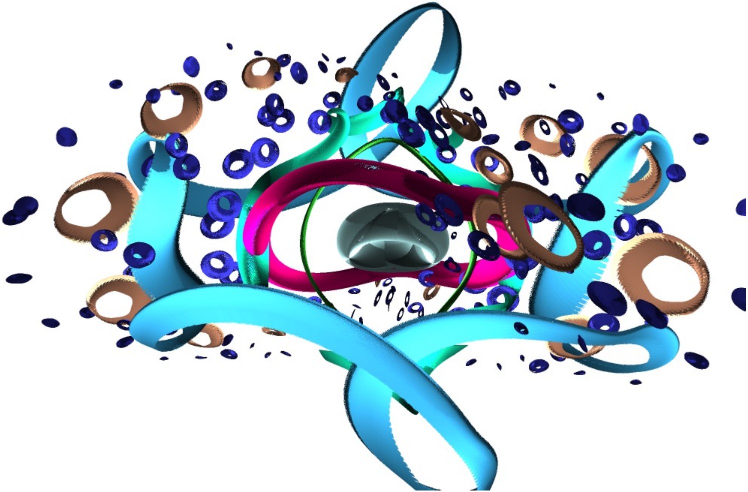
Invariant curves
in Complex Surface Automorphisms



Shigehiro Ushiki Kyoto, July, 3rd, 2020

Abstract

Automorphisms of complex surfaces can have various invariant curves. In this note, we consider families of rational surface automorphisms with invariant cubic curve.

Such rational automorphism can have, at the same time, an invariant line, or an invariant quadratic curve, or a pair of lines intersecting at a point, which are disjoint from the invariant cubic curve.

Contents

0. Fatou domain

Fatou set

Let $f : X \to X$ be an automorphism of a compact complex manifold X.

A point $p \in X$ is a point of the **forward Fatou set** F_f^+ if there exists an open neighborhood U of p on which the sequence $\{f^n\}_{n \in \mathbb{N}}$ forms a normal family of holomorphic mappings from U to X.

Define the **backward Fatou set** F_f^- and the **Fatou set** F_f by

$$F_f^- = F_{f^{-1}}^+, \qquad F_f = F_f^+ \cap F_f^-.$$

Dissipative Hénon map

In the Hénon map case, Bedford and Smilie [BS2] proved :

THEOREM (Bedford-Smilie, 1991) Suppose that f is dissipative, and $f(\Omega) = \Omega$ is an invariant Fatou component satisfying $\overline{\{f^n(p) : n \geq 0\}} \subset \Omega$ for some $p \in \Omega$. Then one of the following occurs:

1. $\Omega =$ basin of an attracting fixed point, and $\Omega \cong \mathbb{C}^2$.
2. $\Omega =$ basin of a rotational disk, and $\Omega \cong \mathbb{D} \times \mathbb{C}$.
3. $\Omega =$ basin of a rotational annulus, and $\Omega \cong \mathbb{A} \times \mathbb{C}$.

REM. Existence of case 3 is an open problem.

REM. Existence of parabolic basins is proved by T. Ueda (1986, 1991).

Dissipative surface automorphism

Followings are found.
(no rigorous proof for case 2)

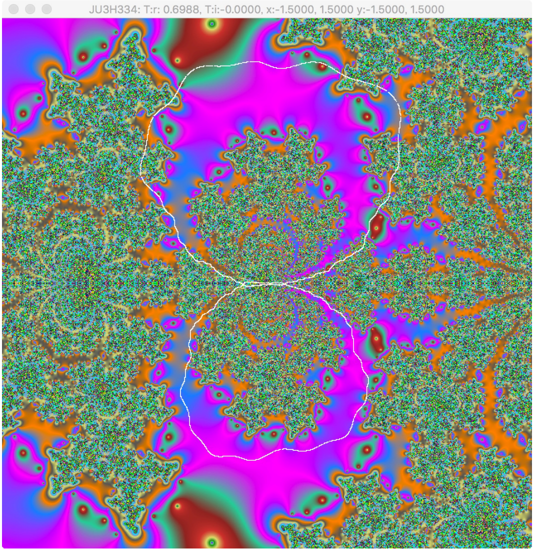
1. basin of an attracting periodic/fixed point $\cong \mathbb{C}^2$.
2. basin of a rotational annulus $\cong$ "$\mathbb{A} \times \mathbb{C}$". (?)
3. basin of a rotational Riemann sphere $\cong$ "$\mathbb{P} \times \mathbb{C}$". (??)
4. basin of a curve of periodic points $\cong$ "$\mathbb{P} \times \mathbb{C}$". (??)

Followings are not found yet.

