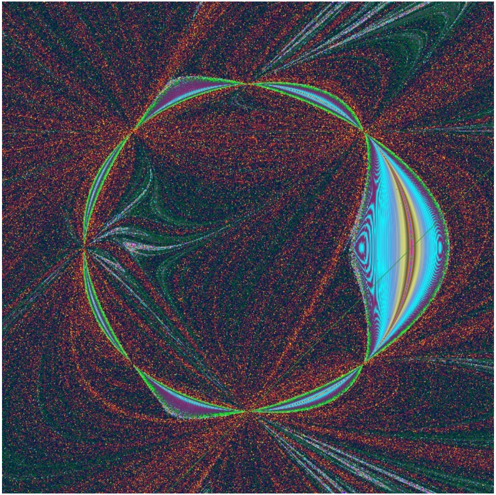
Elementary construction of Complex Surface Automorphisms



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Abstract

Diller gave a method for constructing many examples of surface automorphisms of positive entropy.

Uehara gave explicit formulas for Cremona transformations with invariant cubic curves.

In this note, following their methods, we construct rational surface automorphisms with invariant cubic curves.

Explicit formulas of quadratic Cremona transformations are obtained by elementary calculations.

Contents

1. Birational map

Birational map

Let $f_1(x, y, z), f_2(x, y, z), f_3(x, y, z)$ be homogeneous polynomials of same degree. In this note, we consider only the case of degree 2.

Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a map defined by

$$f([x : y : z]) = [f_1(x, y, z) : f_2(x, y, z) : f_3(x, y, z)],$$

except for the set of indeterminacy points

$$I(f) = \{[x : y : z] \mid f_1(x, y, z) = f_2(x, y, z) = f_3(x, y, z) = 0\}.$$

Such mapping is said to be a rational map.

Rational map f is said to be birational if its inverse map f^{-1} is also a rational map defined by homogeneous polynomials, except for the set $I(f^{-1})$ of its indeterminacy points.

Affine coordinates

Although considerations in $\mathbb{P}^2$ is preferable, mostly in the followings, we discuss in the affine coordinates.

$\mathbb{C}^2$ can be considered as an open and dense subset of $\mathbb{P}^2$ by

$$(x, y) \leftrightarrow [x : y : 1].$$

Let $f_i(x, y) = f_i(x, y, 1)$, $i = 1, 2, 3$.
Define rational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ by

$$f(x, y) \ = \ (\frac{f_1(x, y)}{f_3(x, y)}, \frac{f_2(x, y)}{f_3(x, y)}).$$

$$I(f) = \{[x, y, z] \mid f_1(x, y, z) = f_2(x, y, z) = f_3(x, y, z) = 0\}.$$

2. Cuspidal cubic curve

Cubic curve

Let C denote the cubic curve $\{y = x^3\}$ in $\mathbb{P}^2$.

This curve has a parametrization

$$p : \mathbb{C} \to C, \quad p(t) = (t, t^3).$$

We want to find birational maps $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, which maps C onto itself.

$$f(C) = C.$$

f has indeterminate points $I(f)$. The equality should be understood "modulo exceptional points".

$$f(C) = \overline{f(C \setminus I(f))}.$$

f induces an automorphism of the cubic curve C, which can be described by an affine map $t \mapsto \lambda(t + \mu)$ for some constants $\lambda \in \mathbb{C}^{\times}, \mu \in \mathbb{C}$.

PROPOSITION. For $\lambda \in \mathbb{C}^{\times}$ and $a_1, a_2, a_3 \in \mathbb{C}$ with $a_1 + a_2 + a_3 \neq 0$, there exists a quadratic birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, such that

$$f(C) = C, \quad I(f) = \{p(a_1), p(a_2), p(a_3)\},$$

inducing $t \mapsto \lambda(t + \frac{\nu_1}{3})$, with $\nu_1 = a_1 + a_2 + a_3$.

PROOF. Let $\nu_2 = a_1 a_2 + a_2 a_3 + a_3 a_1$ and $\nu_3 = a_1 a_2 a_3$. The indeterminate points, $p(a_i) = (a_i, a_i^3), i = 1, 2, 3$, are common zeros of the system of equations

$$\begin{cases} y - x^3 & = & 0 \\ x^3 - \nu_1 x^2 + \nu_2 x - \nu_3 & = & 0 \end{cases}.$$

As quadratic polynomial $f_3(x, y)$ must vanish in these indeterminacy points, we can choose

$$f_3(x, y) = \nu_1 x^2 - \nu_2 x + \nu_3 - y.$$

Since $f(p(t)) = p(\lambda(t + \frac{\nu_1}{3}))$ for $t \in \mathbb{C}$, $f : (x, y) \mapsto (X, Y)$ can be written as

$$X = \lambda \left(x + \frac{\nu_1}{3} + \frac{(y - x^3)U(x, y)}{f_3(x, y)} \right),$$

$$Y = \lambda^3 \left((x + \frac{\nu_1}{3})^3 + \frac{(y - x^3)V(x, y)}{f_3(x, y)} \right),$$

where polynomials $U(x, y), V(x, y)$ are chosen so that f becomes a quadratic rational map.

To determine polynomials $U(x,y)$ and $V(x,y)$ we require that

$$f_1(x,y) = \lambda\left((x+\frac{\nu_1}{3})f_3(x,y) + (y-x^3)U(x,y)\right),$$

$$f_2(x,y) = \lambda^3\left((x+\frac{\nu_1}{3})^3 f_3(x,y) + (y-x^3)V(x,y)\right)$$

are quadratic polynomials. We get

$$U(x,y) = \nu_1,$$

$$V(x,y) = \nu_1 x^2 + (\nu_1^2 - \nu_2)x - y + \frac{\nu_1^3}{3} - \nu_1\nu_2 + \nu_3.$$

This gives the explicit formula for the quadratic birational map f.

Explicit formula for invariant cuspidal cubic curve case

PROPOSITION. The quadratic birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ in the previous proposition is given by

$$X = \lambda \left(x + \frac{\nu_1}{3} + \frac{\nu_1(y - x^3)}{\nu_1 x^2 - \nu_2 x + \nu_3 - y} \right),$$

$$Y = \lambda^3 \left((x + \frac{\nu_1}{3})^3 + (y - x^3)(1 + \frac{\nu_1^2 x + \frac{\nu_1^3}{3} - \nu_1 \nu_2}{\nu_1 x^2 - \nu_2 x + \nu_3 - y}) \right).$$

Exceptional lines

A quadratic birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ always acts by blowing up three indeterminacy points in $\mathbb{P}^2$ and blowing down the three exceptional lines joining them.

The inverse map f^{-1} is also quadratic and the images of three exceptional lines of f are the indeterminacy points of f^{-1}.

Parametrization and lines

Our parametrization $p : \mathbb{C} \to C$ of the invariant cubic curve has a nice property.

If three points $p(t_1), p(t_2), p(t_3)$ are on a line, say $\{y = ax + b\}$, then

$$t_i^3 - at_i - b = 0, \qquad i = 1, 2, 3,$$

which shows that t_1, t_2, t_3 are three roots of cubic equation $t^3 - at + b = 0$, hence $t_1 + t_2 + t_3 = 0$.

Conversely, if $t_1 + t_2 + t_3 = 0$, then t_1, t_2, t_3 are the three roots of cubic equation in t :

$$t^3 + (t_1 t_2 + t_2 t_3 + t_3 t_1)t - t_1 t_2 t_3 = 0,$$

which implies that $p(t_1), p(t_2), p(t_3)$ are on a line.

Inverse map

In order to compute the inverse map of f, we need to find the indeterminacy points of f^{-1}, which are the images of the exceptional lines of f.

Suppose the exceptional line passing through indeterminacy points $p(a_j)$ and $p(a_k)$ is mapped to $p(b_i)$, for $\{i, j, k\} = \{1, 2, 3\}$. This exceptional line intersects with C at $p(-a_j - a_k)$, which is mapped to $p(b_i)$, with

$$b_i = \lambda(-a_j - a_k + \frac{\nu_1}{3}) = \lambda(a_i - \frac{2\nu_1}{3}).$$

The dynamics of f^{-1} in the invariant curve C is

$$t \mapsto \lambda^{-1}(t - \frac{\lambda\nu_1}{3}).$$

Construction of the inverse map is similar.

Inner dynamics

Let $\tau : t \mapsto \lambda(t + \nu_1/3)$ denote the dynamics in C.
τ has a unique fixed point $t_0 = \frac{1}{3}\frac{\lambda\nu_1}{1-\lambda}$.
By linear change of variables $t = rt'$, where $r = \frac{\lambda\nu_1}{1-\lambda}$, τ is conjugate to

$$\tau' : t' \mapsto \lambda(t' + \frac{1-\lambda}{3\lambda}),$$

whose fixed point is $\frac{1}{3}$.
So, by linear change of coordinates $x = rx'$, and $y = r^3 y'$, with $a_i = ra_i', i = 1, 2, 3$, birational map f has fixed point $(\frac{1}{3}, \frac{1}{27})$.

To construct surface automorphisms by blow-ups, we may suppose that f fixes $(\frac{1}{3}, \frac{1}{27})$.

Surface automorphism

We have

$$I(f) = \{p(a_1), p(a_2), p(a_3)\}$$

and

$$I(f^{-1}) = \{p(b_1), p(b_2), p(b_3)\}$$

If, for some positive integers n_1, n_2, n_3, and permutation $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$,

$$p(a_{\sigma(i)}) = f^{\circ(n_i - 1)} p(b_i), \qquad i = 1, 2, 3,$$

holds, then f lifts to a surface automorphism by blowing up $(n_1 + n_2 + n_3)$ points (provided they are all distinct)

$$p(b_i), f(p(b_i)), \cdots, f^{\circ(n_i-1)}(p(b_i)), \qquad i = 1, 2, 3.$$

Orbit data

Positive integers (n_1, n_2, n_3) with permutation σ is said an **orbit data**.

Following Diller, we look for determinant λ and a quadratic birational transformation f, which maps C onto itself and realizes the prescribed orbit data.

conditions

In terms of inner dynamics, the conditions are as follows.

$$a_{\sigma(i)} = \lambda^{n_i - 1}(b_i - \frac{1}{3}) + \frac{1}{3}, \quad i = 1, 2, 3,$$

$$b_i = \lambda a_i + \frac{2}{3}(\lambda - 1), \qquad i = 1, 2, 3,$$

$$a_1 + a_2 + a_3 = \frac{1}{\lambda} - 1.$$

Eliminate $a_i, b_i, i = 1, 2, 3$, to obtain an equation in λ, which is a necessary condition.

Polynomial equations for orbit data n_1, n_2, n_3, σ

Necessary condition $P(\lambda) = 0$ is given by followings.

