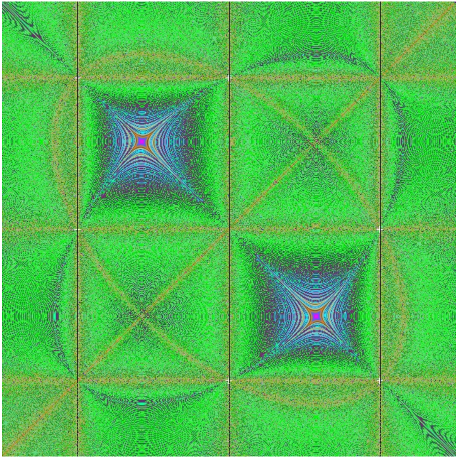
Multiple Dynamics on Elliptic Surface



Shigehiro Ushiki April 22, 2022

Abstract

We construct an elliptic surface, such that the fibration structure and the group of automorphisms can be determined.

A pencil of cubic curves passing through nine base points defines an elliptic surface by blowing up the base points.

Inspecting the pencil of cubic curves, singular fibers and a section of the elliptic fibration are detected.

We construct four birational maps preserving the cubic pencil.

They induce automorphisms of the elliptic surface.

The symmetries of the elliptic fibration and these automorphisms generate the group of automorphisms.

Contents

Acknowledgement

The author is grateful to Prof. Uehara for advices and ideas related to the study of dynamics of elliptic fibrations.

0. Introduction

Birational map

Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map. Under certain conditions, birational map induces a holomorphic automorphism $F : S \to S$ of rational surface S, which is obtained by successive blowing ups of $\mathbb{P}^2$, with projection $\pi : S \to \mathbb{P}^2$.

$$\begin{array}{ccc} S & \xrightarrow{F} & S \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{P}^2 & \overset{f}{\dashrightarrow} & \mathbb{P}^2. \end{array}$$

Elliptic fibration

A surjective holomorphic map $\psi : S \to \mathbb{P}^1$ is an elliptictic fibration if almost all fibers are smooth curves of genus 1, and no fiber contains an exceptional (-1)-curve.

An **elliptic surface** S over $\mathbb{P}^1$ is a smooth projective surface with an elliptic fibration over $\mathbb{P}^1$.

Preservation of elliptic fibration

We say that automorphism $F : S \to S$ preserves elliptic fibration $\psi : S \to \mathbb{P}^1$, if

$$\begin{array}{ccc} S & \xrightarrow{F} & S \\ \downarrow \psi & & \downarrow \psi \\ \mathbb{P}^1 & \xrightarrow{\Omega} & \mathbb{P}^1. \end{array}$$

holds for some Möbius transformation $\Omega : \mathbb{P}^1 \to \mathbb{P}^1$.

Theorem of Gizatullin

The **dynamical degree** λ_1 of F is defined as

$$\lambda_1 = \lim_{n\to\infty} ||(F^n)^*||^{1/n}.$$

THEOREM(Gizatullin [1980], Cantat [1999])
Assume $F \in \mathrm{Aut}(S)$, $\lambda_1 = 1$, and $\{||(F^n)^*||\}_{n\in\mathbb{N}}$ is unbounded. Then F preserves an elliptic fibration.

Kodaira names

Singular fibers are classified by Kodaira. (smooth fiber is indicated by I_0)

$$I_n, \; n \geq 1, \quad II, \quad III, \quad IV,$$
$$I_n^*, \; n \geq 0, \quad IV^*, \quad III^*, \quad II^*.$$