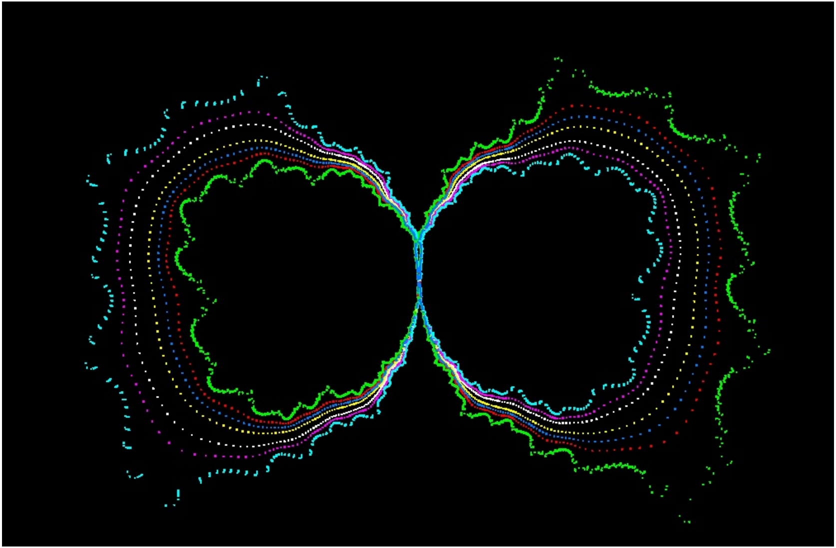
5. basin of a parabolic periodic/fixed point $\cong \mathbb{C}^2$. (???)
6. basin of a rotational disk $\cong \mathbb{D} \times \mathbb{C}$. (???)

REM. "$\mathbb{P} \times \mathbb{C}$" is the normal line bundle.

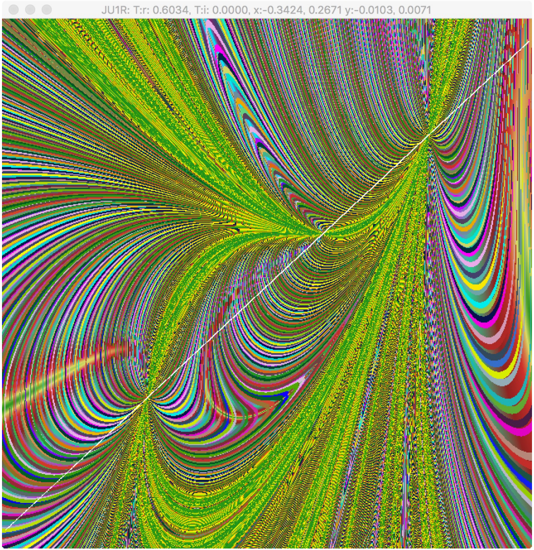
Example : (case 2) orbit data (3,3,4), cyclic, horizontal slice



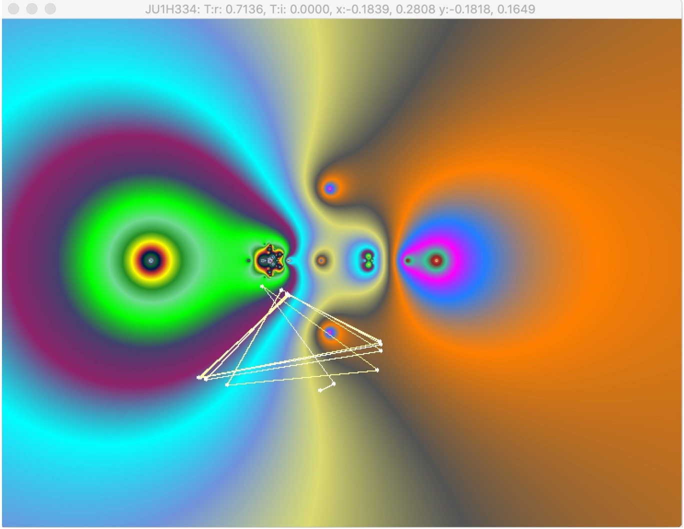
Example : (case 2) orbit data (3,3,4), cyclic, in surface



Example: (case 3) orbit data (3,4,5), id, real slice



Example: (case 4) orbit data (3,3,4), id, horizontal slice



Volume preserving automorphism

Suppose Ω is a Fatou component of a volume preserving automorphism f with $f(\Omega) = \Omega$. Define the set of all limits of convergent subsequences $\mathcal{G}$ by

$$\mathcal{G} = \left\{ g = \lim_{n_j \to \infty} f^{n_j} : \Omega \to \overline{\Omega} \right\}.$$

If $g = \lim_{n_j \to \infty} f^{n_j}$ is such a limit, then g must preserve volume, and thus it is locally invertible. It follows that $g : \Omega \to \Omega$.

It is known that $\mathcal{G}$ is a compact Lie group, by a theorem of H. Cartan. The connected component $\mathcal{G}_0$ of the identity must be a (real) torus.

Rank of a rotation domain

In the volume preserving Hénon map case, known result is as follows.

THEOREM (Bedford-Smilie 1991).
$\mathcal{G}_0$ is isomorphic to $\mathbb{T}^\rho$ with $\rho = 1$ or 2.

Same result should hold for surface automorphism case.

Such a domain is called a **rotation domain**, and we refer to ρ as the **rank** of the rotation domain.

Reinhardt domain

Let $D \subset \mathbb{C}^2$ be a connected open set. We say that D is a **Reinhardt domain** if $(e^{i\theta}z, e^{i\phi}w) \in D$ for all $(z, w) \in D$ and all $\theta, \phi \in \mathbb{R}$.

If Ω is a rank 2 rotation domain, then the $\mathcal{G}$-action on Ω may be conjugated to the standard linear action on $\mathbb{C}^2$.

THEOREM. (Barrettt-Bedford-Dadok 1989) There are a Reinhardt domain $D \subset \mathbb{C}^2$, a linear map $L : (x, y) \mapsto (\alpha x, \beta y)$, $|\alpha| = |\beta| = 1$, and a biholomorphic map $\Phi : \Omega \to D$ such that $\Phi \circ f = L \circ \Phi$.