(case id) $\sigma = id$.

$$P(\lambda) = (\lambda - 2)\lambda^{n_1+n_2+n_3} + \lambda^{n_1+n_2} + \lambda^{n_2+n_3} + \lambda^{n_3+n_1}$$

$$- \lambda^{n_1+1} - \lambda^{n_2+1} - \lambda^{n_3+1} + 2\lambda - 1.$$

(case tr) σ is a transposition $(\sigma(1) = 2, \sigma(2) = 1, \sigma(3) = 3)$.

$$P(\lambda) = (\lambda - 2)\lambda^{n_1+n_2+n_3} + \lambda^{n_1+n_2} + (\lambda - 1)(\lambda^{n_1+n_3} + \lambda^{n_2+n_3})$$

$$-(\lambda - 1)(\lambda^{n_1} + \lambda^{n_2}) + \lambda^{n_3+1} - 2\lambda + 1.$$

(case cy) σ is a cyclic permutation $(\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1)$.

$$P(\lambda) = (\lambda - 2)\lambda^{n_1+n_2+n_3} + (\lambda - 1)(\lambda^{n_1+n_2} + \lambda^{n_2+n_3} + \lambda^{n_3+n_1})$$

$$+(\lambda - 1)(\lambda^{n_1} + \lambda^{n_2} + \lambda^{n_3}) + 2\lambda - 1.$$

Picard coordinate of indeterminate points

(case id) $\sigma = id$.

$$a_i = -\frac{\lambda^{n_i-1}(\lambda-1)}{\lambda^{n_i}-1} + \frac{1}{3} \qquad (i = 1, 2, 3).$$

(case tr) $\sigma = (1, 2)$

$$a_i = -\frac{\lambda^{n_j-1}(\lambda^{n_i}+1)(\lambda-1)}{\lambda^{n_i+n_j}-1} + \frac{1}{3} \qquad ((i, j) = (1, 2), (2, 1)).$$

$$a_k = -\frac{\lambda^{n_k-1}(\lambda-1)}{\lambda^{n_k}-1} + \frac{1}{3} \qquad (k = 3).$$

(case cy) $\sigma = (1, 2, 3)$

$$a_i = -\frac{\lambda^{n_k-1}(\lambda^{n_j}(\lambda^{n_i}+1)+1)(\lambda-1)}{\lambda^{n_i+n_j+n_k}-1} + \frac{1}{3}$$

$$((i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)).$$

Characteristic polynomial

Orbit data determines the characteristic polynomial $P(\lambda)$ of $f^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$.

Bedford and Kim [BK1] have computed explicitly for any orbit data n_1, n_2, n_3, σ.

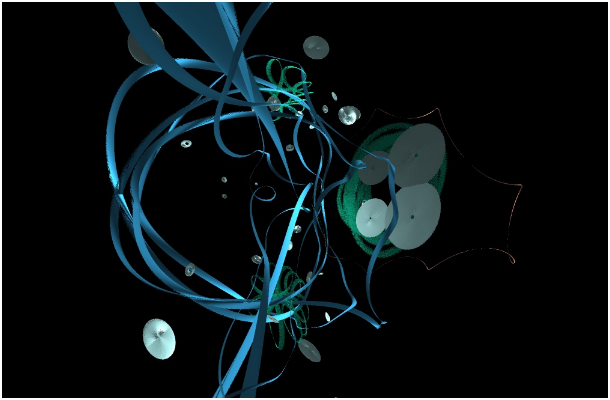
$$P(\lambda) = \lambda^{1+\Sigma n_j} p(\frac{1}{\lambda}) + (-1)^{\mathrm{ord}\sigma} p(\lambda),$$

where

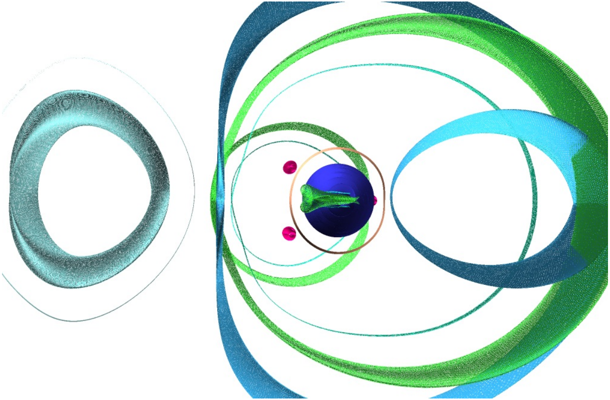
$$p(\lambda) = 1 - 2\lambda + \sum_{j=\sigma_j} \lambda^{1+n_j} + \sum_{j \neq \sigma_j} \lambda^{n_j}(1 - \lambda).$$

The polynomial $P(\lambda)$ obtained as a necessary condition and the characteristic polynomial $P(\lambda)$ coincide (not by chance).

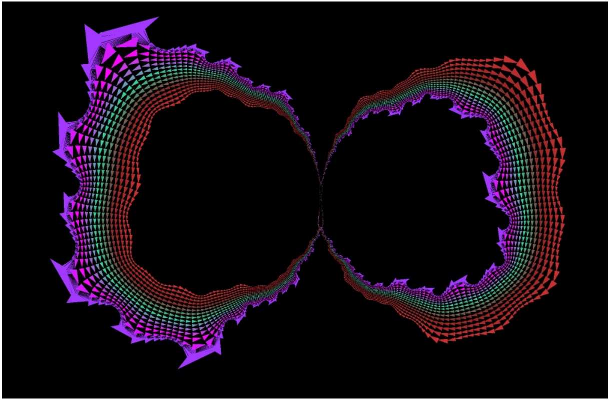
CSPmap (CSPt182st2)



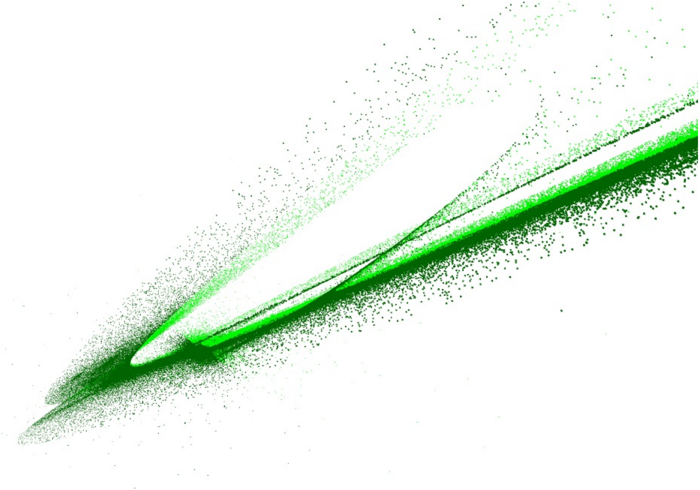
CSPmap (CSPi244t2)



CSPmap (CSPc334)



CSPmap (CSPIc543r)



3. Three lines through a point

Three lines

In this section, we consider the case where the invariant cubic curve is three lines passing through a point (in $\mathbb{P}^2$).

Let $C_L = \{(x, y) \in \mathbb{C}^2 \mid x(x^2 - 1) = 0\}$.
Cubic curve C_L has three components.
We consider three cases.

(case L3I) f maps each line to itself.
(case L3T) f transposes lines $\{x = 1\}$ and $\{x = -1\}$, while line $\{x = 0\}$ is mapped to itself.
(case L3C) f permutes three lines cyclically.

Parametrization

Let

$$p_1 : \mathbb{C} \to \mathbb{P}^2, \qquad p_1(t) = (0, \frac{t}{2}),$$
$$p_2 : \mathbb{C} \to \mathbb{P}^2, \qquad p_2(t) = (1, -t),$$
$$p_3 : \mathbb{C} \to \mathbb{P}^2, \qquad p_3(t) = (-1, -t),$$

be parametrizations of the lines.

Three points $p_1(t_1), p_2(t_2), p_3(t_3)$ are on a line if and only if $t_1 + t_2 + t_3 = 0$.

PROPOSITION. For $\lambda \in \mathbb{C}^\times$ and $a_1, a_2, a_3 \in \mathbb{C}$ with $a_1 + a_2 + a_3 \neq 0$, there exists a quadratic birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, such that

$$f(C_L) = C_L, \quad I(f) = \{p_1(a_1), p_2(a_2), p_3(a_3)\},$$

inducing $t_i \mapsto \lambda(t_{\eta(i)} + \frac{\nu_1}{3})$, $i = 1, 2, 3$, with $\nu_1 = a_1 + a_2 + a_3$. Here $\eta : \{1, 2, 3\} \to \{1, 2, 3\}$ indicates the permutation of three lines.

case L3I

PROOF. In the case of $L3I$, the denominator $f_3(x,y)$ is a quadratic polynomial representing a parabolic curve passing through the three indeterminacy points

$$(0, \frac{1}{2}a_1), (1, -a_2), (-1, -a_3),$$

given by

$$f_3(x,y) = \nu_1 x^2 + (a_2 - a_3)x + 2y - a_1.$$

$f_1(x,y)$ is a quadratic polynomial given by

$$X_I = \frac{f_1(x,y)}{f_3(x,y)} = x - \frac{\nu_1 x(x^2-1)}{f_3(x,y)},$$

which preserves three lines. (coefficient ν_1 is chosen so that $f_1(x,y)$ is quadratic.)

case L3I

Dynamics in invariant lines are as follows.

$$(0, y) \mapsto (0, \lambda(y + \frac{\nu_1}{6})),$$

$$(1, y) \mapsto (1, \lambda(y - \frac{\nu_1}{3})),$$

$$(-1, y) \mapsto (-1, \lambda(y - \frac{\nu_1}{3})).$$

So, we can arrange as follows.

$$Y_I = \frac{f_2(x, y)}{f_3(x, y)} = \lambda \left\{ (y - \frac{\nu_1}{3}) - \frac{(x^2 - 1)\nu_1(y - \frac{a_1}{2})}{\nu_1 x^2 + (a_2 - a_3)x + 2y - a_1} \right\}.$$

cases L3I

We obtained in the case of $\eta = id.$,

$$X_I = x - \frac{\nu_1 x(x^2-1)}{\nu_1 x^2 + (a_2 - a_3)x + 2y - a_1},$$

$$Y_I = \lambda \left\{ (y - \frac{\nu_1}{3}) - \frac{(x^2-1)\nu_1(y - \frac{a_1}{2})}{\nu_1 x^2 + (a_2 - a_3)x + 2y - a_1} \right\}.$$

case L3T

In the case of transposition $\eta = (2,3)$, we have

$$X_T = -x + \frac{\nu_1 x(x^2-1)}{\nu_1 x^2 + (a_2 - a_3)x + 2y - a_1},$$

$$Y_T = \lambda \left\{ (y - \frac{\nu_1}{3}) - \frac{(x^2-1)\nu_1(y - \frac{a_1}{2})}{\nu_1 x^2 + (a_2 - a_3)x + 2y - a_1} \right\}.$$

case L3C

In the case of L3C, the indeterminacy points are same as in the case L3I.

The dynamics in the invariant lines $t_i \mapsto \lambda(t_{\eta(i)} + \frac{\nu_1}{3})$, $i = 1, 2, 3$, give

$$(0, y) \mapsto (1, -2\lambda(y + \frac{\nu_1}{6})),$$

$$(1, y) \mapsto (-1, \lambda(y - \frac{\nu_1}{3})),$$

$$(-1, y) \mapsto (0, -\frac{\lambda}{2}(y - \frac{\nu_1}{3})).$$

case L3C

Denominator $f_3(x, y)$ vanishes at the indeterminate points, we set

$$f_3(x, y) = k(x^2 + x)(y + a_2) + \ell(x^2 - x)(y + a_3) - (k + \ell)(x^2 - 1)(y - \frac{a_1}{2}),$$

for some constants k and ℓ, so that $f_3(x, y)$ is a quadratic polynomial.