Euler number:

$$\chi(I_n) = n, \quad \chi(II) = 2, \quad \chi(III) = 3, \quad \chi(IV) = 4,$$
$$\chi(I_n^*) = n + 6, \quad \chi(IV^*) = 8, \quad \chi(III^*) = 9, \quad \chi(II^*) = 10.$$

$$\sum_{F_v:\text{singular fiber}} \chi(F_v) = 12.$$

Classical example

Lyness ([L], 1942, 1945, 1961) found an invariant function

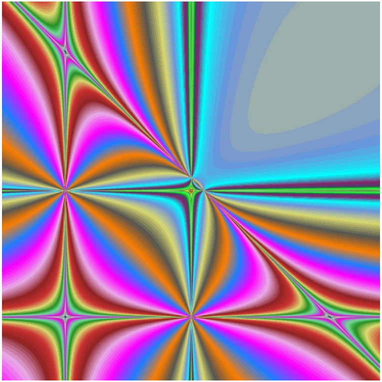
$$r_a(x, y) = \frac{(x + y + a)(x + 1)(y + 1)}{xy}$$

for a family of birational automorphisms

$$f_a(x, y) = (y, \frac{y + a}{x}).$$

f_a has an invariant pencil of cubic curves.

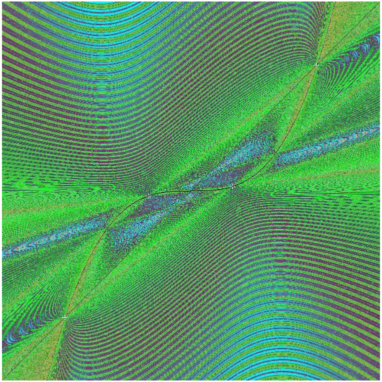
Lyness example



Picture of $\log|r_a(x,y)|$.

Single dynamics

In the case of orbit data $(1, 1, 7)$, *cyclic*, with invariant cuspidal curve, the configuration of singular fibers of the invariant elliptic fibration is II, I_5, I_3, I_2.



In this case, we have $\mathrm{rank}(MW(S)) = 1$, $\mathrm{Aut}_S(\mathbb{P}^1) = \{id.\}$, and

$$\mathrm{Aut}(S) \ \simeq \ < F > \rtimes \mathbb{Z}/2\mathbb{Z}.$$

Multiple dynamics

The elliptic surface S itself is determined by the nine base points of successive blowups.

In this note, we consider surface automorphisms sharing a rational surface S.

Together with the symmetries of S, we try to understand the group of automorphisms $\mathrm{Aut}(S)$.

We construct an elliptic surface S with

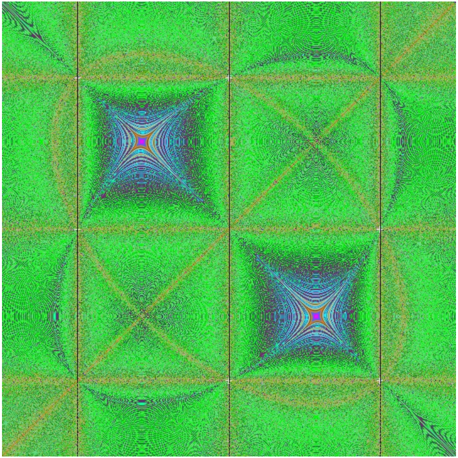
$$\mathrm{Aut}(S) \ \simeq \ < H_f, H_g > \rtimes D_4.$$

Oguiso and Shioda ([OS], 1991) have calculated $MW(S)$ for each configuration of singular fibers. $\{H_f(\sigma), H_g(\sigma)\}$ is an orthonormal basis of $MW(S) \simeq < 1/6 >^{\oplus 2}$.

Our surface S provides a concrete and explicit example.

1. An Elliptic Surface

Base points



Let $\Sigma \subset \mathbb{P}^2$ denote the set of nine points $\{(n, m) | n, m = 0, \pm 1\}$.

Pencil of cubic curves

Cubic curves $\{x^3 - x = 0\}$ and $\{y^3 - y = 0\}$ pass through all points of Σ.

The family of cubic curves

$$y^3 - y = v(x^3 - x), \qquad v \in \mathbb{P}^1$$

defines a pencil of cubic curves.

Three lines passing through a point (IV) : $v = 0, \infty$.
Quadric and a line intersecting in two points (I_2) : $v = \pm 1$.

