$$y_* = \frac{\Omega + \bar{\Omega}}{2}, \qquad \alpha_0 = 2y_* - y_*^2 = \frac{-7 \pm \sqrt{5}}{8}.$$

THEOREM A A pair of cycles of period 5 bifurcates from the fixed point P_* for $\alpha < \alpha_0$ near α_0. One of the cycles is saddle type and the other is center type.

Let $\gamma = \omega_2 - \omega_1 = \mp\sqrt{5}$ and

$$\rho_1(n) = \begin{cases} 2 & (n \equiv 0, \mod 5) \\ \omega_1 & (n \equiv 1 \text{ or } 4) \\ \omega_2 & (n \equiv 2 \text{ or } 3) \end{cases}, \quad \rho_2(n) = \begin{cases} 2 & (n \equiv 0, \mod 5) \\ \omega_2 & (n \equiv 1 \text{ or } 4) \\ \omega_1 & (n \equiv 2 \text{ or } 3) \end{cases}.$$

Theorem B There exists a function

$$\alpha(\varepsilon) = \alpha_0 + \gamma\omega_1\varepsilon^2 + \gamma\varepsilon^3 + \cdots$$

and a family of periodic sequences

$$y_n = y_* - \frac{\gamma}{5}\varepsilon^2 + \rho_1(n)(\varepsilon - \frac{\gamma}{4}\varepsilon^2 + \cdots) + \rho_2(n)(\frac{\gamma}{5}\varepsilon^2 - \frac{1}{10}\varepsilon^3 + \cdots)$$

holomorphic in ε near $0 \in \mathbb{C}$, such that for each ε, $H_{\alpha(\varepsilon)}$ has a cycle $\{P_n = (y_n, y_{n+1})\}$ of period 5.

Theorem C The trace function of the cycle

$$\tau(\varepsilon) = \text{ trace } DH^{\circ 5}_{\alpha(\varepsilon)}\Big|_{(y_0,y_1)}$$

is real analytic in ε and not constant near $\varepsilon = 0$, with $\tau(0) = 2$.

5. Discrete Fourier expansion and the principal part of the solution

Apply the discrete Fourier expansion method

$$y_n = u_0 + \Omega^n u_1 + \Omega^{2n} u_2 + \bar{\Omega}^{2n} u_3 + \bar{\Omega}^n u_4,$$

to the periodic sequence $\{y_n\}$ satisfying $y_{n+1} + y_{n-1} = y_n^2 + \alpha$. We get a system of equations:

$$\begin{aligned}
(F_0) \quad 2u_0 &= u_0^2 + 2u_1u_4 + 2u_2u_3 + \alpha, \\
(F_1) \quad \omega_1 u_1 &= 2u_0u_1 + u_3^2 + 2u_2u_4, \\
(F_2) \quad \omega_2 u_2 &= 2u_0u_2 + u_1^2 + 2u_3u_4, \\
(F_3) \quad \omega_2 u_3 &= 2u_0u_3 + u_4^2 + 2u_1u_2, \\
(F_4) \quad \omega_1 u_4 &= 2u_0u_4 + u_2^2 + 2u_1u_3.
\end{aligned}$$

Here, α appears only in the first equation (F_0). Let δ denote a constant, which will be determined as $\delta = \frac{2}{\gamma}$ later, and let

$$u_0 = y_* - \frac{\delta}{2}\varepsilon^2 = \frac{\omega_1}{2} - \frac{\delta}{2}\varepsilon^2.$$

Replace u_0 in equations $(F_1), \cdots, (F_4)$ to obtain a system of algebraic equations parametrized by ε:

$$\begin{aligned}
(F_{\varepsilon,1}) \quad & \delta\varepsilon^2 u_1 &=& \ u_3^2 + 2u_2u_4, \\
(F_{\varepsilon,2}) \quad & (\gamma + \delta\varepsilon^2)u_2 &=& \ u_1^2 + 2u_3u_4, \\
(F_{\varepsilon,3}) \quad & (\gamma + \delta\varepsilon^2)u_3 &=& \ u_4^2 + 2u_1u_2, \\
(F_{\varepsilon,4}) \quad & \delta\varepsilon^2 u_4 &=& \ u_2^2 + 2u_1u_3.
\end{aligned}$$

The difference of the both sides defines a polynomial mapping:

$$F_\varepsilon : \mathbb{C}^4 \to \mathbb{C}^4, \qquad F_\varepsilon(\mathbf{u}) = \mathbf{0}, \qquad \mathbf{u} = (u_1, u_2, u_3, u_4).$$

Clearly, $\mathbf{u} = \mathbf{0}$ is always a solution. The solutions of our system of algebraic equations $F_\varepsilon(\mathbf{u}) = \mathbf{0}$ are defined as an algebraic set. So, the solving process is essentially a resolution of singularities.