Then the numerator $f_2(x, y)$ must be as

$$f_2(x, y) = k(x^2 + x)(y + a_2)\lambda(y - \frac{\nu_1}{3})$$

$$+ \ell(x^2 - x)(y + a_3)(-\frac{\lambda}{2})(y - \frac{\nu_1}{3})$$

$$- (k + \ell)(x^2 - 1)(y - \frac{a_1}{2})(-2\lambda)(y + \frac{\nu_1}{6}).$$

And we get $\ell = -2k$, for $f_2(x, y)$ to be quadratic.

case L3C

We set $k = 1$, $\ell = -2$.
Now, f_1 takes the form

$$f_1(x,y) = -\frac{x+1}{3x-1} f_3(x,y) - x(x^2-1)\frac{Q}{P},$$

for some polynomials P, Q. As f_1 has no poles, P is a multiple of $3x-1$. We can set

$$\frac{f_1(x,y)}{f_3(x,y)} = -\frac{x+1}{3x-1}\left(1 + \frac{x(x-1)Q}{f_3(x,y)}\right).$$

By posing this to be a quadratic rational function, we require that

$$f_3(x,y) + x(x-1)Q$$

has factor $3x-1$. And we conclude $Q = 2\nu_1$.

case L3C

In the case of L3C,

$$X_C = -\frac{x+1}{3x-1}\left(1 + \frac{2\nu_1 x(x-1)}{f_3(x,y)}\right),$$

$$Y_C = \frac{\lambda\left((y-\frac{\nu_1}{3})(2y+\nu_1 x^2+(a_2-a_3)x-a_1)-\nu_1(x^2-1)(y-\frac{a_1}{2})\right)}{f_3(x,y)},$$

where

$$f_3(x,y) = (3x-1)y + (-\frac{a_1}{2}+a_2-2a_3)x^2 + (a_2+2a_3)x + \frac{a_1}{2}.$$

Orbit data for three lines

Orbit data $(n_1, n_2, n_3), \sigma$ must be compatible with the permutation η of the three lines L_1, L_2, L_3.

If

$$p_i(a_i) \in L_i, \qquad i = 1, 2, 3,$$

then

$$p_{\eta(i)}(b_i) \in L_{\eta(i)}, \qquad i = 1, 2, 3.$$

And

$$\eta^{n_i - 1}(i) = \sigma(i).$$

Orbit data $(n_1, n_2, n_3), \sigma$

(case L3I) $\eta = id$.

$\sigma = id$.

(case L3T) $\eta = (i, j)$ (transposition),

$\sigma = id$, and n_i, n_j are even.

$\sigma = \eta$, and n_i, n_j are odd.

(case L3C) $\eta = (i, j, k)$ (cyclic), $\{i, j, k\} = \{1, 2, 3\}$.

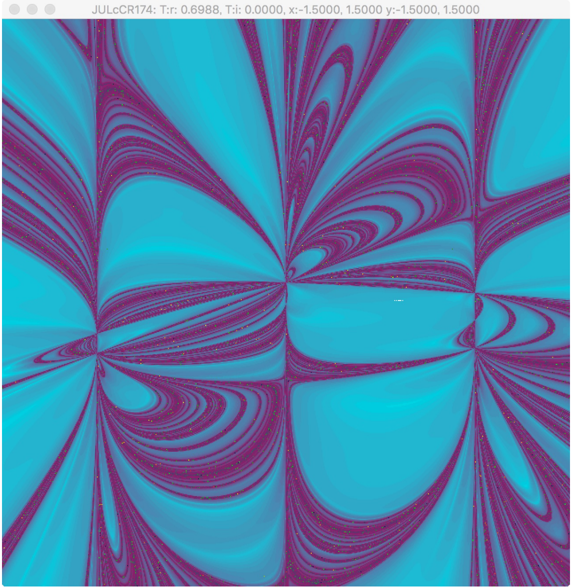
$\sigma = id$, and $n_i \equiv n_j \equiv n_k \equiv 0 \pmod 3$.

$\sigma = (i, j)$, and n_i, n_j, n_k are distinct $\pmod 3$ with $n_k \equiv 0 \pmod 3$.

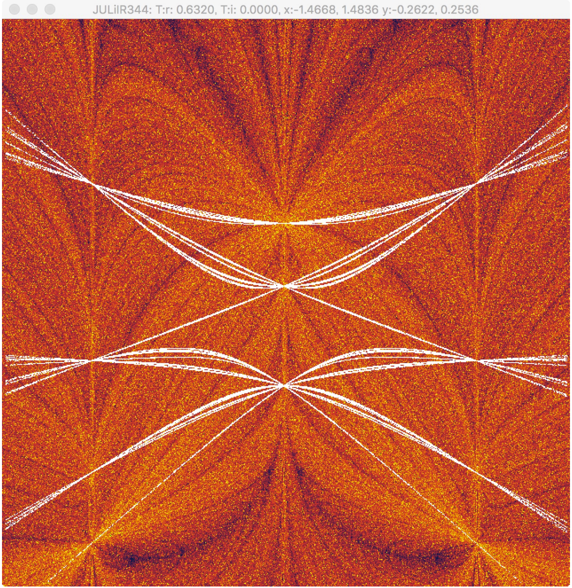
$\sigma = \eta$ and $n_i \equiv n_j \equiv n_k \equiv 1 \pmod 3$.

$\sigma = \eta^{-1}$ and $n_i \equiv n_j \equiv n_k \equiv 2 \pmod 3$.

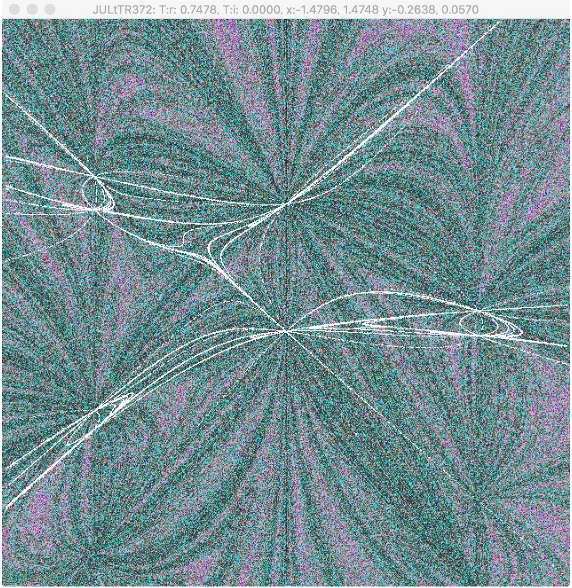
L3Cmap (L3Cc174r)



L3Imap (L3Ii344r)



L3Tmap (L3Tt372r)



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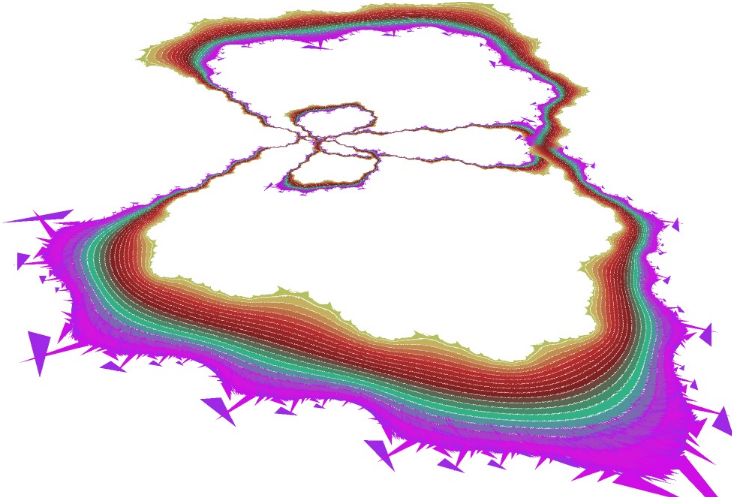


4. Conic and a tangent line

Conic and a tangent line

In this section, we consider the case where the invariant cubic curve is a conic with a tangent line.

Let $C_Q = \{(x, y) \in \mathbb{C}^2 \mid x(xy - 1) = 0\}$.
And let $Q = \{(x, y) \in \mathbb{C}^2 \mid xy = 1\}$, $L = \{(x, y) \in \mathbb{C}^2 \mid x = 0\}$.
We consider two cases.

(case QQ) $f(Q) = Q$, and $f(L) = L$.

(case QL) $f(Q) = L$ and $f(L) = Q$.

Parametrization

Let

$$p_Q : \mathbb{C} \to \mathbb{P}^2, \qquad p_Q(t) = (t^{-1}, t),$$
$$p_L : \mathbb{C} \to \mathbb{P}^2, \qquad p_L(t) = (0, -t),$$

be parametrisations.

Three points $p_Q(t_1), p_Q(t_2)$ and $p_L(t_3)$ are on a line if and only if $t_1 + t_2 + t_3 = 0$.

case QQ

PROPOSITION. For $\lambda \in \mathbb{C}^\times$ and $a_1, a_2, a_3 \in \mathbb{C}$ with $a_1 + a_2 + a_3 \neq 0$, there exists a quadratic birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, such that

$$f(Q) = Q, f(L) = L \quad I(f) = \{p_Q(a_1), p_Q(a_2), p_L(a_3)\},$$

inducing $t \mapsto \lambda(t + \frac{\nu_1}{3})$, with $\nu_1 = a_1 + a_2 + a_3$.

Proof

PROOF. Dynamics in Q and L are as follows.

$$(y^{-1}, y) \mapsto (\lambda^{-1}(y + \frac{\nu_1}{3})^{-1}, \lambda(y + \frac{\nu_1}{3})), \quad \text{in } Q,$$

$$(0, y) \mapsto (0, \lambda(y - \frac{\nu_1}{3})), \quad \text{in } L.$$

From the dynamics in L, we can assume

$$f_1(0, y) = 0,$$

$$f_2(0, y) = \lambda(y - \frac{\nu_1}{3})(y + a_3),$$

$$f_3(0, y) = y + a_3.$$

Then set

$$f_1(x,y) = \lambda^{-1}(Ax + By + C)x,$$

for some A, B, C.

Along Q, by setting $x = y^{-1}$, we must have

$$f_1(y^{-1}, y) = \lambda^{-1}y^{-2}(By^2 + Cy + A) = \lambda^{-1}y^{-2}(y - a_1)(y - a_2).$$

which gives $A = a_1a_2, B = 1, C = -(a_1 + a_2)$.