If $v = 1$, then

$$y^3 - y - (x^3 - x) = (y - x)(y^2 + xy + x^2 - 1).$$

If $v = -1$, then

$$y^3 - y + (x^3 - x) = (y + x)(y^2 - xy + x^2 - 1).$$

Blowups

By blowing up the nine points in Σ, we obtain a rational surface, S, with projection $\pi : S \to \mathbb{P}^2$.

Define rational function $\varphi : \mathbb{P}^2 \to \mathbb{P}^1$ by

$$\varphi(x, y) = \frac{y^3 - y}{x^3 - x}.$$

Then $\psi =$ (extention of) $\varphi \circ \pi : S \to \mathbb{P}^1$ defines an elliptic fibration.

Singular fibers are (in Kodaira's notation):

$$IV(v = 0), \quad IV(v = \infty), \quad I_2(v = 1), \quad I_2(v = -1).$$

This configuration has symmetries $v \mapsto \frac{1}{v}$, and $v \mapsto -v$. We denote this group by $\mathrm{Aut}_S(\mathbb{P}^1) \simeq (\mathbb{Z}/2\mathbb{Z})^2$.

Section

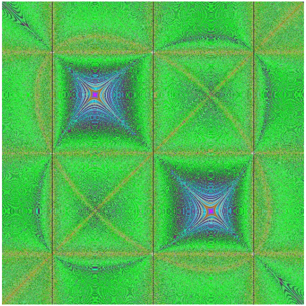
Let $\sigma : \mathbb{P}^1 \to S$ be a section of the fibration $\psi : S \to \mathbb{P}^1$, defined by

$$\sigma(v) = "v" \in \pi^{-1}(O) \simeq \mathbb{P}^1.$$

After Karayayla([K],2011), we denote, by $\mathrm{Aut}_\sigma(S)$, the group of automorphisms of S leaving the image of this section invariant. He showed a short exact sequence of groups

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}_\sigma(S) \to \mathrm{Aut}_S(\mathbb{P}^1) \to 1.$$

$\mathrm{Aut}_\sigma(S)$



The group $\mathrm{Aut}_\sigma(S) \simeq D_4$ is generated by involutions

$$(x, y) \mapsto (y, x) \quad \Rightarrow \quad v \mapsto \frac{1}{v},$$

$$(x, y) \mapsto (-x, y) \quad \Rightarrow \quad v \mapsto -v,$$

$$(x, y) \mapsto (-x, -y) \quad \Rightarrow \quad v \mapsto v.$$

2. Group of Automorphisms

Mordell-Weil group

By specifying a section $\sigma : \mathbb{P}^1 \to S$, the set of sections of fibration $\psi : S \to \mathbb{P}^1$ form an additive group, regarding the specified section σ as the origin of each fiber. The addition of sections is defined by the group law in each smooth fiber as elliptic curve, and by taking the closure for a section. This group is called the Mordell-Weil group, $MW(S)$, of S.

Karayayla([K], 2011) proved :

THEOREM.

$$\mathrm{Aut}(S) = MW(S) \rtimes \mathrm{Aut}_\sigma(S).$$

$$(t_{\zeta_1} \circ \alpha_1)(t_{\zeta_2} \circ \alpha_2) = (t_{\zeta_1 + \alpha_1(\zeta_2)} \circ (\alpha_1 \circ \alpha_2)).$$

Mordell-Weil rank

It is known ([Gi], 1980) that in the case of rational surface,

$$\operatorname{rank}(MW(S)) = 8 - \sum_{v \in R}(m_v - 1).$$

Where, R is the set of points $v \in \mathbb{P}^1$, such that $F_v = \psi^{-1}(v)$ is not smooth, and m_v is the number of irreducible components of the singular fiber.

In our case,

$$R = \{0, \infty, 1, -1\},$$

$$m_0 = m_\infty = 3, \qquad m_1 = m_{-1} = 2.$$

So, we have

$$\operatorname{rank}(MW(S)) = 2.$$

In the next section, we look for two automorphisms.