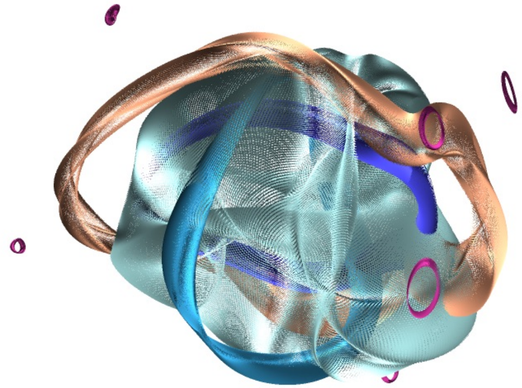
Volume preserving Hénon map

Volume preserving Hénon map can have rotation domains.

1. Rotation domain of rank 1 (not well understood).
2. Siegel disk $\cong$ Reinhardt domain $\supset \mathbb{D} \times \mathbb{D}$.
3. Exotic rotation domain $\cong \mathbb{A} \times \mathbb{D}$. (?)

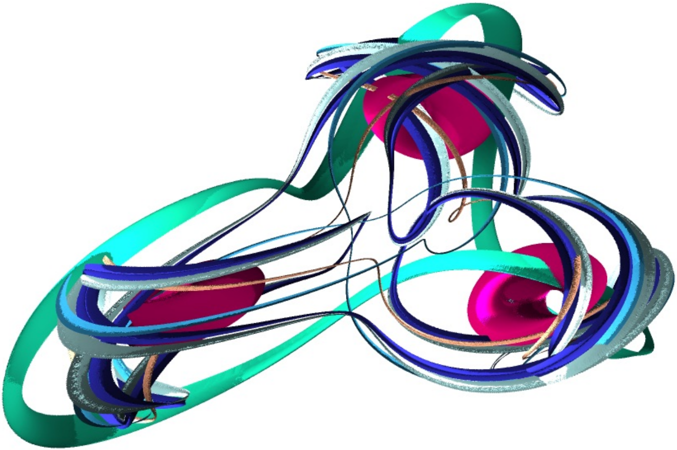
REM. Case 3 is numerically found. We don't have a proof.

Example: (case 2) Siegel disk

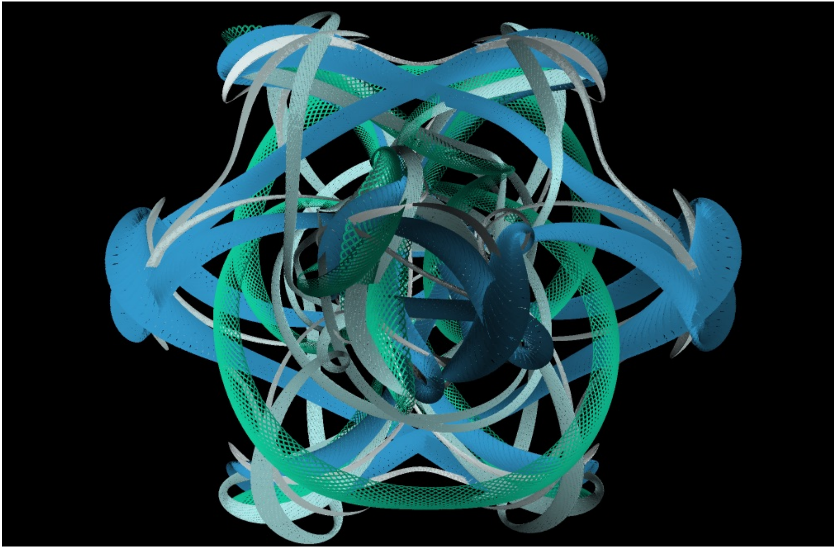


Example: (case 2) Siegel disk

Example: (case 3) exotic rotation domain (?)



Example: (case 3) exotic rotation domain (?)



Volume preserving surface automorphism

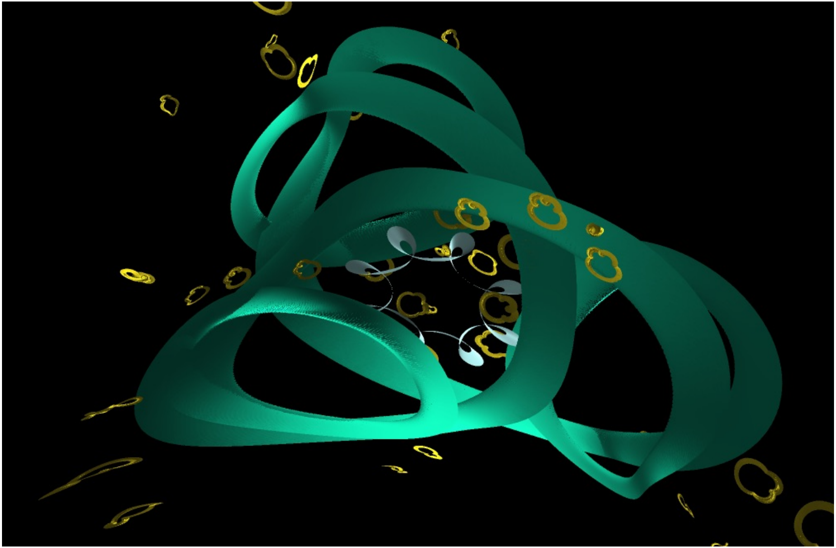
Volume preserving surface automorphisms can have various kinds of rotation domains.