Hence we find

$$f_1(x,y) = \lambda^{-1}(a_1a_2x^2 + xy - (a_1 + a_2)x).$$

Next, set

$$f_3(x,y) = (Px + Qy + R)x + y + a_3.$$

Along Q, by setting $x = y^{-1}$, we require

$$\frac{f_1(y^{-1},y)}{f_3(y^{-1},y)} = \lambda^{-1}(y + \frac{\nu_1}{3})^{-1}.$$

This gives

$$P = \frac{\nu_1}{3}a_1a_2, \quad Q = -\frac{2}{3}\nu_1, \quad R = a_1a_2 - \frac{\nu_1}{3}(a_1 + a_2).$$

We get

$$f_3(x,y) = \frac{\nu_1}{3}a_1a_2x^2 - \frac{2}{3}\nu_1xy + (a_1a_2 - \frac{\nu_1}{3}(a_1 + a_2))x + y + a_3.$$

Similarly, we find

$$f_2(x,y) = \lambda\left(Sx^2 + Txy + Ux + (y+a_3)(y-\frac{\nu_1}{3})\right),$$

where

$$S = \frac{\nu_1^2}{9}a_1a_2,$$

$$T = \frac{4}{9}\nu_1^2 + a_1a_2 - \nu_1(a_1+a_2),$$

$$U = \frac{2}{3}\nu_1a_1a_2 - \frac{1}{9}\nu_1^2(a_1+a_2).$$

case QQ

Summing up.

$$X_{QQ} = \frac{\lambda^{-1}(a_1 a_2 x^2 + xy - (a_1 + a_2)x)}{\frac{\nu_1}{3} a_1 a_2 x^2 - \frac{2}{3}\nu_1 xy + (a_1 a_2 - \frac{\nu_1}{3}(a_1 + a_2))x + y + a_3},$$

$$Y_{QQ} = \frac{\lambda\left(Sx^2 + Txy + Ux + (y + a_3)(y - \frac{\nu_1}{3})\right)}{\frac{\nu_1}{3} a_1 a_2 x^2 - \frac{2}{3}\nu_1 xy + (a_1 a_2 - \frac{\nu_1}{3}(a_1 + a_2))x + y + a_3}.$$

where

$$S = \frac{\nu_1^2}{9} a_1 a_2,$$

$$T = \frac{4}{9}\nu_1^2 + a_1 a_2 - \nu_1(a_1 + a_2),$$

$$U = \frac{2}{3}\nu_1 a_1 a_2 - \frac{1}{9}\nu_1^2(a_1 + a_2).$$

case QL

PROPOSITION. For $\lambda \in \mathbb{C}^\times$ and $a_1, a_2, a_3 \in \mathbb{C}$ with $a_1 + a_2 + a_3 \neq 0$, there exists a quadratic birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, such that

$$f(Q) = L, \quad f(L) = Q, \quad I(f) = \{p_Q(a_1), p_Q(a_2), p_Q(a_3)\},$$

inducing $t \mapsto \lambda(t + \frac{\nu_1}{3})$, with $\nu_1 = a_1 + a_2 + a_3$.

Proof

PROOF. Dynamics in Q and L are as follows.

$$(y^{-1}, y) \mapsto (0, -\lambda(y + \frac{\nu_1}{3})), \quad \text{in } Q,$$

$$(0, y) \mapsto (-\lambda^{-1}(y - \frac{\nu_1}{3})^{-1}, -\lambda(y - \frac{\nu_1}{3})), \quad \text{in } L.$$

From the dynamics in Q, we assume

$$f_1(x, y) = xy - 1,$$

From the dynamics in L we can assume

$$f_3(0, y) = \lambda(y - \frac{\nu_1}{3}),$$

$$f_2(0, y) = -\lambda^2(y - \frac{\nu_1}{3})^2.$$

Set

$$f_3(x,y) = \lambda((Ax + By + C)x + y - \frac{\nu_1}{3}).$$

Then along Q, we have

$$f_3(y^{-1}, y) = \frac{\lambda}{y^2}(A + By^2 + Cy + y^3 - \frac{\nu_1}{3}y^2),$$

which must be factorized by

$$y^3 - \nu_1 y^2 + \nu_2 y - \nu_3 = (y - a_1)(y - a_2)(y - a_3).$$

Hence

$$A = -\nu_3, \quad B = -\frac{2}{3}\nu_1, \quad C = \nu_2,$$

which gives

$$f_3(x,y) = \lambda(-\nu_3 x^2 - \frac{2}{3}\nu_1 xy + \nu_2 x + y - \frac{\nu_1}{3}).$$

Next, set

$$f_2(x,y) = -\lambda^2((Dx + Ey + F)x + (y - \frac{\nu_1}{3})^2).$$

Then along Q we must have

$$\frac{f_2(y^{-1},y)}{f_3(y^{-1},y)} \;=\; -\lambda(y + \frac{\nu_1}{3}),$$

which gives

$$D = -\frac{1}{3}\nu_1\nu_3, \quad E = \nu_2 - \frac{4}{9}\nu_1^2, \quad F = \frac{1}{3}\nu_1\nu_2 - \nu_3,$$

$$f_2(x,y) = -\lambda^2(-\frac{\nu_1\nu_3}{3}x^2 + (\nu_2 - \frac{4\nu_1^2}{9})xy + (\frac{\nu_1}{3}\nu_2 - \nu_3)x + (y - \frac{\nu_1}{3})^2).$$

case QL

Summing up :

$$X_{QL} = \frac{xy-1}{\lambda(-\nu_3 x^2 - \frac{2}{3}\nu_1 xy + \nu_2 x + y - \frac{\nu_1}{3})},$$

$$Y_{QL} = \frac{-\lambda(-\frac{\nu_1\nu_3}{3}x^2 + (\nu_2 - \frac{4\nu_1^2}{9})xy + (\frac{\nu_1}{3}\nu_2 - \nu_3)x + (y - \frac{\nu_1}{3})^2)}{(-\nu_3 x^2 - \frac{2}{3}\nu_1 xy + \nu_2 x + y - \frac{\nu_1}{3})}.$$

Orbit data for conic and a tangent line

Orbit data $(n_1, n_2, n_3), \sigma$ must be compatible with the permutation η of Q and L.

(case QQ) $f(Q) = Q$, $f(L) = L$.

$$p_Q(a_1) \in Q, \quad p_Q(a_2) \in Q, \quad p_L(a_3) \in L,$$

$$p_Q(b_1) \in Q, \quad p_Q(b_2) \in Q, \quad p_L(b_3) \in L.$$

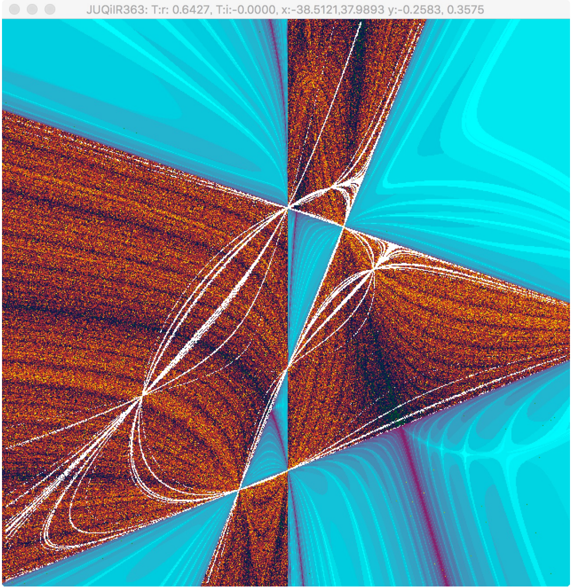
$\sigma = id.$ or $\sigma = (1, 2)$.

(case QL) $f(Q) = L$, $f(L) = Q$.

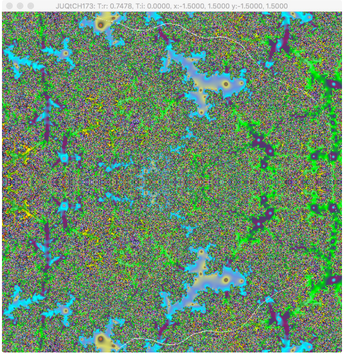
$$p_Q(a_i) \in Q, \quad p_Q(b_i) \in Q, \qquad i = 1, 2, 3.$$

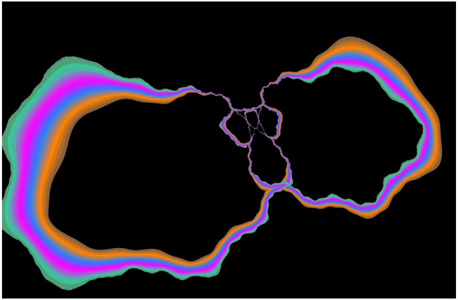
σ any, n_1, n_2, n_3 odd.

QQmap (QQi363r)



QLmap (QLc173r)





5. Nodal cubic

Cubic curve with nodes

In this section, we consider the case where invariant cubic curve has nodal singularities.

In this section, it is somewhat inconvenient to work with our affine coordinates. We also use homogenious coordinates with $(x, y) \leftrightarrow [x : y : 1]$, if necessary.

(case ND3) cubic curve consists of three lines $L_x = \{y = 0\}$, $L_y = \{x = 0\}$, and the line at infinity $L_z \subset \mathbb{P}^2$.

(case ND2) cubic curve consists of conic $Q = \{xy = 1\}$ and the line at infinity L_z.

(case ND1) cubic curve has one node. In this case the surface automorphism obtained by blow-ups has entropy zero.

case ND3

Parametrization of the cubic curve in case ND3 is as follows.

Let $t \in \mathbb{C}/\mathbb{Z}$.

$$\begin{aligned} p_x(t) &= (e^{2\pi i t}, 0) \in L_x, \\ p_y(t) &= (0, e^{2\pi i t}) \in L_y, \\ p_z(t) &= [1 : -e^{2\pi i t} : 0] \in L_z. \end{aligned}$$

Three points $p_x(t_1), p_y(t_2), p_z(t_3)$ are on a line if and only if $t_1 + t_2 + t_3 \equiv 0 \mod 1$.

case ND3

PROPOSITION. For $a_1, a_2, a_3 \in \mathbb{C}/\mathbb{Z}$ and $b_1, b_2, b_3 \in \mathbb{C}/\mathbb{Z}$, with

$$a_1 + a_2 + a_3 \equiv b_1 + b_2 + b_3 \not\equiv 0 \mod 1,$$

there exists a quadratic birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, such that

$$f(L_x) = L_x, \quad f(L_y) = L_y, \quad f(L_z) = L_z$$

$$\text{and} \quad I(f) = \{p_x(a_1), p_y(a_2), p_z(a_3)\},$$

inducing $t_i \mapsto t_i + b_i$, $i = 1, 2, 3$, in L_x, L_y, L_z respectively.

REM. Multiplier is always 1. Do not confuse with b_i in cuspidal case.

ND3

PROOF. Let $A_1 = e^{2\pi i a_1}$, $A_2 = e^{2\pi i a_2}$, $A_3 = e^{2\pi i a_3}$, and $B_1 = e^{2\pi i b_1}$, $B_2 = e^{2\pi i b_2}$, $B_3 = e^{2\pi i b_3}$. Then,

$$p_x(a_1) = (A_1, 0) \in L_x, \;\; p_y(a_2) = (0, A_2^{-1}) \in L_y,$$

$$p_z(a_3) = [1 : -A_3 : 0] \in L_z.$$

We construct quadratic birational transformation of the form

$$f(x,y) = \left(\frac{f_1(x,y)}{f_3(x,y)}, \frac{f_2(x,y)}{f_3(x,y)}\right).$$

The line at infinity L_z is mapped to itself, the denominator $f_3(x,y)$ must be a polynomial of degree 1, which defines a line passing through indeterminacy points $p_x(a_1)$ and $p_y(a_2)$, we set

$$f_3(x,y) \;\; = \;\; -A_1^{-1}x - A_2 y + 1.$$

ND3

As the y-axis is mapped to itself, and in the x-axis L_x, f induces

$$x = e^{2\pi i t_1} \mapsto X = e^{2\pi i (t_1 + b_1)} = B_1 x,$$

we can set

$$f_1(x, y) \;=\; B_1 x(-A_1^{-1} x + \alpha y + 1),$$

for some constant α.
As the x-axis is mapped to itself, and in the y-axis L_y, f induces

$$y = e^{-2\pi i t_2} \mapsto Y = e^{-2\pi i (t_2 + b_2)} = B_2^{-1} y,$$

we can set

$$f_2(x, y) \;=\; B_2^{-1} y(\beta x - A_2 y + 1),$$

for some constant β.