3. Multiple Dynamics

Birational map yielding S

Let us consider quadratic birational map

$$f(x, y) = \left(\frac{2x^2 - xy - x + y - 1}{3xy + x + y - 1}, \quad \frac{-2y^2 + xy - x + y + 1}{3xy + x + y - 1} \right).$$

Its inverse map is

$$f^{-1}(x, y) = \left(\frac{-2x^2 - xy + x + y + 1}{3xy - x + y + 1}, \quad \frac{-2y^2 - xy - x - y + 1}{3xy - x + y + 1} \right).$$

Indeterminacy points

Quadratic birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ acts by blowing up three indeterminacy points in $\mathbb{P}^2$ and blowing down the three exceptional lines joining them.

The inverse map f^{-1} is also quadratic and the images of three exceptional lines of f are the indeterminacy points of f^{-1}.

Let

$$I(f) = \{p_1^+, p_2^+, p_3^+\}$$

and

$$I(f^{-1}) = \{p_1^-, p_2^-, p_3^-\},$$

with

$$p_i^- = f(\ell(p_j^+, p_k^+)), \quad \{i, j, k\} = \{1, 2, 3\}.$$

Orbit data

In our case of f,

$$p_1^+ = (0,1), \quad p_2^+ = (1,0), \quad p_3^+ = (-1,-1),$$

$$p_1^- = (1,0), \quad p_2^- = (0,-1), \quad p_3^- = (-1,1).$$

And

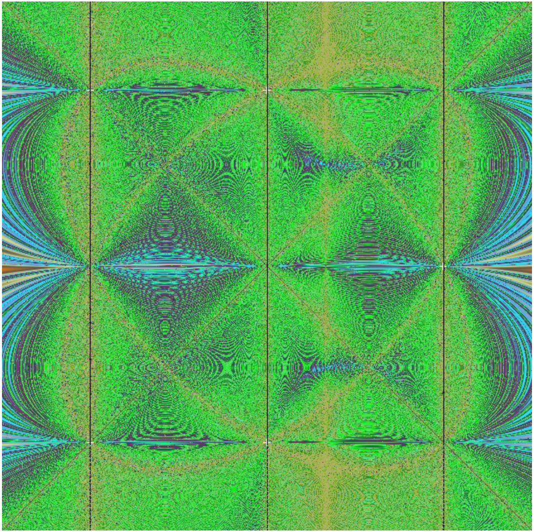
$$p_2^+ = p_1^-, \qquad p_1^+ = f^4(p_2^-), \qquad p_3^+ = f^2(p_3^-).$$

$$p_2^- \mapsto (1,1) \mapsto (0,0) \mapsto (1,-1) \mapsto p_1^+,$$

$$p_3^- \mapsto (-1,0) \mapsto p_3^+.$$

The orbit data of f is $(1,5,3)$, *transposition*$(1,2)$, and Σ is the set of nine basepoints for f.

$F : S \to S$



Grouth of F^*

Quadratic birational map f lifts to a surface automorphism $F : S \to S$.

The characteristic polynomial of the cohomology homomorphism $F^* : H^2(S, \mathbb{Z}) \to H^2(S, \mathbb{Z})$ is

$$\chi(d) = (d-1)(d^2-1)(d^3-1)(d^4-1).$$

And

$$||(F^*)^n|| \sim Cn^2.$$

Inner dynamics in vertical $IV(v=\infty)$

F preserves the elliptic fibration on S. Observe the dynamics in cubic curve $\{x^3 - x = 0\}$.

Let

$$t_1 \mapsto (0, \frac{1}{2}t_1), \qquad t_2 \mapsto (1, -t_2), \qquad t_3 \mapsto (-1, -t_3)$$

be the Picard parametrization.