By weighted scaling of variables:

$$u_1 = \varepsilon v_1, \quad u_2 = \varepsilon^2 v_2, \quad u_3 = \varepsilon^2 v_3, \quad u_4 = \varepsilon v_4,$$

and weighted scaling of equations, assuming $\varepsilon \neq 0$, we obtain:

$$\begin{aligned}
(G_{\varepsilon,1}) \quad & \delta v_1 &=& \ 2v_2v_4 + \varepsilon v_3^2, \\
(G_{\varepsilon,2}) \quad & (\gamma + \delta\varepsilon^2)v_2 &=& \ v_1^2 + 2\varepsilon v_3v_4, \\
(G_{\varepsilon,3}) \quad & (\gamma + \delta\varepsilon^2)v_3 &=& \ v_4^2 + 2\varepsilon v_1v_2, \\
(G_{\varepsilon,4}) \quad & \delta v_4 &=& \ 2v_1v_3 + \varepsilon v_2^2.
\end{aligned}$$

These define an equation $G_\varepsilon(\mathbf{v}) = \mathbf{0}$, with $\mathbf{v} = \mathbf{v}(\varepsilon) = (v_1, v_2, v_3, v_4)$. Let $\mathbf{a} = \mathbf{v}(0)$. Then $\mathbf{a} = (a_1, a_2, a_3, a_4)$ is a solution of (G_0):

$$\delta a_1 = 2a_2a_4, \quad \gamma a_2 = a_1^2, \quad \gamma a_3 = a_4^2, \quad \delta a_4 = 2a_1a_3.$$

Now, determine the constant $\delta = \frac{2}{\gamma}$, as noticed above, to obtain non-trivial solutions in a simple form. Suppose $a_1 \neq 0$. then we have

$$a_2 = \frac{a_1^2}{\gamma}, \quad a_3 = \frac{a_1^{-2}}{\gamma}, \quad a_4 = a_1^{-1}.$$

Here, $\mathbf{a}$ is not uniquely determined. In the next section, we see a_1 must satisfy another condition to have a nontrivial family of solutions $\mathbf{v}(\varepsilon)$.

6. Second jet

In the previous section, $\mathbf{a}$ was not uniquely determined. Let $a_1 = \sigma$, and $\mathbf{a} = \mathbf{a}(\sigma) = (\sigma, \frac{\sigma^2}{\gamma}, \frac{\sigma^{-2}}{\gamma}, \sigma^{-1})$. Then $G_0(\mathbf{a}(\sigma)) = 0$ holds for $\sigma \in \mathbb{C} \setminus \{0\}$.

Let $\mathbf{v} = \mathbf{a} + \varepsilon \mathbf{w}$, $\mathbf{v} = (v_1, v_2, v_3, v_4)$,$\mathbf{w} = (w_1, w_2, w_3, w_4)$, $v_i = a_i + \varepsilon w_i$, and rewrite the equation (G_ε).

$$(M_{\varepsilon,1}) \quad \frac{2}{\gamma} w_1 = \frac{\sigma^{-4}}{\gamma^2} + \frac{2\sigma^2}{\gamma} w_4 + 2\sigma^{-1} w_2 + \varepsilon(\frac{2\sigma^{-2}}{\gamma} w_3 + 2w_2 w_4) + \varepsilon^2 w_3^2,$$

$$(M_{\varepsilon,2}) \quad \gamma w_2 + \varepsilon(\frac{2\sigma^2}{\gamma^2} + \frac{2\varepsilon}{\gamma} w_2) = 2\sigma w_1 + \frac{2\sigma^{-3}}{\gamma} + \varepsilon(w_1^2 + \frac{2\sigma^{-2}}{\gamma} w_4 + 2\sigma^{-1} w_3) + 2\varepsilon^2 w_3 w_4,$$