1. Rotation domain of rank 1.
2. Siegel disk $\cong$ Reinhardt domain $\supset \mathbb{D} \times \mathbb{D}$.
3. Exotic rotation domain $\cong$ "$\mathbb{A} \times \mathbb{D}$". (?)
4. Super-exotic rotation domain $\cong$ "$\mathbb{P} \times \mathbb{D}$". (??)
5. Ultra-exotic rotation domain $\cong$ "$\mathbb{A} \times \mathbb{A}$". (???)

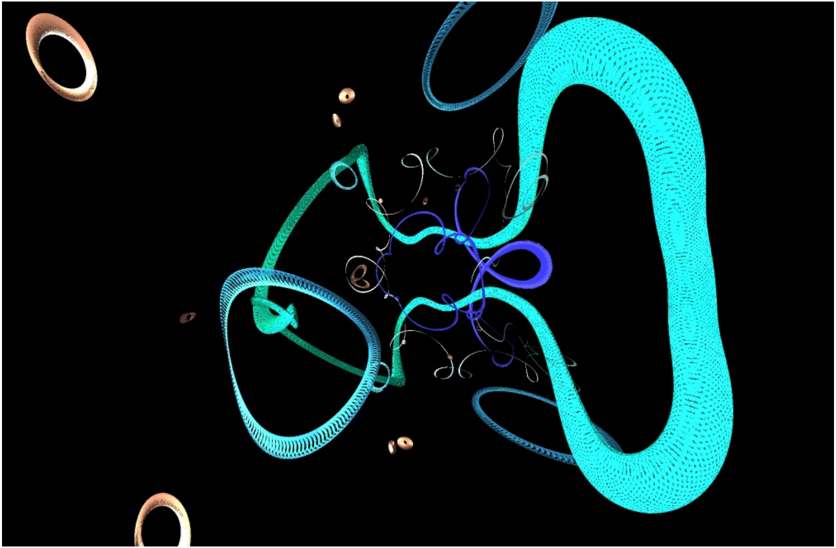
REM. In case 1, there are various types, not well understood.
REM. Case 3 is numerically found without proof.
REM. Case 4 is numerically observed.
REM. Case 5 is not found yet.

REM. "$\mathbb{P} \times \mathbb{D}$" is the normal disk bundle.

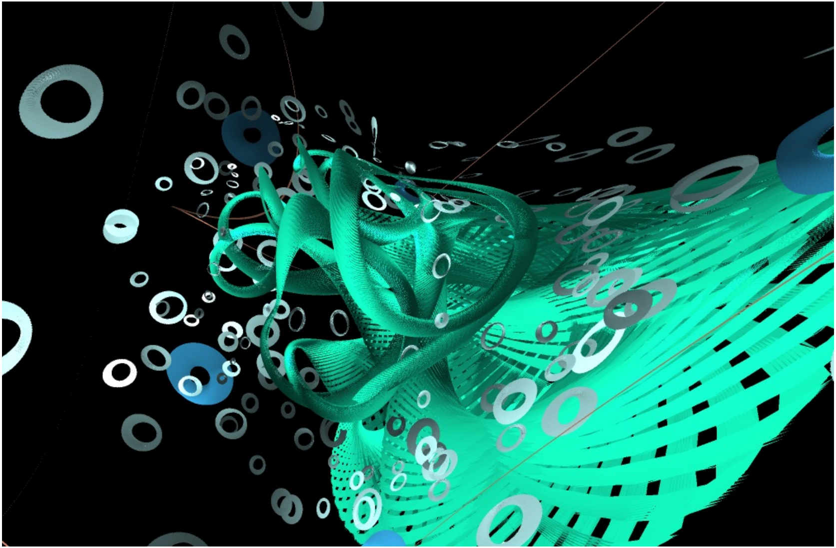
Example: (case 3) exotic rotation domain (?)



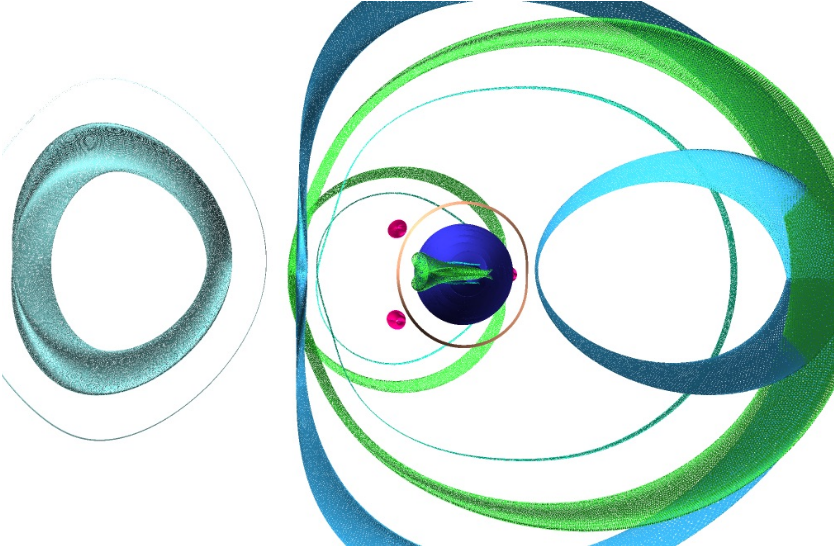
Example: (case 3) exotic rotation domain (?)