ND3

The mapping induced on the line at infinity L_z is

$$z = \frac{y}{x} \mapsto Z = \frac{Y}{X} \simeq B_1^{-1} B_2^{-1} z \frac{\beta - A_2 z}{-A_1^{-1} + \alpha z}.$$

As the induced dynamics in L_z must be as $z \mapsto B_3 z$, we get

$$B_3 = -B_1^{-1} B_2^{-1} \beta A_1 = -B_1^{-1} B_2^{-1} A_2 \alpha^{-1}.$$

From the condition $a_1 + a_2 + a_3 \equiv b_1 + b_2 + b_3$, we have $A_1 A_2 A_3 = B_1 B_2 B_3$. Hence

$$\alpha = -A_1^{-1} A_3^{-1}, \quad \beta = -A_2 A_3.$$

ND3

We got

$$
\begin{aligned}
X_{ND3} &= \frac{B_1 x(-A_1^{-1}x - A_1^{-1}A_3^{-1}y + 1)}{-A_1^{-1}x - A_2 y + 1}, \\
Y_{ND3} &= \frac{B_2^{-1} y(-A_2 A_3 x - A_2 y + 1)}{-A_1^{-1}x - A_2 y + 1}.
\end{aligned}
$$

Or

$$
\begin{aligned}
X_{ND3} &= B_1 \left(x + \frac{(A_2 - A_1^{-1}A_3^{-1})xy}{-A_1^{-1}x - A_2 y + 1} \right), \\
Y_{ND3} &= B_2^{-1} \left(y + \frac{(A_1^{-1} - A_2 A_3)xy}{-A_1^{-1}x - A_2 y + 1} \right).
\end{aligned}
$$

Orbit data for ND3 map

In this case, as each component of the smooth part of the invariant cubic curve is isomorphic to $\mathbb{C}/\mathbb{Z}$, possible determinant λ is ± 1. Possible combination of the permutation τ of the three lines and the determinant τ is $\tau = id$. and $\lambda = 1$ as a candidate for surface automorphism of positive entropy.

Let $(n_1, n_2, n_3), \sigma = id$ be an orbit data. Suppose the characteristic polynomial of this orbit data has a real root greater than 1.

We need $(m_1, m_2, m_3) \in \mathbb{Z}^3$ to determine a quadratic birational transformation f preserving the nodal cubic curve $\{xyz = 0\}$, and having the prescribed orbit data.

Conditions

Let $p_1^+, p_2^+, p_3^+ \in \mathbb{C}/\mathbb{Z}$ denote the parameter values of the indeterminacy points $I(f) = \{p_x(p_1^+), p_y(p_2^+), p_z(p_3^+)\}$.

And let $p_1^-, p_2^-, p_3^- \in \mathbb{C}/\mathbb{Z}$ denote the parameter values of the indeterminacy points $I(f^{-1}) = \{p_x(p_1^-), p_y(p_2^-), p_z(p_3^-)\}$.

We look for translations $b_1, b_2, b_3 \in \mathbb{C}/\mathbb{Z}$, together with $\{p_i^\pm\}$. Set $b = b_1 + b_2 + b_3$.

The conditions to be satisfied are as follows.

$$p_1^+ + p_2^+ + p_3^+ \equiv b \mod 1,$$

$$p_j^- \equiv p_j^+ + b_j - b \mod 1, \qquad j = 1, 2, 3,$$

$$p_j^+ \equiv p_j^- + (n_j - 1)b_j \mod 1, \qquad j = 1, 2, 3.$$

Translations

From the second and third conditions, we have

$$n_j b_j \equiv b \mod 1, \qquad j = 1, 2, 3.$$

To solve these, we set (abusing notation)

$$n_j b_j = b + m_j, \qquad j = 1, 2, 3.$$

And with

$$b_j = \frac{1}{n_j}(b + m_j), \qquad j = 1, 2, 3,$$

$$b = b_1 + b_2 + b_3 = (\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3})b + \frac{m_1}{n_1} + \frac{m_2}{n_2} + \frac{m_3}{n_3},$$

we get

$$b = \frac{m_1 n_2 n_3 + m_2 n_1 n_3 + m_3 n_1 n_2}{n_1 n_2 n_3 - n_1 n_2 - n_2 n_3 - n_3 n_1},$$

$$b_j = \frac{m_j}{n_j} + \frac{1}{n_j} b, \qquad j = 1, 2, 3.$$

Choice of indeterminate points

We can choose

$$p_1^+ = \frac{b_2 + b_3}{2} + r_1, \quad p_2^+ = \frac{b_1 + b_3}{2} + r_2, \quad p_3^+ = \frac{b_1 + b_2}{2} - r_1 - r_2,$$

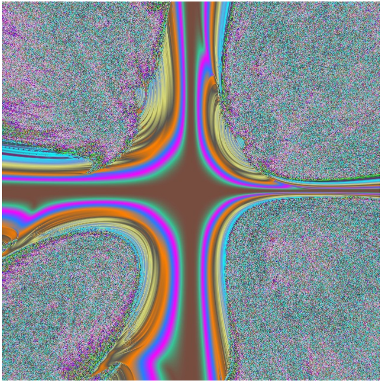
for any $r_1, r_2 \in \mathbb{C}$.

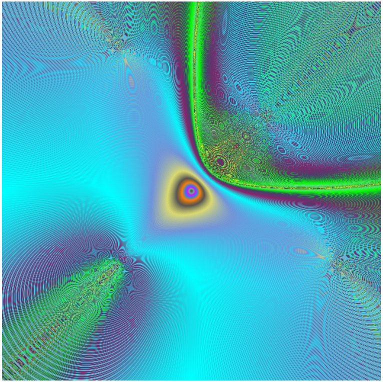
Choice of r_1 and r_2 induces change of coordinates $(x, y) \mapsto (e^{2\pi i r_1} x, e^{-2\pi i r_2} y)$. Dynamical systems are all conjugate to each other.

If $r_1 = r_2 = 0$, the obtained map has a symmetry

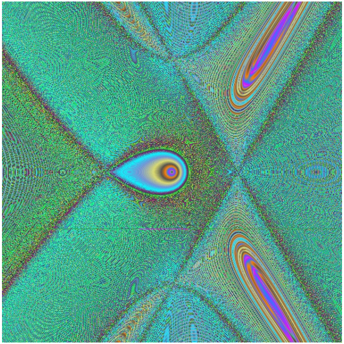
$$p_i^- = -p_i^+.$$

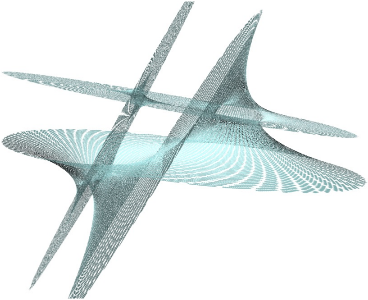
Pictures from NO3map (ND3i445R122,ND3i445R111)



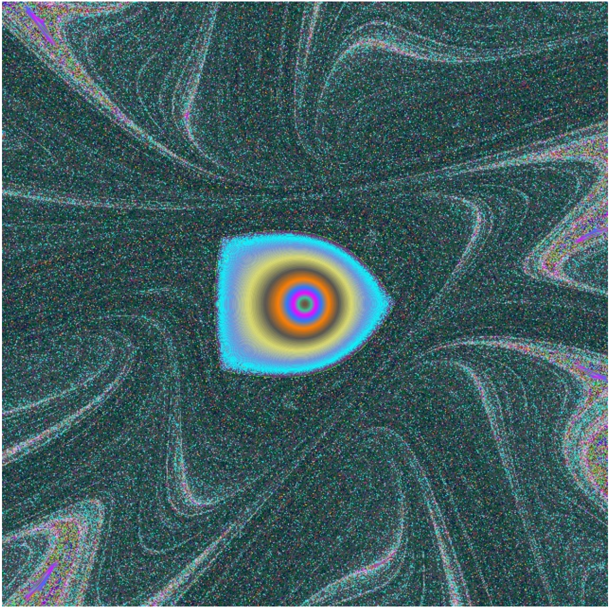


Pictures from NO3map (ND3i445S111,ND3i445D111)





ND3map (ND3i556S110)



case ND2

The case ND2 is treated as follows.

Take parametrization in curve $\{z(xy - z^2) = 0\}$ as follows
Let $t \in \mathbb{C}/\mathbb{Z}$.

$$p_Q(t) = (e^{2\pi it}, e^{-2\pi it}) \in Q = \{xy = 1\}.$$

$$p_L(t) = [1 : -e^{2\pi it} : 0] \in L = \text{line at infinity}.$$

Let $p_j^+ \in \mathbb{C}/\mathbb{Z}$, and set $A_j = e^{2\pi i p_j^+}$, $j = 1, 2, 3$.

For translation b in the hyperbola Q, set $B = e^{2\pi ib}$.
For translation c in the line at infinity L, set $C = e^{2\pi ic}$.

Picard coordinates

Suppose $t_1 \not\equiv t_2$.

If three points $p_Q(t_1), p_Q(t_2) \in Q$, $p_L(t_3) \in L$ are on a line, then

$$\frac{e^{-2\pi i t_2} - e^{-2\pi i t_1}}{e^{2\pi i t_2} - e^{2\pi i t_1}} = -e^{2\pi i t_3}.$$

This gives

$$(1 - e^{2\pi i (t_1 + t_2 + t_3)})(e^{-2\pi i t_2} - e^{-2\pi i t_1}) = 0,$$

and

$$t_1 + t_2 + t_3 \equiv 0.$$

And if $t_1 + t_2 + t_3 \equiv 0$, then the three points are on a line.

case ND2

PROPOSITION. For $p_1^+, p_2^+, p_3^+ \in \mathbb{C}/\mathbb{Z}$ and $b, c \in \mathbb{C}/\mathbb{Z}$, with

$$p_1^+ + p_2^+ + p_3^+ \equiv 2b + c \not\equiv 0 \mod 1,$$

there exists a quadratic birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, such that

$$f(Q) = Q, \quad f(L) = L,$$

$$\text{and} \quad I(f) = \{p_Q(p_1^+), p_Q(p_2^+), p_L(p_3^+)\},$$

inducing

$$t \mapsto t + b, \quad \text{in} \quad Q,$$

$$t \mapsto t + c, \quad \text{in} \quad L.$$

REM. Multiplier is always 1. Do not confuse with b_i in cuspidal case.

case ND2

We construct birational map $f : (x, y) \mapsto (X, Y)$, as follows. As the line at infinity is mapped to itself, the denominator must be of degree 1 defining the line passing through the indeterminacy points $p_Q(p_1^+) = (A_1, A_1^{-1})$ and $p_Q(p_2^+) = (A_2, A_2^{-1})$, the denominator can be set to

$$f_3(x, y) \;=\; x - A_1 - A_2 + A_1 A_2 y.$$

Recall $f(p_Q(t)) = p_Q(t + b)$, *i.e.*, $X = Bx$, $Y = B^{-1}y$ in Q. The dynamics in the hyperbola $\{xy = 1\}$ is $(x, y) \mapsto (Bx, B^{-1}y)$. Let

$$X = B\left(x + \frac{U(xy-1)}{x - (A_1 + A_2) + A_1 A_2 y}\right),$$

$$Y = B^{-1}\left(y + \frac{V(xy-1)}{y - (A_1^{-1} + A_2^{-1}) + A_1^{-1} A_2^{-1} x}\right),$$

for some $U, V \in \mathbb{C}$.

case ND2

Recall $f(p_L(t)) = p_L(t+c)$, *i.e.*, $-Z = -Cz$ in L.