$$(M_{\varepsilon,3}) \quad \gamma w_3 + \varepsilon(\frac{2\sigma^{-2}}{\gamma^2} + \frac{2\varepsilon}{\gamma} w_3) = 2\sigma^{-1} w_4 + \frac{2\sigma^3}{\gamma} + \varepsilon(w_4^2 + \frac{2\sigma^2}{\gamma} w_1 + 2\sigma w_2) + 2\varepsilon^2 w_1 w_2,$$

$$(M_{\varepsilon,4}) \quad \frac{2}{\gamma} w_4 = \frac{\sigma^4}{\gamma^2} + \frac{2\sigma^{-2}}{\gamma} w_1 + 2\sigma w_3 + \varepsilon(\frac{2\sigma^2}{\gamma} w_2 + 2w_1 w_3) + \varepsilon^2 w_2^2.$$

Here, $\mathbf{w}$ is supposed to be an analytic function of ε, and let $\mathbf{w} = \mathbf{b} + O(\varepsilon)$, $\mathbf{b} = (b_1, b_2, b_3, b_4)$, with $w_i = b_i + O(\varepsilon)$. The principal part of (M_ε) is obtained by letting $\varepsilon \to 0$.

$$(M_{0,1}) \quad \frac{2}{\gamma} b_1 = \frac{\sigma^{-4}}{\gamma^2} + \frac{2\sigma^2}{\gamma} b_4 + 2\sigma^{-1} b_2,$$

$$(M_{0,2}) \quad \gamma b_2 = 2\sigma b_1 + \frac{2\sigma^{-3}}{\gamma},$$

$$(M_{0,3}) \quad \gamma b_3 = 2\sigma^{-1} b_4 + \frac{2\sigma^3}{\gamma},$$

$$(M_{0,4}) \quad \frac{2}{\gamma} b_4 = \frac{\sigma^4}{\gamma^2} + \frac{2\sigma^{-2}}{\gamma} b_1 + 2\sigma b_3.$$

Rewrite this system of equations as follows.

$$\begin{pmatrix} -\frac{2}{\gamma} & 2\sigma^{-1} & 0 & \frac{2\sigma^2}{\gamma} \\ 2\sigma & -\gamma & 0 & 0 \\ 0 & 0 & -\gamma & 2\sigma^{-1} \\ \frac{2\sigma^{-2}}{\gamma} & 0 & 2\sigma & -\frac{2}{\gamma} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} -\frac{\sigma^{-4}}{\gamma^2} \\ -\frac{2\sigma^{-3}}{\gamma} \\ -\frac{2\sigma^3}{\gamma} \\ -\frac{\sigma^4}{\gamma^2} \end{pmatrix}.$$

Here, the rank the coefficient matrix is 3. So, to have a non-trivial solution of $\mathbf{b}$, it is necessary that the rank of the extended matrix is 3, too. This condition gives $\sigma^5 - \sigma^{-5} = 0$. Note that $\mathbf{b}$ is not uniquely determined here.

Proposition. Without loss of generalities, we can choose $\sigma = 1$. Other choices of σ give the same family of cycles.

Proof. As replacing a_1 by $\Omega^k a_1$ changes the initial point of the periodic orbit, we only need to examine the cases $\sigma = \pm 1$. Observe the equation (M_ε) carefully. The equation has a kind of symmetry.

Let $\sigma' = -\sigma$, $\varepsilon' = -\varepsilon$, and

$$w_1'(\varepsilon') = w_1(-\varepsilon'), \quad w_2'(\varepsilon') = -w_2(-\varepsilon'),$$

$$w_3'(\varepsilon') = -w_3(-\varepsilon'), \quad w_4'(\varepsilon') = w_4(-\varepsilon').$$

Then if

$$u_0 = y_* - \frac{1}{\gamma}\varepsilon^2, \quad u_1 = \varepsilon\sigma + \varepsilon^2 w_1(\varepsilon), \quad u_2 = \frac{1}{\gamma}\varepsilon^2 + \varepsilon^3 w_2(\varepsilon),$$

$$u_3 = \frac{1}{\gamma}\varepsilon^2 + \varepsilon^3 w_3(\varepsilon), \quad u_4 = \varepsilon\sigma + \varepsilon^2 w_4(\varepsilon)$$

is a solution of (M_ε^σ), then

$$u_0 = y_* - \frac{1}{\gamma}(\varepsilon')^2, \quad u_1 = \varepsilon'\sigma' + (\varepsilon')^2 w_1'(\varepsilon'), \quad u_2 = \frac{1}{\gamma}(\varepsilon')^2 + (\varepsilon')^3 w_2'(\varepsilon'),$$

$$u_3 = \frac{1}{\gamma}(\varepsilon')^2 + (\varepsilon')^3 w_3'(\varepsilon'), \quad u_4 = \varepsilon'\sigma' + (\varepsilon')^2 w_4'(\varepsilon')$$

is a solution of $(M_{\varepsilon'}^{\sigma'})$. Equations (M_ε^σ) and $(M_{\varepsilon'}^{\sigma'})$ are equivalent.