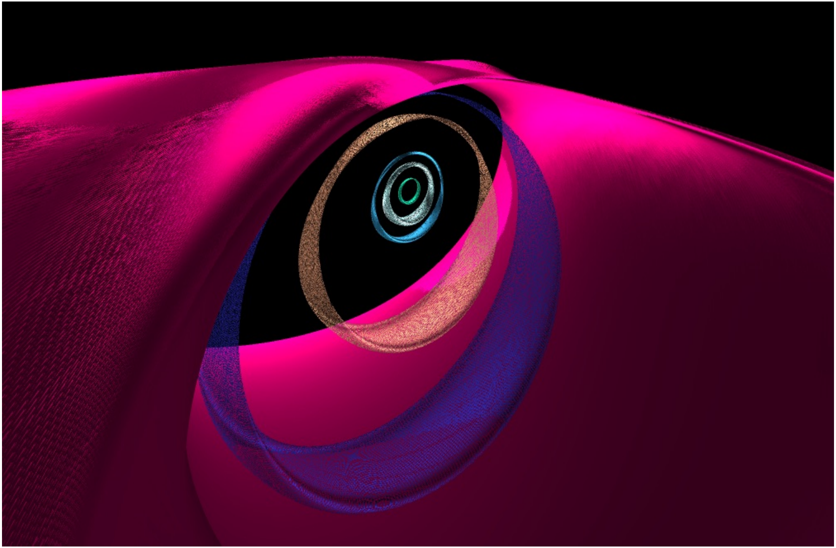
Example: (case 3) exotic rotation domain (?)



Example: (case 4) super exotic rotation domain (??)



Example: (case 4) super exotic rotation domain (??)

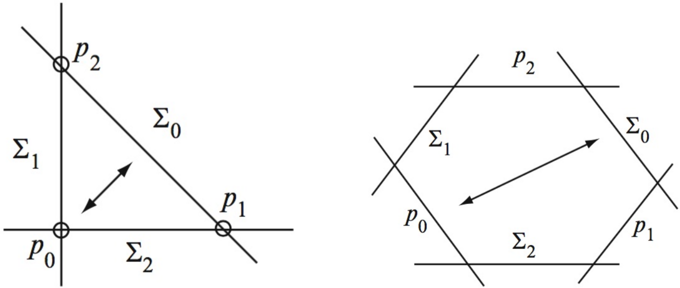


1. Cremona transformation

Cremona involution

Cremona involution $J : \mathbb{P}^2 \to \mathbb{P}^2$ is defined by

$$J[x : y : z] = [x^{-1} : y^{-1} : z^{-1}] = [yz : zx : xy].$$



For linear transformations $L_1, L_2 \in PGL(\mathbb{P}^2)$,

$$f = L_1 \circ J \circ L_2$$

is a birational transformation.

Cremona transformations with invariant cubic curve

A birational transformation $f : \mathbb{P}^2 \to \mathbb{P}^2$ is called a **Cremona transformation**.

A quadratic transformation $f : \mathbb{P}^2 \to \mathbb{P}^2$ always acts by blowing up three (indeterminacy) points $I(f) = \{p_1^+, p_2^+, p_3^+\}$ in $\mathbb{P}^2$ and blowing down the (exceptional) lines joining them. The inverse map f^{-1} is also a quadratic transformation and $I(f^{-1}) = \{p_1^-, p_2^-, p_3^-\}$ consists of the images of the three exceptional lines.

$$p_i^- = f(\ell(p_j^+, p_k^+)) \quad \text{for} \quad \{i, j, k\} = \{1, 2, 3\}.$$

Here, $\ell(p, q)$ denotes the line passing through p and q.

Orbit data

Suppose that for natural numbers n_1, n_2, n_3, and a permutation $\sigma : \{1,2,3\} \to \{1,2,3\}$, f satisfies

$$f^{n_i - 1}(p_i^-) = p_{\sigma(i)}^+, \qquad i = 1,2,3.$$

$$\ell(p_j^+, p_k^+) \to p_i^- \to f(p_i^-) \to \cdots \to p_{\sigma(i)}^+ \to \ell(p_{\sigma(j)}^-, p_{\sigma(k)}^-).$$

By blowing up in $n_1 + n_2 + n_3$ points

$$p_1^-, f(p_1^-), \cdots, f^{n_1-1}(p_1^-) = p_{\sigma(1)}^+,$$

$$p_2^-, f(p_2^-), \cdots, f^{n_2-1}(p_2^-) = p_{\sigma(2)}^+,$$

$$p_3^-, f(p_3^-), \cdots, f^{n_3-1}(p_3^-) = p_{\sigma(3)}^+,$$

f lifts to a surface automorphism.

2. Surface automorphism

Quadratic Cremona transformation

Let C be a cubic curve of one of the following :

(case C) a caspidal cubic curve
(case L) three lines passing through a point,
(case Q) a conic and a tangent line.

THEOREM. (Diller 2011) Let orbit data $n_1, n_2, n_3, \sigma \in \Sigma_3$ be given. Except for some specific cases, there exists an automorphism f for each root of $P(\lambda)$ that is not a root of unity, which realize the orbit data, with determinant λ.

Such f is unique up to conjugacy of linear transformation preserving C.

Uehara's formula of birational transformation

Uehara(2016) obtained an explicit formula for Cremona transformations with an invariant cuspidal cubic curve.