As the dynamics in the line at infinity is $z \mapsto Cz$, with $z = y/x$,

$$Z = \lim_{x,y\to\infty} Y/X = \frac{B^{-2}}{1 + A_1^{-1}A_2^{-1}U} z \frac{z + (1+V)A_1^{-1}A_2^{-1}}{z + (A_1A_2 + U)^{-1}},$$

gives $U = A_3^{-1} - A_1A_2$ and $V = A_1A_2A_3 - 1$.
(Used $B^2C = A_1A_2A_3$.)

The Cremona transformation $F : (x,y) \mapsto (X,Y)$ is given by

$$X = B\left(x + \frac{(A_3^{-1} - A_1A_2)(xy-1)}{x - (A_1 + A_2) + A_1A_2 y}\right),$$

$$Y = B^{-1}\left(y + \frac{(A_3 - A_1^{-1}A_2^{-1})(xy-1)}{y - (A_1^{-1} + A_2^{-1}) + A_1^{-1}A_2^{-1} x}\right).$$

Orbit data for ND2

Relation between parameters of indeterminate points and translation b and c :

$$p_1^+ + p_2^+ + p_3^+ \equiv 2b + c \mod 1.$$

Relation between parameters of indeteminate points :

$$p_1^- \equiv p_1^+ - b - c, \quad p_2^- \equiv p_2^+ - b - c, \quad p_3^- \equiv p_3^+ - 2b \mod 1.$$

For orbit data $(n_1, n_2, n_3), \sigma$, parameters must satisfy followings.

$$p_{\sigma(j)}^+ \equiv p_1^- + (n_j - 1)b \mod 1, \quad j = 1, 2,$$

$$p_3^+ \equiv p_3^- + (n_3 - 1)c \mod 1.$$

Here, σ is either *id*. or transposition $(1, 2)$.

extra data

Choose integers m_1, m_2, m_3. Take representatives $b, c \in \mathbb{C}$ (abusing notations), and compute in $\mathbb{C}$.

$$p_1^+ + p_2^+ + p_3^+ = 2b + c,$$

$$p_1^- = p_1^+ - b - c, \quad p_2^- = p_2^+ - b - c, \quad p_3^- = p_3^+ - 2b,$$

$$p_{\sigma(j)}^+ = p_j^- + (n_j - 1)b + m_j, \quad j = 1, 2,$$

$$p_3^+ = p_3^- + (n_3 - 1)c + m_3.$$

Maybe other choice of extra data, case by case.

(ND2) transposition case

For integers m_1, m_2, m_3, and a complex number s, we get

$$b \equiv \frac{(n_3 - 1)(m_1 + m_2) + 2m_3}{(n_1 + n_2 - 4)(n_3 - 1) - 4} \mod 1,$$

$$c \equiv \frac{2(m_1 + m_2) + (n_1 + n_2 - 4)m_3}{(n_1 + n_2 - 4)(n_3 - 1) - 4} \mod 1,$$

and

$$p_1^+ \equiv \frac{n_2 - 1}{2} b + s - \frac{m_1}{2} \mod 1,$$

$$p_2^+ \equiv \frac{n_1 - 1}{2} b + s + \frac{m_2}{2} \mod 1,$$

$$p_3^+ \equiv b - 2s \mod 1.$$

Parameter s gives choice of coordinates. When $s = 0$, the map has symmetries. It is reversible by the complex conjugation, and it is symmetric with respect to the conjugate diagonal. It is also reversible by swapping involution $(x, y) \mapsto (y, x)$.

(ND2) case $\sigma = id$.

In the case of $\sigma = id.$, we need $n_1 = n_2$.
For $m_1, m_3 \in \mathbb{Z}$, $\ell \in \mathbb{Z}$ and $\zeta_1, \zeta_2 \in \mathbb{C}$, we get

$$b \equiv \frac{(n_3 - 1)m_1 + m_3}{(n_1 - 2)(n_3 - 1) - 2} \mod 1,$$

$$c \equiv \frac{2m_1 + (n_1 - 2)m_3}{(n_1 - 2)(n_3 - 1) - 2} \mod 1,$$

and

$$p_1^+ \equiv \frac{2b + c + \ell}{3} + \zeta_1 + \zeta_2 \mod 1,$$

$$p_2^+ \equiv \frac{2b + c + \ell}{3} + \zeta_1 - \zeta_2 \mod 1,$$

$$p_3^+ \equiv \frac{2b + c + \ell}{3} - 2\zeta_1 \mod 1.$$

Parameters ζ_1, ζ_2 gives choice of coordinates.

example

In the case of orbit data $(n_1, n_2, n_3) = (4, 3, 5)$, $\sigma = (1, 2)$, and $(m_1, m_2, m_3) = (1, 1, 1)$, with $s = 0$, we have

$$b \equiv \frac{1}{4}, \quad c \equiv \frac{7}{8},$$

$$p_1^+ \equiv \frac{3}{4}, \quad p_2^+ \equiv \frac{3}{8}, \quad p_3^+ \equiv \frac{1}{4}.$$

And

$$p_1^- \equiv \frac{5}{8}, \quad p_2^- \equiv \frac{1}{4}, \quad p_3^- \equiv \frac{3}{4}.$$

Observe the symmetries of the Cremona transformation.

$$X = B\left(x + \frac{(A_3^{-1} - A_1A_2)(xy - 1)}{x - (A_1 + A_2) + A_1A_2y}\right),$$

$$Y = B^{-1}\left(y + \frac{(A_3 - A_1^{-1}A_2^{-1})(xy - 1)}{y - (A_1^{-1} + A_2^{-1}) + A_1^{-1}A_2^{-1}x}\right).$$

When $s = 0$,

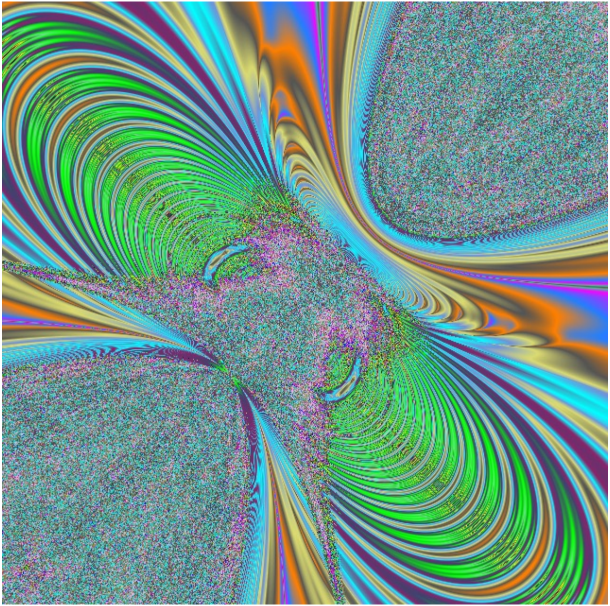
$$p_1^- \equiv -p_2^+, \quad p_2^- \equiv -p_1^+, \quad p_3^- \equiv -p_3^+,$$

we see

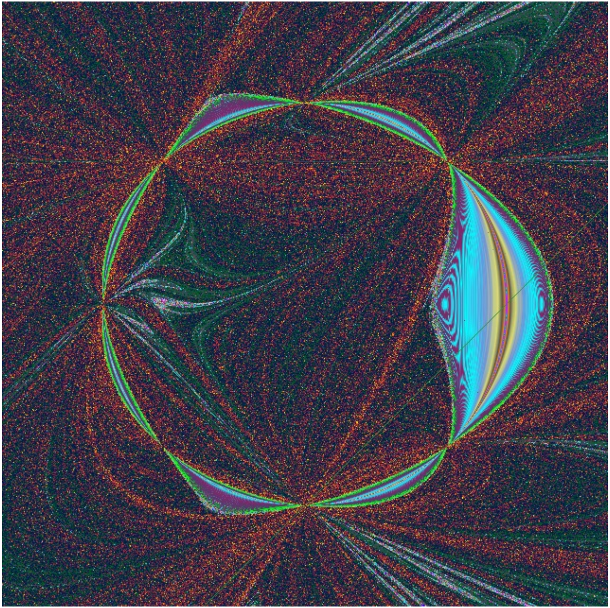
$$\overline{f} = f^{-1} = S \circ f \circ S, \qquad T \circ f \circ T = f,$$

where $S : (x, y) \mapsto (y, x)$, $T : (x, y) \mapsto (\bar{y}, \bar{x})$, are involutions. Therefore, $f : (x, y) \mapsto (X, Y)$ is reversible with respect to involution S, and involution by the complex conjugation. It is symmetric with respect to involution T.

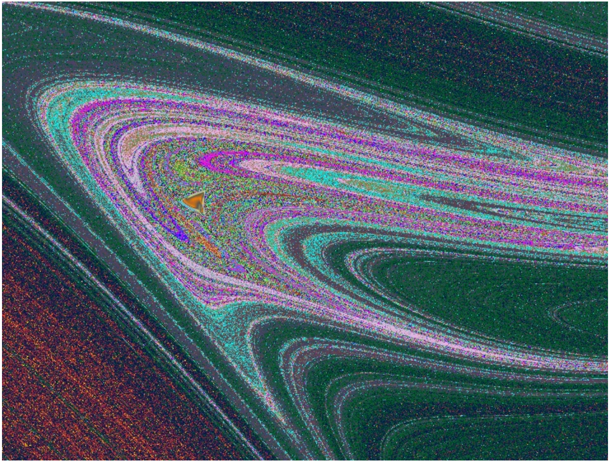
Real slice for ND2map (4,3,5),$\sigma = (1,2)$



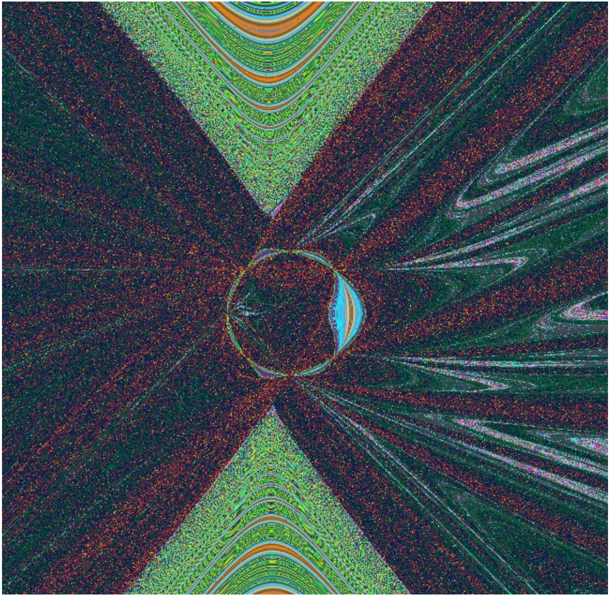
Conjugate diagonal slice for ND2map (4,3,5),$\sigma = (1, 2)$



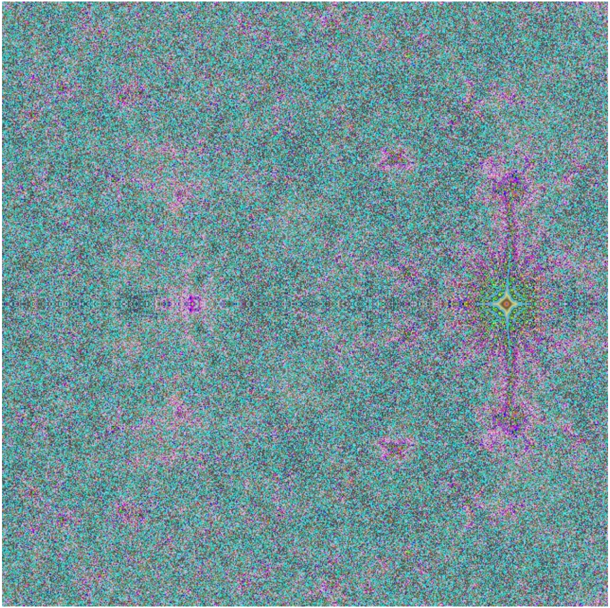
Conjugate diagonal slice for ND2map (4,3,5),$\sigma = (1, 2)$, some part