We see

$$f(0, \frac{1}{2}t_1) = (1, -(t_1+1)),$$

$$f(1, -t_2) = (0, \frac{1}{2}(t_2+1)),$$

$$f(-1, -t_3) = (-1, -(t_3+1)).$$

The dynamics in $\{x^3 - x = 0\}$ is,

transposition of two components, $\quad \eta_\infty = \mathrm{tr}(0,1)$.

translation by 1, $\quad \tau_\infty : t \mapsto t+1$.

Inner dynamics in horizontal $IV(v=0)$

F also preserves cubic curve $\{y^3 - y = 0\}$.
Let

$$u_1 \mapsto (\frac{1}{2}u_1, 0), \qquad u_2 \mapsto (-u_2, 1), \qquad u_3 \mapsto (-u_3, -1).$$

We have

$$f(\frac{1}{2}u_1, 0) = (-(-u_1 - 1), -1),$$

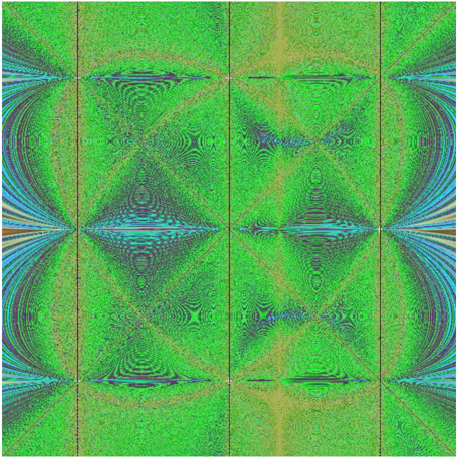
$$f(-u_2, 1) = (\frac{1}{2}(-u_2 - 1), 0),$$
$$f(-u_3, -1) = (-(-u_3 - 1), 1).$$

The dynamics in $\{y^3 - y = 0\}$ is

cyclic permutation of components, $\quad \eta_0 = \mathrm{cy}(0, -1, 1)$.

affine map, $\quad \tau_0 : u \mapsto -u - 1$.

$F : S \to S$



$IV(v = \infty): \ \eta_\infty = \mathrm{tr}(0,1), \ \tau_\infty : t \mapsto t+1, \ (1,5,3)tr.$
$IV(v = 0): \ \eta_0 = \mathrm{cy}(0,-1,1), \ \tau_0 : u \mapsto -u-1, \ (1,5,3)tr.$

Mordell-Weil automorphism

Our automorphism $F : S \to S$ preserves singular fibers $IV(v = 0)$, and $IV(v = \infty)$, but permutes other singular fibers $I_2(v = 1)$ and $I_2(v = -1)$.

Involution $(x, y) \mapsto (-x, y)$ preserves singular fibers $IV(v = 0)$, $IV(v = \infty)$ and permutes $I_2(v = 1)$ and $I_2(v = -1)$.

Composition of F with this involution preserves each singular fibers and hence each leaf of the elliptic fibration. And the image of the specific section σ determines an element in the Mordell-Weil group.

Birational map preserving every fiber

Recall birational map

$$f(x,y)=\left(\frac{2x^2-xy-x+y-1}{3xy+x+y-1},\ \ \frac{-2y^2+xy-x+y+1}{3xy+x+y-1}\right).$$

Let

$$h_f(x,y)=\left(\frac{-2x^2+xy+x-y+1}{3xy+x+y-1},\ \ \frac{-2y^2+xy-x+y+1}{3xy+x+y-1}\right).$$

$$h_f^{-1}(x,y)=\left(\frac{2x^2-xy+x-y-1}{3xy-x-y-1},\ \ \frac{2y^2-xy-x+y-1}{3xy-x-y-1}\right).$$