For $\lambda \in \mathbb{C}^\times$ and $a_1, a_2, a_3 \in \mathbb{C}$ with $a_1 + a_2 + a_3 \neq 0$,

$$\begin{cases} X_C &= \lambda \left(x + \frac{\nu_1}{3} + \frac{\nu_1 (y - x^3)}{\nu_1 x^2 - \nu_2 x + \nu_3 - y} \right) \\ Y_C &= \lambda^3 \left((x + \frac{\nu_1}{3})^3 + y - x^3 + \frac{\nu_1 (y - x^3)}{\nu_1 x^2 - \nu_2 x + \nu_3 - y} (\nu_1 (x + \frac{\nu_1}{3}) - \nu_2) \right) \end{cases}$$

where $\nu_1 = a_1 + a_2 + a_3$, $\nu_2 = a_1 a_2 + a_2 a_3 + a_3 a_1$, and $\nu_3 = a_1 a_2 a_3$.

Three lines passing through a point

In the case of three lines passing through a point, Uehara(2019) obtained an explicit formula. There are three cases for the permutation of three lines $\{x(x^2-1)=0\}$.

$$\begin{cases} X_{LI} &= \quad x - \frac{\nu_1 x(x^2-1)}{\nu_1 x^2+(a_2-a_3)x+2y-a_1} \\ Y_{LI} &= \lambda(y-\frac{\nu_1}{3}) + \frac{(\lambda-1)(x^2-1)(y-\frac{a_1}{2})}{\nu_1 x^2+(a_2-a_3)x+2y-a_1} \end{cases}$$

$$\begin{cases} X_{LT} &= \quad -x + \frac{\nu_1 x(x^2-1)}{\nu_1 x^2+(a_2-a_3)x+2y-a_1} \\ Y_{LT} &= \lambda(y-\frac{\nu_1}{3}) + \frac{(\lambda-1)(x^2-1)(y-\frac{a_1}{2})}{\nu_1 x^2+(a_2-a_3)x+2y-a_1} \end{cases}$$

$$\begin{cases} X_{LC} &= -\frac{x+1}{3x-1}\left(1+\frac{2\nu_1 x(x-1)}{(-\frac{a_1}{2}+a_2-2a_3)x^2+(a_2+2a_3)x+(3x-1)y+\frac{a_1}{2}}\right) \\ Y_{LC} &= \frac{\lambda(y-\frac{\nu_1}{3})(2y+\nu_1 x^2+(a_2-a_3)x-a_1)-\lambda\nu_1(x^2-1)(y-\frac{a_1}{2})}{(-\frac{a_1}{2}+a_2-2a_3)x^2+(a_2+2a_3)x+(3x-1)y+\frac{a_1}{2}} \end{cases}$$

Conic and a tangent line

Uehara(2019) obtained an explicit formula also in the case of conic and a tangent line. There are two cases for the permutation of components : conic $\{xy = 1\}$ and a tangent line $\{x = 0\}$.

$$\begin{cases} X_{QQ} &= \dfrac{a_1 a_2 x^2 - (a_1 + a_2)x + xy}{\lambda\left(\frac{\nu_1}{3} a_1 a_2 x^2 - \frac{2}{3}\nu_1 xy + (a_1 a_2 - \frac{\nu_1}{3}(a_1 + a_2))x + y + a_3\right)} \\ Y_{QQ} &= \dfrac{\lambda\left(\frac{\nu^2}{9} a_1 a_2 x^2 + (\frac{4}{9}\nu_1^2)xy + y^2 + (\frac{2}{3}\nu_1 a_1 a_2 - \frac{\nu_1}{9}(a_1 + a_2))x + (a_3 - \frac{\nu_1}{3})y - \frac{\nu_1}{3} a_3\right)}{\frac{\nu_1}{3} a_1 a_2 x^2 - \frac{2}{3}\nu_1 xy + (a_1 a_2 - \frac{\nu_1}{3}(a_1 + a_2))x + y + a_3} \end{cases}$$

$$\begin{cases} X_{QL} &= \dfrac{xy - 1}{-\lambda\left(\nu_3 x^2 + \frac{2}{3}\nu_1 xy - \nu_2 x - y + \frac{\nu_1}{3}\right)} \\ Y_{QL} &= \dfrac{-\lambda\left(\frac{\nu_1\nu_3}{3} x^2 + (\frac{4}{9}\nu_1^2 - \nu_2)xy + (\nu_3 - \frac{\nu_1\nu_2}{3})x - (y - \frac{\nu_1}{3})^2\right)}{\nu_3 x^2 + \frac{2}{3}\nu_1 xy - \nu_2 x - y + \frac{\nu_1}{3}} \end{cases}$$

Characteristic polynomial

Orbit data determines the characteristic polynomial $P(\lambda)$ of $f^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$.

Bedford and Kim [BK1] have computed explicitly for any orbit data n_1, n_2, n_3, σ.

$$P(\lambda) = \lambda^{1+\Sigma n_j} p(\frac{1}{\lambda}) + (-1)^{\mathrm{ord}\sigma} p(\lambda),$$

where

$$p(\lambda) = 1 - 2\lambda + \sum_{j=\sigma_j} \lambda^{1+n_j} + \sum_{j\neq\sigma_j} \lambda^{n_j}(1-\lambda).$$

Characteristic polynomial for orbit data n_1, n_2, n_3, σ

(case id) $\sigma = id$.

$$P(\lambda) = (\lambda - 2)\lambda^{n_1+n_2+n_3} + \lambda^{n_1+n_2} + \lambda^{n_2+n_3} + \lambda^{n_3+n_1}$$

$$- \lambda^{n_1+1} - \lambda^{n_2+1} - \lambda^{n_3+1} + 2\lambda - 1.$$

(case tr) σ is a transposition ($\sigma(1) = 2, \sigma(2) = 1, \sigma(3) = 3$).