In the following, we treat only the case of $\sigma = 1$.

7. Further change of variables and rescaling of equations

From equations $(M_{\varepsilon,1}), \cdots, (M_{\varepsilon,4})$, we obtain, by setting $\sigma = 1$, and putting all terms on the righthand side,

$$
\begin{aligned}
(M_1') \quad 0 &= \frac{1}{\gamma^2} - \frac{2}{\gamma} w_1 + 2w_2 + \frac{2}{\gamma} w_4 + \varepsilon(\frac{2}{\gamma} w_3 + 2w_2 w_4) + \varepsilon^2 w_3^2, \\
(M_2') \quad 0 &= \frac{2}{\gamma} + 2w_1 - \gamma w_2 + \varepsilon(-\frac{2}{\gamma^2} + w_1^2 + 2w_3 + \frac{2}{\gamma} w_4) \\
&\qquad + \varepsilon^2(-\frac{2}{\gamma} w_2 + 2w_3 w_4), \\
(M_3') \quad 0 &= \frac{2}{\gamma} - \gamma w_3 + 2w_4 + \varepsilon(-\frac{2}{\gamma^2} + \frac{2}{\gamma} w_1 + 2w_2 + w_4^2) \\
&\qquad + \varepsilon^2(-\frac{2}{\gamma} w_3 + 2w_1 w_2), \\
(M_4') \quad 0 &= \frac{1}{\gamma^2} + \frac{2}{\gamma} w_1 + 2w_3 - \frac{2}{\gamma} w_4 + \varepsilon(\frac{2}{\gamma} w_2 + 2w_1 w_3) + \varepsilon^2 w_2^2.
\end{aligned}
$$

Observe the symmetry of the equations with respect to the variables. By change of variables:

$$p = w_1 + w_4, \quad q = w_2 + w_3, \quad r = w_2 - w_3, \quad s = w_1 - w_4,$$

and change of equations:

$$
\begin{aligned}
(P) = (M_1') + (M_4'), \quad (Q) = (M_2') + (M_3'), \\
(R) = (M_2') - (M_3'), \quad (S) = (M_1') - (M_4'),
\end{aligned}
$$

we obtain the following system of equations. (Terms as $O(\varepsilon)$ will be computed later.)

$$
\begin{aligned}
(P) &\quad \frac{2}{\gamma^2} + 2q + O(\varepsilon) = 0, \\
(Q) &\quad \frac{4}{\gamma} + 2p - \gamma q + O(\varepsilon) = 0, \\
(R) &\quad 2s - \gamma r + \varepsilon(ps - 2r - \frac{2}{\gamma} s) + O(\varepsilon^2) = 0,
\end{aligned}
$$

$$(S) \qquad -\frac{4}{\gamma}s + 2r + \varepsilon(-\frac{2}{\gamma}r + pr - qs) + O(\varepsilon^2) = 0.$$

Now, let $\varepsilon \to 0$, to have:

$$\frac{2}{\gamma^2} + 2q_0 = 0, \quad \frac{4}{\gamma} + 2p_0 - \gamma q_0 = 0, \quad 2s_0 - \gamma r_0 = 0, \quad -\frac{4}{\gamma}s_0 + 2r_0 = 0.$$

The last two equations are equivalent. We have:

$$q_0 = -\frac{1}{\gamma^2}, \quad p_0 = -\frac{5}{2\gamma}, \quad \text{and} \;\; 2s_0 - \gamma r_0 = 0.$$

Here, s_0 and r_0 are not uniquely determined. Remember that $b_1, \cdots, b_4$ were not uniquely determined.

$$p_0 = b_1 + b_4, \quad q_0 = b_2 + b_3, \quad r_0 = b_2 - b_3, \quad s_0 = b_1 - b_4.$$