Dynamics of h_f

$$I(h_f) = \{(1,0),\ (0,1),\ (-1,-1)\}.$$

$$I(h_f^{-1}) = \{(1,1),\ (0,-1),\ (-1,0)\}.$$

In vertical IV,

$$\{x=0\} \mapsto \{x=-1\} \mapsto \{x=1\} \mapsto \{x=0\},$$

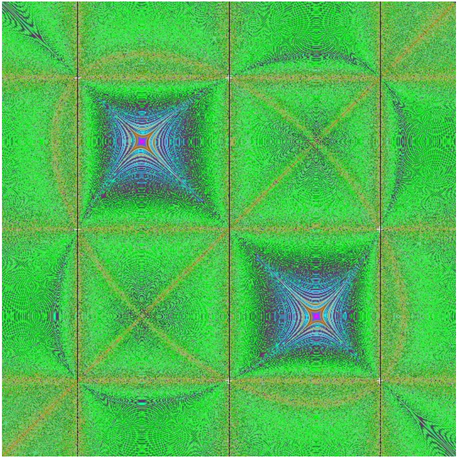
$$t \mapsto t+1.$$

In horizontal IV,

$$\{y=0\} \mapsto \{y=-1\} \mapsto \{y=1\} \mapsto \{y=0\},$$

$$u \mapsto u+1.$$

$H_f : S \to S$, (Mordell-Weil map)



$IV(v=\infty):\ \eta_\infty = \mathrm{cy}(0,-1,1),\ \tau_\infty : t \mapsto t+1,\ (3,3,3)id.$
$IV(v=0):\ \eta_0 = \mathrm{cy}(0,-1,1),\ \tau_0 : u \mapsto u+1,\ (3,3,3)id.$

Another automorphism

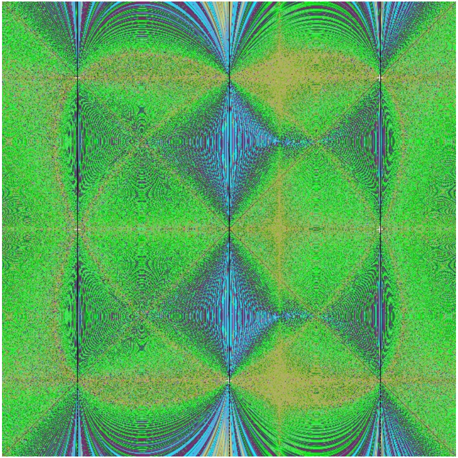
There are similar automorphisms with different dynamics.

$$g(x,y)=\left(\frac{2x^2-xy+x-y-1}{3xy-x-y-1},\frac{-2y^2+xy+x-y+1}{3xy-x-y-1}\right).$$

$$g^{-1}(x,y)=\left(\frac{2x^2+xy-x-y-1}{3xy-x+y+1},\frac{2y^2+xy+x+y-1}{3xy-x+y+1}\right).$$

This automorphism also defines a Mordell-Weil section by composing involution $(x,y)\mapsto(x,-y)$.

$G : S \to S$



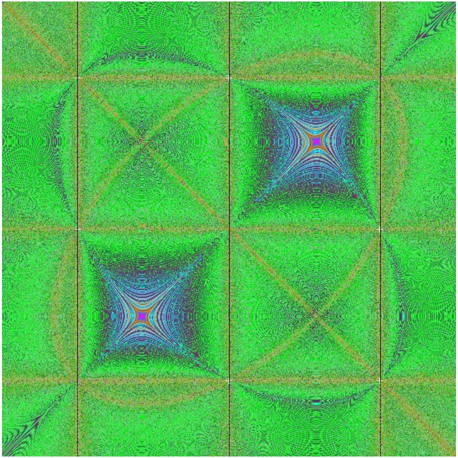
$IV(v = \infty): \ \eta_\infty = \mathrm{cy}(0, 1, -1), \ \tau_\infty : t \mapsto -t - 1, \ (1, 5, 3)tr.$
$IV(v = 0): \ \eta_0 = \mathrm{tr}(0, -1), \ \tau_0 : u \mapsto u + 1, \ (1, 5, 3)tr.$

Let

$$h_g(x,y) = \left(\frac{2x^2 - xy + x - y - 1}{3xy - x - y - 1}, \;\; \frac{2y^2 - xy - x + y - 1}{3xy - x - y - 1} \right).$$