$$P(\lambda) = (\lambda - 2)\lambda^{n_1+n_2+n_3} + \lambda^{n_1+n_2} + (\lambda - 1)(\lambda^{n_1+n_3} + \lambda^{n_2+n_3})$$

$$-(\lambda - 1)(\lambda^{n_1} + \lambda^{n_2}) + \lambda^{n_3+1} - 2\lambda + 1.$$

(case cy) σ is a cyclic permutation ($\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1$).

$$P(\lambda) = (\lambda - 2)\lambda^{n_1+n_2+n_3} + (\lambda - 1)(\lambda^{n_1+n_2} + \lambda^{n_2+n_3} + \lambda^{n_3+n_1})$$

$$+(\lambda - 1)(\lambda^{n_1} + \lambda^{n_2} + \lambda^{n_3}) + 2\lambda - 1.$$

Orbit data to parameters

From orbit data (n_1, n_2, n_3),σ, parameters a_1, a_2, a_3 are determined by the followings. To simplify the computations, the Picard coordinate of the fixed point is fixed to $\frac{1}{3}$.

$$a_1 + a_2 + a_3 = \frac{1}{\lambda} - 1.$$

$$a_{\sigma(i)} - \frac{1}{3} = \lambda^{n_i - 1}(b_i - \frac{1}{3}),$$
$$b_i - \frac{1}{3} = \lambda \cdot (a_i - \frac{1}{3}) + \lambda - 1,$$
$$\text{for} \quad i = 1, 2, 3.$$

These equations have a solution iff $P(\lambda) = 0$ (assuming λ is not a root of unity).

Picard coordinate of indeterminate points

(case id) $\sigma = id$.

$$a_i = -\frac{\lambda^{n_i-1}(\lambda-1)}{\lambda^{n_i}-1} + \frac{1}{3} \qquad (i = 1, 2, 3).$$

(case tr) $\sigma = (1, 2)$

$$a_i = -\frac{\lambda^{n_j-1}(\lambda^{n_i}+1)(\lambda-1)}{\lambda^{n_i+n_j}-1} + \frac{1}{3} \qquad ((i, j) = (1, 2), (2, 1)).$$

$$a_k = -\frac{\lambda^{n_k-1}(\lambda-1)}{\lambda^{n_k}-1} + \frac{1}{3} \qquad (k = 3).$$

(case cy) $\sigma = (1, 2, 3)$

$$a_i = -\frac{\lambda^{n_k-1}(\lambda^{n_j}(\lambda^{n_i}+1)+1)(\lambda-1)}{\lambda^{n_i+n_j+n_k}-1} + \frac{1}{3}$$

$$((i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)).$$

Indeterminate points

(case C) caspidal cubic curve $\{y = x^3\}$:

$$p_i^+ = (a_i, a_i^3), \quad i = 1, 2, 3,$$

$$p_i^- = (b_i, b_i^3), \quad i = 1, 2, 3.$$

(case LI) three lines $\{x(x^2 - 1) = 0\}$, each line mapped to itself.

$$p_1^+ = (0, -\frac{1}{2}a_1), \quad p_2^+ = (1, a_2), \quad p_3^+ = (-1, a_3).$$

$$p_1^- = (0, -\frac{1}{2}b_1), \quad p_2^- = (1, b_2), \quad p_3^- = (-1, b_3).$$

Indeterminate points

(case LT) three lines $\{x(x^2-1)=0\}$, $\{x=\pm 1\}$ mapped to each other.

$$p_1^+ = (0, -\frac{1}{2}a_1), \quad p_2^+ = (1, a_2), \quad p_3^+ = (-1, a_3).$$

$$p_1^- = (0, -\frac{1}{2}b_1), \quad p_2^- = (1, b_3), \quad p_3^- = (-1, b_2).$$

(case LC) three lines $\{x(x^2-1)=0\}$, mapped cyclically.

$$p_1^+ = (0, -\frac{1}{2}a_1), \quad p_2^+ = (1, a_2), \quad p_3^+ = (-1, a_3).$$

$$p_1^- = (0, -\frac{1}{2}b_3), \quad p_2^- = (1, b_1), \quad p_3^- = (-1, b_2).$$

Indeterminate points

(case QQ) conic $\{xy = 1\}$ and a tangent line $\{x = 0\}$, each mapped to itself.

$$p_1^+ = (a_1, a_1^{-1}), \quad p_2^+ = (a_2, a_2^{-1}), \quad p_3^+ = (0, -a_3).$$

$$p_1^- = (b_1, b_1^{-1}), \quad p_2^- = (b_2, b_2^{-1}), \quad p_3^- = (0, -b_3).$$

(case QL) conic $\{xy = 1\}$ and a tangent line $\{x = 0\}$, mapped to each other.

$$p_1^+ = (a_1, a_1^{-1}), \quad p_2^+ = (a_2, a_2^{-1}), \quad p_3^+ = (a_3, a_3^{-1}).$$

$$p_1^- = (b_1, b_1^{-1}), \quad p_2^- = (b_2, b_2^{-1}), \quad p_3^- = (b_3, b_3^{-1}).$$

Determinant and Eigen meromorphic form

In our cases of cubic curve, each component of regular part is isomorphic to $\mathbb{C}$. Automorphism $f : S \to S$ restricted to the invariant cubic curve is an "affine" map. The "derivative" $D(f|_C)$ is called the **determinant** of f.