In order to extract further information from the equation, eliminate the constant terms from (R) and (S), by a new equation $(U) = (2(R)+\gamma(S))/\varepsilon$, to get:

$$(U) \qquad (\gamma p - 6)r + (2p - \gamma q - \frac{4}{\gamma})s + O(\varepsilon) = 0.$$

We supposed that our equations holds for all ε near 0. So we assume, by letting $\varepsilon \to 0$,

$$(\gamma p_0 - 6)r_0 + (2p_0 - \gamma q_0 - \frac{4}{\gamma})s_0 = 0$$

holds. This turns out to:

$$-\frac{17}{2}r_0 - \frac{8}{\gamma}s_0 = 0.$$

Hence together with $2s_0 - \gamma r_0 = 0$, we determine $r_0 = s_0 = 0$.

8. Analytic family of cycles of period 5

Now, we have a system of algebraic equations $\{(P), (Q), (R), (U)\}$, in variables (p, q, r, s), analytically parametrized by ε. This system of algebraic equations has a solution $(p_0, q_0, r_0, s_0) = (-\frac{5}{2\gamma}, -\frac{1}{\gamma}, 0, 0)$ for $\varepsilon = 0$.

In order to apply the implicit function theorem to have solutions for small ε, we compute the Jacobian at the solution.

$$\det \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & -\gamma & 0 & 0 \\ 0 & 0 & -\gamma & 2 \\ 0 & 0 & -\frac{17}{2} & -\frac{8}{\gamma} \end{pmatrix} = -100 \neq 0.$$

By the implicit function theorem, our system has a family of solutions. We have the following proposition.

PROPOSITION System of equations $\{(P),(Q),(R),(S)\}$ has a family of solutions $(p(\varepsilon),q(\varepsilon),r(\varepsilon),s(\varepsilon))$, analytic near $\varepsilon=0$, satisfying $p(0)=p_0$, $q(0)=q_0$, and $r(\varepsilon)\equiv 0$, $s(\varepsilon)\equiv 0$.

PROOF System of equations $\{(P),(Q),(R),(S)\}$ is equivalent to the system of equations $\{(P),(Q),(R),(U)\}$, which has the solution. System of equations $\{(P),(Q),(R),(U)\}$ has a solution $(p(\varepsilon),q(\varepsilon),r(\varepsilon),s(\varepsilon))$, analytic near $\varepsilon=0$, satisfying $p(0)=p_0$, $q(0)=q_0$, $r(0)=0$, $s(0)=0$. The terms $O(\varepsilon^2)$ in equations (R) and (S) are computed as follows.

$$-\varepsilon^2(pr+qs+\frac{1}{\gamma}r), \qquad -\varepsilon^2(qr).$$

Hence, (R) and (S) always hold if $r=s=0$.

By assuming $r=s=0$, we see our system of equations reduces to the following system of equation in p and q only.

$$(P_0) \qquad \frac{2}{\gamma^2}+2q+\varepsilon(\frac{2}{\gamma}q+pq)+\varepsilon^2\frac{q^2}{2}=0,$$

$$(Q_0) \qquad \frac{4}{\gamma}+2p-\gamma q+\varepsilon(\frac{1}{2}p^2+\frac{2}{\gamma}p+2q-\frac{4}{\gamma^2})+\varepsilon^2(pq-\frac{1}{\gamma}q)=0,$$

which has a family of solutions $p(\varepsilon)$ and $q(\varepsilon)$, near $\varepsilon=0$, satisfying $p(0)=p_0$ and $q(0)=q_0$.

By the uniqueness of the solutions given by the implicit function theorem, these solutions are the same.

9. Proof of theorem B

As stated in the above, our system of equations has a family of solutions parametrized by ε. Obviously, our solutions give the followings.

$$a_1 = a_4 = 1, \quad a_2 = a_3 = \frac{1}{\gamma}, \quad b_1 = b_4 = -\frac{5}{4\gamma}, \quad b_2 = b_3 = -\frac{1}{2\gamma^2}.$$

$$u_0 = y_* - \frac{\varepsilon^2}{\gamma},$$

$$u_1 = u_4 = \varepsilon - \frac{5}{4\gamma}\varepsilon^2 + \cdots,$$

$$u_2 = u_3 = \frac{1}{\gamma}\varepsilon^2 - \frac{1}{2\gamma^2}\varepsilon^3 + \cdots.$$