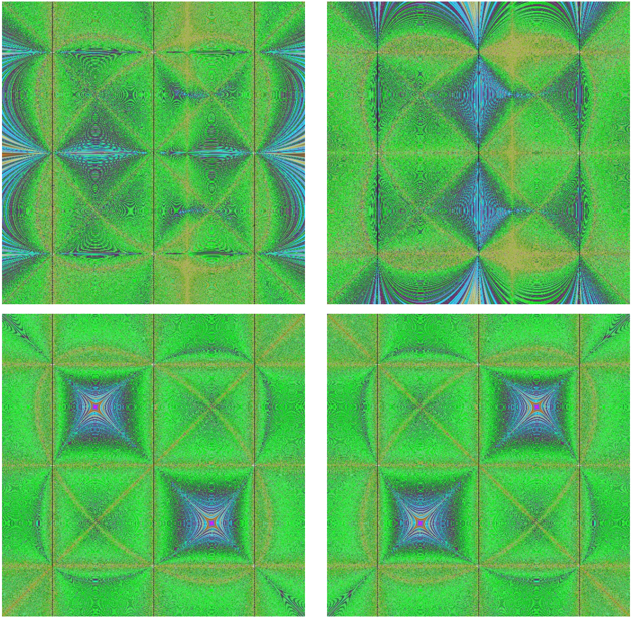
$$h_g^{-1}(x,y) = \left(\frac{-2x^2 - xy - x + y + 1}{3xy + x - y + 1}, \;\; \frac{-2y^2 - xy + x + y + 1}{3xy + x - y + 1} \right).$$

$H_g : S \to S$ Mordell-Weil map derived from G



$IV(v = \infty) : \ \eta_\infty = \mathrm{cy}(0, 1, -1), \ \tau_\infty : t \mapsto t + 1, \ (3, 3, 3) id.$
$IV(v = 0) : \ \eta_0 = \mathrm{cy}(0, -1, 1), \ \tau_0 : u \mapsto u - 1, \ (3, 3, 3) id.$

Automorphisms of S



Recall Karayayla's theorem:

THEOREM.

$$\mathrm{Aut}(S) = MW(S) \rtimes \mathrm{Aut}_\sigma(S).$$

With Gizatullin's estimate

$$\mathrm{rank}(MW(S)) = 2$$

for our surface S.

Additive group $MW(S)$ is generated by two automorphisms above.

H_f and H_g commutes, and $H_f^{\circ p} \circ H_g^{\circ q}$ induces $\tau_\infty : t \mapsto t + p + q$, and $\tau_0 : u \mapsto u + p - q$. Hence $\mathrm{rank}(< H_f, H_g >) = 2$.

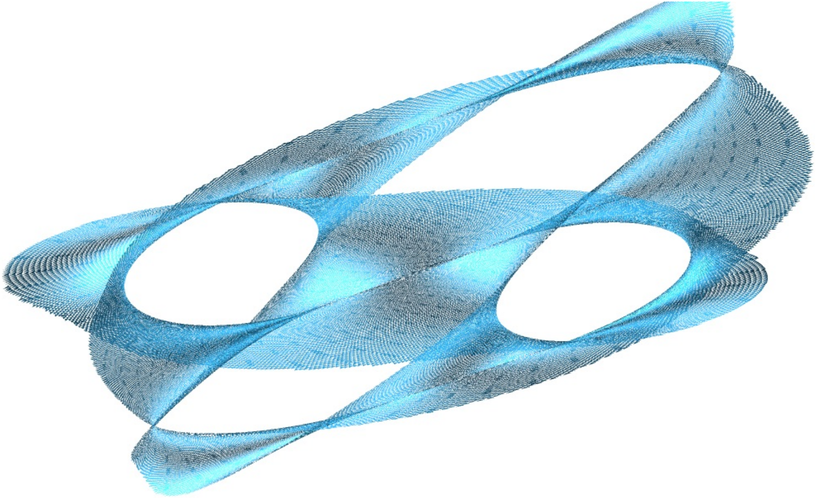
$\mathrm{Aut}_\sigma(S) \simeq D_4$ is generated by involutions.