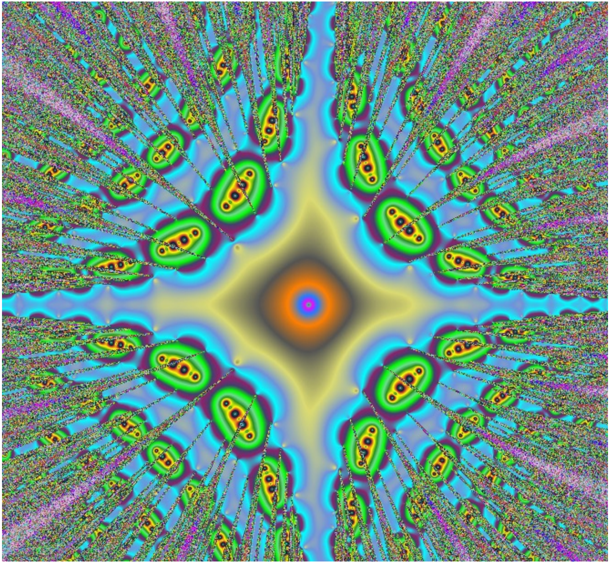
Conjugate diagonal slice for ND2map (4,3,5),$\sigma = (1, 2)$, zoomed out



Diagonal slice for ND2map (4,3,5),$\sigma = (1, 2)$



Diagonal slice for ND2map (4,3,5),$\sigma = (1, 2)$, zoomed in



6. Elliptic curve

Elliptic curve

Diller [D] stated the existence of surface automorphisms with positive entropy preserving a smooth cubic curve.

PROPOSITION(Diller, 2011). Suppose that f is a quadratic transformation properly fixing a smooth cubic curve C. If f has positive entropy and lifts to an automorphism of some modification $X \to \mathbb{P}^2$, then either

$C \cong \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ and the multiplier for $f|_C$ is $\pm i$; or

$C \cong \mathbb{C}/(\mathbb{Z} + e^{\pi i/3}\mathbb{Z})$ and the multiplier for $f|_C$ is a prime cube root of -1.

Weierstraß $\wp$-function

We use Weierstraß $\wp$-function as parametrization of invariant smooth cubic curve.

Let $\tau \in \mathbb{C} \setminus \mathbb{R}$ and $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ be a lattice.

Weierstraß $\wp$-function $\wp : \mathbb{C}/\Lambda_\tau \to \mathbb{P}$ is defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda'_\tau} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right),$$

where $\Lambda'_\tau = \Lambda_\tau \setminus \{0\}$.

THEOREM The Weierstraß $\wp$-function satisfies a Weierstraß equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3,$$

$$\text{with} \quad g_2 = 60 \sum_{\omega \in \Lambda'_\tau} \omega^{-4}, \quad \text{and} \quad g_3 = 140 \sum_{\omega \in \Lambda'_\tau} \omega^{-6}.$$

parametrization

The parametrization of elliptic curve $\{y^2 = 4x^3 - g_2 x - g_3\}$ is given by

$$p(t) = (\wp(t), \wp'(t)), \qquad t \in \mathbb{C}/\Lambda_\tau.$$

THEOREM(Diller, 2011) Let $C \subset \mathbb{P}^2$ be a smooth cubic curve. Suppose we are given points $p(p_1^+), p(p_2^+), p(p_3^+) \in C_{reg}$, a multiplier $a \in \mathbb{C}^\times$, and a translation $b \in \mathbb{C}/\Lambda$. Then there exists at most one quadratic transformation f properly fixing C with $I(f) = \{p(p_1^+), p(p_2^+), p(p_3^+)\}$ and $f(p(t)) = p(at+b)$. This f exists if and only if the following hold.

$p_1^+ + p_2^+ + p_3^+ \not\equiv 0$;
a is a multiplier for C_{reg};
$a(p_1^+ + p_2^+ + p_3^+) \equiv 3b$.

Finally, the points of indeterminacy for f^{-1} are given by $p_j^- = ap_j^+ - 2b$, $j = 1, 2, 3$.

Cremona transformation

The most basic non-linear birational transformation $J : \mathbb{P}^2 \to \mathbb{P}^2$ can be expressed as

$$[x : y : z] \mapsto [yz : zx : xy].$$

J acts by blowing up points $e_1 = [1 : 0 : 0]$, $e_2 = [0 : 1 : 0]$,$e_3 = [0 : 0 : 1]$ and then collasping the lines $\{x = 0\}$,$\{y = 0\}$,$\{z = 0\}$ to e_1, e_2, e_3 respectively.

A generic quadratic Cremona transformation can be obtained from J by pre- and post- composing with linear transformations $f = L_1 \circ J \circ L_2^{-1}$.

conditions

We see that

$$I(f) = \{L_2(e_1), L_2(e_2), L_2(e_3)\}, \quad I(f^{-1}) = \{L_1(e_1), L_1(e_2), L_1(e_3)\}.$$

The choice of L_1 and L_2 is not unique, since specification of three points does not determine a linear transformation uniquely. We need a supplementary condition to determine the transformation with uniqueness.

A unique biquadratic transformation $f = L_1 \circ K \circ J \circ L_2^{-1}$ is obtained by specifying a linear transformation $K : \mathbb{P}^2 \to \mathbb{P}^2$, which fixes e_1,e_2, e_3, and setting

$$\tilde{K} = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix},$$

$$\tilde{L}_1 = \begin{pmatrix} \wp(p_1^-) & \wp(p_2^-) & \wp(p_3^-) \\ \wp'(p_1^-) & \wp'(p_2^-) & \wp'(p_3^-) \\ 1 & 1 & 1 \end{pmatrix},$$

$$\tilde{L}_2 = \begin{pmatrix} \wp(p_1^+) & \wp(p_2^+) & \wp(p_3^+) \\ \wp'(p_1^+) & \wp'(p_2^+) & \wp'(p_3^+) \\ 1 & 1 & 1 \end{pmatrix}.$$

Take a fixed point t_0 of the inner dynamics $t \mapsto at + b$. Then point $p(t_0)$ must be a fixed point of f. We can choose $\tilde{K}$ by

$$\tilde{L}_1^{-1}(p(t_0)) = \tilde{K} \circ \tilde{J} \circ \tilde{L}_2^{-1}(p(t_0)).$$

Obtained biquadratic transformation $f = L_1 \circ K \circ J \circ L_2^{-1}$ is the unique one satisfying

$$I(f) = \{p(p_1^+), p(p_2^+), p(p_3^+)\}, \quad I(f^{-1}) = \{p(p_1^-), p(p_2^-), p(p_3^-)\},$$

$$\text{and} \quad f(p(t_0)) = p(t_0).$$

orbit data to transformation

As at most one quadratic transformation properly fixing C, this f is the quadratic transformation described in the above theorem.

We have two cases.

(case ELI) $C \cong \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ and the multiplier for $f|_C$ is $\pm i$;

(case ELW) $C \cong \mathbb{C}/(\mathbb{Z} + e^{\pi i/3}\mathbb{Z})$ and the multiplier for $f|_C$ is a prime cube root of -1.

case ELI

In case ELI, let $\Lambda = \mathbb{Z} + i\mathbb{Z}$, and we suppose the multiplier for $f|_C$ is i.
The case of $-i$ is similar.

Suppose the translation of the inner dynamics is $b \in \mathbb{C}/\Lambda$.

Conditions for orbit data $(n_1, n_2, n_3), \sigma$ are as follows.

$$p_1^+ + p_2^+ + p_3^+ \equiv -3ib \not\equiv 0 \mod \Lambda,$$

$$p_j^- \equiv ip_j^+ - 2b \mod \Lambda, \quad j = 1, 2, 3.$$

$$p_{\sigma(j)}^+ \equiv i^{n_j - 1}(p_j^- - \frac{1+i}{2}b) + \frac{1+i}{2}b \mod \Lambda, \quad j = 1, 2, 3.$$

case ELI

The inner dynamics $z \mapsto iz + b$ is periodic of period 4. We don't have $n_j \geq 5$ as a realizable orbit data. So necessarily, to have a transformation of positive entropy, we need

$$n_j \leq 4, \quad j = 1, 2, 3.$$

If $n_j = 4$ for some j, with $\sigma(j) = j$, we have

$$p_j^+ \equiv p_j^+ + 3ib \mod \Lambda,$$

which gives $3ib \equiv 0$. This case is not allowed.

If $n_1 + n_2 + n_3 \leq 9$, then the topological entropy of the surface automorphism is 0.

case ELI

Therefor in the case of $\sigma = id.$, we have $n_1 + n_2 + n_3 \leq 9$. In this case, the topological entropy of the quadratic transformation is 0.

In the case of cyclic permutation $\sigma = (1,2,3)$, possible orbit data for quadratic transformation to have positive entropy are:

$$(4,4,4), \quad (4,4,3), \quad (4,3,3).$$

(case (4,4,2) does not have a solution)

In the case of transposition $\sigma = (1,2)$,possible orbit data for quadratic transformation to have positive entropy are:

$$(4,3,3), \quad (4,4,3).$$

case ELI, σ is cyclic

(case $(4,4,4)$, $\sigma=(1,2,3)$)

$$b \equiv \frac{1}{9}\beta,\ p_1^+ \equiv 8ib+\frac{1}{3}\alpha,\ p_2^+ \equiv 5ib+\frac{1}{3}\alpha,\ p_3^+ \equiv 2ib+\frac{1}{3}\alpha,$$

where $\beta \in (\Lambda \setminus 3\Lambda)/9\Lambda$ and $\alpha \in \Lambda/3\Lambda$.

(case $(4,4,3)$, $\sigma=(1,2,3)$)

$$b \equiv \frac{1}{15}\beta+\frac{1}{2}\alpha,\ p_1^+ \equiv (5-4i)b+\frac{1}{2}\alpha,\ p_2^+ \equiv (5-i)b+\frac{1}{2}\alpha,\ p_3^+ \equiv (5+2i)b+\frac{1}{2}\alpha,$$

where $\beta \in (\Lambda \setminus 5\Lambda)/15\Lambda$, $\alpha=0$, or $\beta \in \Lambda/15\Lambda$, $\alpha = 1+i$.