Hence, we have

$$y_n = y_* - \frac{1}{\gamma}\varepsilon^2 + \rho_1(n)(\varepsilon - \frac{5}{4\gamma}\varepsilon^2 + \cdots) + \rho_2(n)(\frac{1}{\gamma}\varepsilon^2 - \frac{1}{2\gamma^2}\varepsilon^3 + \cdots).$$

Furthermore, from (F_0),

$$\alpha(\varepsilon) - \alpha_0 = (\frac{\omega_1}{\gamma} - \frac{2}{\gamma} - 2)\varepsilon^2 + \frac{5}{\gamma}\varepsilon^3 + \cdots,$$

with $\omega_1 - 2 - 2\gamma = \omega_1 - 2 - 2\omega_2 + 2\omega_1 = 5\omega_1$, we get

$$\alpha(\varepsilon) = \alpha_0 + \frac{5}{\gamma}\omega_1\varepsilon^2 + \frac{5}{\gamma}\varepsilon^3 + \cdots.$$

Note that if $\omega_1 = \Omega + \bar{\Omega} > 0$, then $\gamma = \Omega^2 + \bar{\Omega}^2 - (\Omega + \bar{\Omega}) < 0$. And if $\Omega + \bar{\Omega} < 0$, then $\gamma > 0$. So, $\frac{5}{\gamma}\omega_1 < 0$. These prove Theorem B.

10. Proof of Theorem C

The system of equations $\{(P_0), (Q_0)\}$, with conditions $p(0) = p_0$ and $q(0) = q_0$, can be regarded as a real analytic family of systems of real analytic equations. So, for sufficiently small real values of ε, $p(\varepsilon)$ and $q(\varepsilon)$ are real. With real values of $\mathbf{a}$ and $\mathbf{b}$, the corresponding parameter $\alpha(\varepsilon)$ and periodic points are real and real analytic with respect to ε, near $\varepsilon = 0$.

The trace of the Jacobian matrix along the cycle is also real analytic in ε and takes real values. It is also holomorphic in ε, considered as a complex variable, near $\varepsilon = 0$. As $\alpha(0) = \alpha_0$, and the eigenvalues of the fixed point P_* are Ω and $\bar{\Omega}$, we see that $\tau(0) = 2$. On the other hand, the coordinates of the periodic cycle is algebraic with respect to complex parameter α. For sufficiently large value of α, the periodic cycle become hyperbolic, *i.e.*, the absolute value of the analytic continuation of the trace function is larger than 2. Therefore, the trace function is not constant as an algebraic function of α. Hence $\tau(\varepsilon)$ is not constant near $\varepsilon = 0$.

11. Proof of Theorem A

As is shown in the proof of Theorem B, $\frac{\omega_1}{\gamma} < 0$ holds in both cases of Ω. Parameter α is related to ε by a real analytic function

$$\alpha(\varepsilon) = \alpha_0 + \frac{5}{\gamma}\omega_1\varepsilon^2 + \frac{5}{\gamma}\varepsilon^3 + \cdots .$$

If $\alpha < \alpha_0$ and α is sufficiently near α_0, there exist real values ε_- and ε_+ near $\varepsilon = 0$, such that

$$\alpha = \alpha(\varepsilon_-) = \alpha(\varepsilon_+), \qquad \varepsilon_- < 0 < \varepsilon_+,$$

with

$$\tau(\varepsilon_-) \neq 2, \qquad \tau(\varepsilon_+) \neq 2.$$

If $\alpha > \alpha_0$ and sufficiently near α_0, then $\alpha = \alpha(\varepsilon)$ has no solutions near $\varepsilon = 0$.

Index of a fixed point $P \in \mathbb{R}^2$ of mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as follows. Let U denote a small neighborhood of the fixed point. Define a mapping $\varphi : U \setminus \{P\} \to \mathbb{R}^2 \setminus \{O\}$ by $\varphi(X) = f(X) - X$. By an appropriate choice of the neighborhood U, the induced homomorphism, $\varphi_* : \pi_1(U \setminus \{P\}) \to \pi_1(\mathbb{R}^2 \setminus \{O\})$, of the fundamental groups defines an integer. This integer is called the local index of fixed point P.

By Poincaré's index theorem, the sum of the local indices of the fixed points is invariant under continuous perturbations of the mapping f. In the case of area preserving diffeomorphism, the local index of a saddle is -1, and the local index of a center is $+1$. So, the created two cycles cannot be the same type. This proves Theorem A.