$\mathrm{Aut}(S)$

We conclude, in our case,

$$\mathrm{Aut}(S) \simeq \; < H_f, H_g > \rtimes D_4.$$

Thank you.



Appendix

The generator of $E(K)^0$ of S are

$$V_{H_f} = P_{-1,1} + P_{1,0} + P_{0,-1} - P_{1,-1} - P_{0,1} - P_{-1,0},$$

$$V_{H_g} = P_{1,1} + P_{0,-1} + P_{-1,0} - P_{-1,-1} - P_{1,0} - P_{0,1}.$$

And the generator of $MW(S)$ are

$$P_{1,-1}, \quad P_{-1,-1},$$

for H_f, H_g, and

$$P_{-1,1}, \quad P_{1,1},$$

for H_f^{-1}, H_g^{-1}.

$$H_f^*(V_{H_f}) = V_{H_f} - K.$$

References

[BK1] E. Bedford and K. Kim. Dynamics of Rational Surface Automorphisms: Linear Fractional Recurrences. J. Geomet. Anal. **19**(2009), 553-583.
[BK2] E. Bedford and K. Kim. Dynamics of rational surface automorphisms: rotation domains. Amer. J. Math. **134**(2012), no. 2, 379-405.

References

[C1] S. Cantat. Dynamique des automorphisms des surfaces projectives complexes. C.R. Acad. Sci. Paris Sér I Math., 328(10):901-906, 1999.
[C2] S. Cantat. Dynamics of automorphisms of compact complex surfaces. "Frontiers in Complex Dynamics – In Celebration of John Milnor's 80th birthday", Eds. A.Bonifant, M. Lyubich, S. Sutherland, Prinston University Press, Princeton and Oxford, pp. 463-509, 2014

References

[D] J. Diller. Cremona transformations, surface automorphisms, and plane cubics. Michigan Math. J. **60**(2011), no. 2, pp409-440, with an appendix by Igor Dolgachev.
[Gi] M. H. Gizatullin. Rational G-surfaces. Izv. Akad. Nauk SSSR Ser. Mat. **44**(1980), 110-144, 239.
[Gr] J. Grivaux. Parabolic automorphisms of projective surfaces (after M. H. Gizatullin)
https://hal.archives-ouvertes.fr/hal-01301468, Nov. 2019.
[K] T. Karayayla. The Classification of Automorphism Groups of Rational Elliptic Surface With Section,
https://repository.upenn.rdu/edissertations/988, Spring 2011

References

[L] R. C. Lyness. Notes 1581, 1847, and 2952. Math. Gaz. **26**, 62 (1942), **29**, 231 (1945), and **45**, 201 (1961).
[M] C. T. McMullen. Dynamics on blowups of the projective plane. Publ. Sci. IHES, **105**, 49-89(2007).
[N] M. Nagata. On rational surfaces. II. Mem. Coll. Sci. Univ. Kyoto Ser. A Math., 33:271-293, 1960/1961.
[OS] K. Oguiso and T. Shioda. The Mordell-Weil Lattice of a Rational Elliptic Surface. Commentarii Mathematici Universitatis Sancti Pauli **40** (1991), 83-99.

References

[SS] M.Schütt, T. Shioda. Elliptic surfaces, Algebraic Geometry in East Asia – Seoul 2008, pp.51-160, Advanced Strudies in Pure Mathematics 60, 2010.
[U1] T. Uehara. Rational surface automorphisms preserving cuspidal anticanonical curves. Mathematische Annalen, Band 362, Heft 3-4, 2015.
[U2] T. Uehara. Rational surface automorphisms with positive entropy. Ann. Inst. Fourier (Grenoble) **66**(2016), 377-432.