(case $(4,3,3)$, $\sigma=(1,2,3)$)

$$b \equiv \frac{2}{15}\beta+\frac{i}{15}\alpha,\ p_1^+ \equiv (2-\frac{5}{2}i)b+\frac{1}{2}\alpha,\ p_2^+ \equiv (2+\frac{i}{2})b+\frac{1}{2}\alpha,\ p_3^+ \equiv (\frac{7}{2}-i)b-\frac{i}{2}\alpha,$$

where $\beta \in (\Lambda \setminus 5\Lambda)/15\Lambda$, $\alpha \in \{0,1,i,1+i\}$.

case ELI, σ is a transposition

(case (4,4,3), $\sigma = (1,2)$)

$$b \equiv \frac{1}{6}\beta,\ p_1 \equiv (-1+\frac{i}{2})b-\frac{\gamma}{4}+\frac{\alpha}{2},\ p_2^+ \equiv (-1-\frac{5}{2}i)b-\frac{\gamma}{4}+\frac{\alpha}{2},\ p_3^+ \equiv (2-i)b+\frac{\gamma}{2},$$

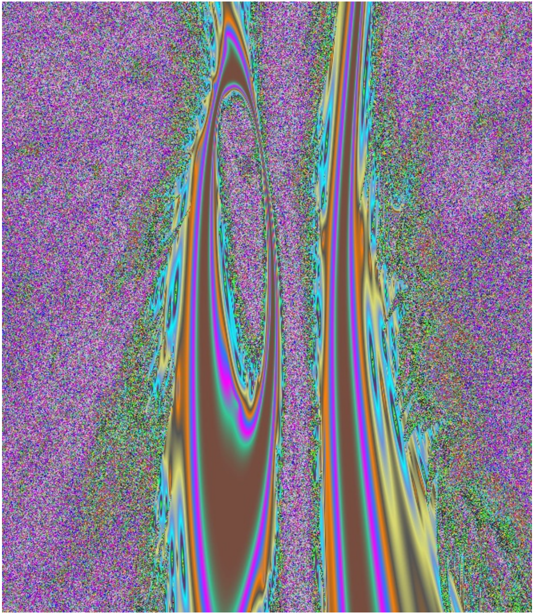
$$\text{where } \beta \in (\Lambda \setminus 2\Lambda)/6\Lambda, \quad \gamma \in (1+i)\Lambda, \quad \alpha \in \Lambda.$$

(case (4,3,3), $\sigma = (1,2)$)

$$b \equiv \frac{1}{18}\beta,\ p_1 \equiv (\frac{7}{2} - \frac{5i}{2}b) + \frac{\alpha}{2},\ p_2^+ \equiv (\frac{7}{2} + \frac{i}{2}i)b + \frac{\alpha}{2},\ p_3^+ \equiv (-7-i)b,$$

$$\text{where } \beta \in ((1+i)\Lambda \setminus 6\Lambda)/18\Lambda, \quad \alpha \in (1+i)\Lambda/2\Lambda.$$

ELImap (ELIc433Rb40)



case ELW

In case ELW, let $\epsilon = e^{\pi i/3}$ and $\Lambda_\epsilon = \mathbb{Z} + \epsilon\mathbb{Z}$. We suppose the mutiplier for $f|_C$ is ϵ.

The case of $\bar{\epsilon}$ is similar.

Suppose the translation of the inner dynamics is $b \in \mathbb{C}/\Lambda_\epsilon$.

Conditions for orbit data $(n_1, n_2, n_3), \sigma$ are as follows.

$$p_1^+ + p_2^+ + p_3^+ \equiv (3 - 3\epsilon)b \not\equiv 0 \mod \Lambda_\epsilon,$$

$$p_j^- \equiv \epsilon p_j^+ - 2b \mod \Lambda_\epsilon, \quad j = 1, 2, 3.$$

$$p_{\sigma(j)}^+ \equiv \epsilon^{n_j - 1}(p_j^- - \epsilon b) + \epsilon b \mod \Lambda_\epsilon, \quad j = 1, 2, 3.$$

notations

By eliminating p_j^-, we get

$$p^+_{\sigma(j)} \equiv \epsilon^{n_j} p_j^+ + (\epsilon - 2\epsilon^{n_j - 1} - \epsilon^{n_j})b.$$

To simplify notations, let

$$\kappa_k = \epsilon - 2\epsilon^{k-1} - \epsilon^k, \quad k = 0, 1, \cdots, 6,$$

$$\delta_k = \frac{1}{1-\epsilon^k}, \quad k = 1, 2, \cdots, 5.$$

Then the equations are :

$$p^+_{\sigma(j)} \equiv \epsilon^{n_j} p_j^+ + \kappa_{n_j} b \mod \Lambda_\epsilon, \quad j = 1, 2, 3.$$

with

$$p_1^+ + p_2^+ + p_3^+ \equiv (3 - 3\epsilon)b \not\equiv 0 \mod \Lambda_\epsilon.$$

case ELW, $\sigma = id$.

As the dynamics $t \mapsto \epsilon t + b \equiv \epsilon(t - \epsilon b) + \epsilon b$ is periodic of period 6, we need

$$n_1, n_2, n_3 \leq 6.$$

The case $\sigma(j) = j$ and $n_j = 6$, is not appropriate, as

$$0 \equiv \kappa_6 b \equiv (3\epsilon - 3)b \mod \Lambda_\epsilon.$$

For orbit data (n_1, n_2, n_3), $\sigma = id.$, $n_1 + n_2 + n_3 \geq 10$, with $n_j < 6, j = 1, 2, 3,$,

$$b \equiv \frac{\beta - \delta_{n_1}\alpha_1 - \delta_{n_2}\alpha_2 - \delta_{n_3}\alpha_3}{\delta_{n_1}\kappa_{n_1} + \delta_{n_2}\kappa_{n_2} + \delta_{n_3}\kappa_{n_3} + 3\epsilon - 3} \mod \Lambda_\epsilon,$$

$$p_j^+ \equiv \delta_{n_j}(\kappa_{n_j} b + \alpha_j) \mod \Lambda_\epsilon, \quad j = 1, 2, 3.$$

where $\alpha_1, \alpha_2, \alpha_3, \beta \in \Lambda_\epsilon$.

case ELW, $\sigma = (1,2)$, transposition

In the transposition case of orbit data $(n_1, n_2, n_3), \sigma = (1,2)$, with $n_1, n_2 \le 6$, $n_3 < 6$, $n_1 + n_2 + n_3 \ge 10$, and $n_1 + n_2 \ne 6$, we have

$$b \equiv \frac{\beta - \delta_{n_1+n_2}((\epsilon^{n_2}+1)\alpha_1 + (\epsilon^{n_1}+1)\alpha_2) - \delta_{n_3}\alpha_3}{\delta_{n_1+n_2}((\epsilon^{n_2}+1)\kappa_{n_1} + (\epsilon^{n_1}+1)\kappa_{n_2}) + \delta_{n_3}\kappa_{n_3} + 3\epsilon - 3} \mod \Lambda_\epsilon,$$

$$p_1^+ \equiv \delta_{n_1+n_2}((\epsilon^{n_2}\kappa_{n_1} + \kappa_{n_2})b + \epsilon^{n_2}\alpha_1 + \alpha_2) \mod \Lambda_\epsilon,$$

$$p_2^+ \equiv \delta_{n_1+n_2}((\epsilon^{n_1}\kappa_{n_2} + \kappa_{n_1})b + \epsilon^{n_1}\alpha_2 + \alpha_1) \mod \Lambda_\epsilon,$$

$$p_3^+ \equiv \delta_{n_3}(\kappa_{n_3} b + \alpha_3) \mod \Lambda_\epsilon,$$

where $\alpha_1, \alpha_2, \alpha_3, \beta \in \Lambda_\epsilon$.

case ELW, $\sigma = (1,2,3)$, cyclic permutation

In the cyclic permutation case of orbit data $(n_1, n_2, n_3), \sigma = (1,2,3)$, $n_1 + n_2 + n_3 \geq 10$, with $n_1, n_2, n_3 \leq 6$, and $n_1 + n_2 + n_3 \neq 12$, we have

$$b \equiv \frac{\beta - A_1\alpha_1 - A_2\alpha_2 - A_3\alpha_3}{\delta_{n_1+n_2+n_3}(A_1\kappa_{n_1} + A_2\kappa_{n_2} + A_3\kappa_{n_3}) + 3\epsilon - 3} \mod \Lambda_\epsilon,$$

where $A_1 = \epsilon^{n_2+n_3} + \epsilon^{n_2} + 1$, $A_2 = \epsilon^{n_3+n_1} + \epsilon^{n_3} + 1$, $A_3 = \epsilon^{n_1+n_2} + \epsilon^{n_1} + 1$.

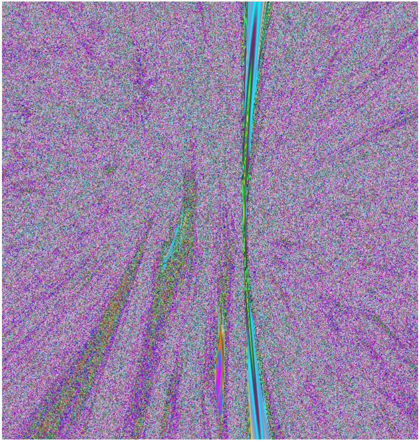
$$p_1^+ \equiv \delta_{n_1+n_2+n_3}((\epsilon^{n_2+n_3}\kappa_{n_1} + \epsilon^{n_3}\kappa_{n_2} + \kappa_{n_3})b + \epsilon^{n_2+n_3}\alpha_1 + \epsilon^{n_3}\alpha_2 + \alpha_3) \mod \Lambda_\epsilon,$$

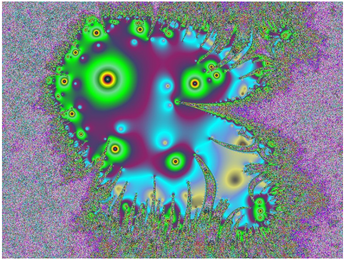
$$p_2^+ \equiv \delta_{n_1+n_2+n_3}((\epsilon^{n_3+n_1}\kappa_{n_2} + \epsilon^{n_1}\kappa_{n_3} + \kappa_{n_1})b + \epsilon^{n_3+n_1}\alpha_2 + \epsilon^{n_1}\alpha_3 + \alpha_1) \mod \Lambda_\epsilon,$$

$$p_3^+ \equiv \delta_{n_1+n_2+n_3}((\epsilon^{n_1+n_2}\kappa_{n_3} + \epsilon^{n_2}\kappa_{n_1} + \kappa_{n_2})b + \epsilon^{n_1+n_2}\alpha_3 + \epsilon^{n_2}\alpha_1 + \alpha_2) \mod \Lambda_\epsilon,$$

where $\alpha_1, \alpha_2, \alpha_3, \beta \in \Lambda_\epsilon$.

ELWmap(ELWi543Rb31a01a10a00)





Appendix

Theta functions. Let $q = e^{\pi i \tau}$ denote the nome.

$$\vartheta_0(z) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2n\pi z,$$

$$\vartheta_1(z) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin(2n+1)\pi z,$$

$$\vartheta_2(z) = 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\cos(2n+1)\pi z,$$

$$\vartheta_3(z) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos 2n\pi z.$$

Weierstraß $\wp$-function.

$$\wp(z) = \pi^2(\vartheta_3^2\vartheta_2^2\frac{\vartheta_0^2(z)}{\vartheta_1^2(z)} - \frac{1}{3}(\vartheta_3^4 + \vartheta_2^4)).$$

$$\wp'(z) = 2\pi^2\vartheta_3^2\vartheta_2^2\vartheta_0(z)\frac{\vartheta_0'(z)\vartheta_1(z) - \vartheta_0(z)\vartheta_1'(z)}{\vartheta_1^3(z)}.$$

$$g_2 = \frac{2}{3}\pi^4(\vartheta_2^8 + \vartheta_3^8 + \vartheta_0^8),$$

$$g_3 = \frac{8}{27}\pi^6(\vartheta_2^{12} - \frac{3}{2}\vartheta_2^8\vartheta_3^4 - \frac{3}{2}\vartheta_2^4\vartheta_3^8 + \vartheta_3^{12}).$$

Here, $\vartheta_i = \vartheta_i(0)$, $i = 0, 2, 3$.

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