Meromorphic $(1,1)$-form η with pole along the invariant curve C is mapped to a scaler multiple of η.

$$f^*\eta = \lambda(f)\eta.$$

THEOREM

$$D(f|_C) = \lambda(f).$$

$\lambda(f)$ is also called the **determinant** of f. If $p \in S \setminus C$ is a periodic point of period k, then

$$\det Df_p^k = \lambda(f)^k.$$

3. Invariant line

Attracting invariant line

In the dissipative case, $(0 < \lambda < 1)$, the determinant with respect to the two-form η is equal to λ.

If there is an invariant curve, disjoint from the cubic curve, and the intrinsic dynamics in the extra invariant curve is neutral, then this extra curve must be an attractor, since η is regular in $S \setminus C$.

According to [DJS], invariant curve must be a tree of genus 0, if it is not contained in the cubic curve.

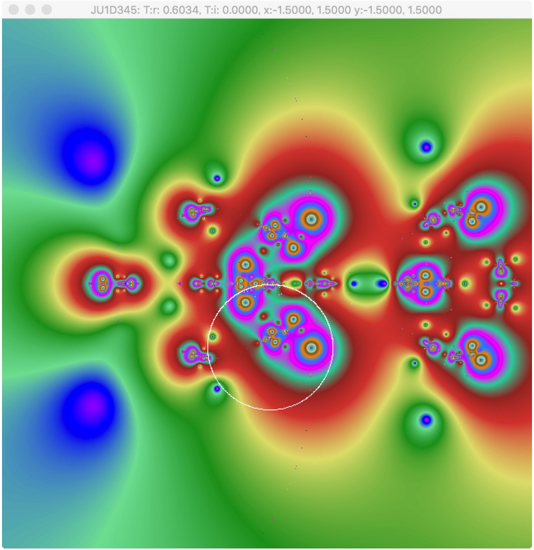
Invariant curve

THEOREM. (Diller-Jackson-Sommese 2007)
Let $f : S \to S$ be an algebraically stable map with $\lambda_f > 1$, and suppose that $C = f(C)$ is a connected curve with $g(C) = 1$. Then by contracting finitely many curves, one may further arrange the following.

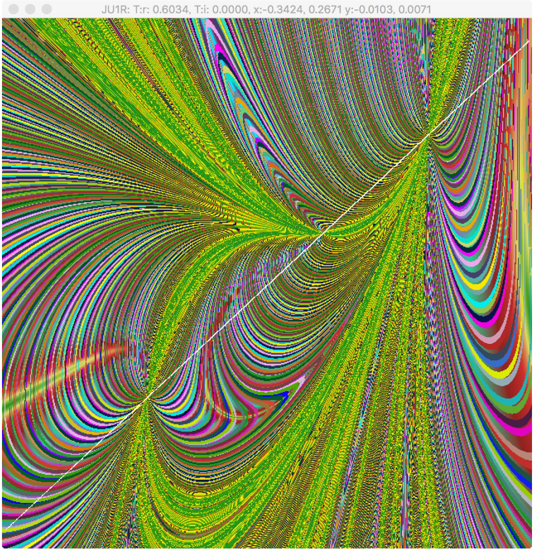
(1) $C \sim -K_S$ is an anticanonical divisor.
(2) $I(f^n) \subset C$ for every $n \in \mathbb{Z}$.
(3) Any connected curve strictly contained in C has genus zero.
(4) If W is a connected f-invariant curve not completely contained in C, then W has genus zero, is disjoint from C, and is equal to a tree of smooth rational curves, each with self-intersection -2.

REM. Here λ_f means the first dynamical degree of f.

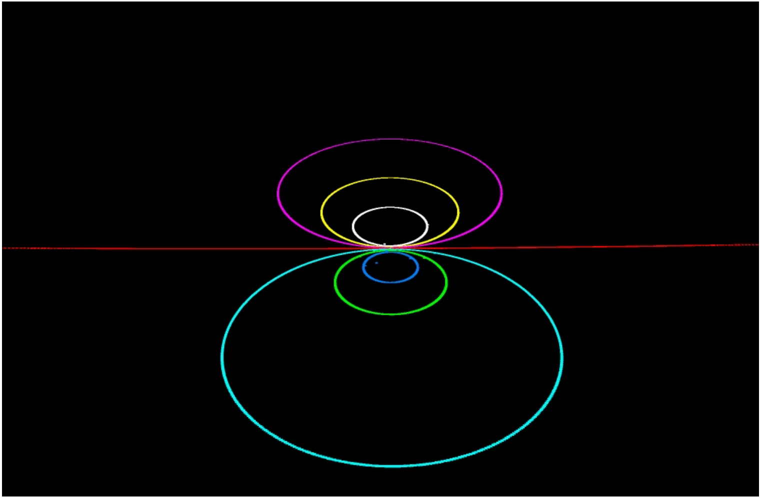
Example : (case C) orbit data (3,4,5), id, diagonal slice



Attracting invariant line with irrational(?) rotation, real slice



Attracting invariant line with irrational(?) rotation



Invariant line (necessary condition)

If there exists an invariant line disjoint from the anticanonical cubic curve, it passes through three points to be blown up, one of which is an indeterminate point of the base birational map.

The sum of the Picard coordinates of the three blowup points vanishes.

The intersection of the invariant line and a component of the anticanonical curve, counted as points in $\mathbb{P}^2$, must be equal to the degree of the component.

This line necessarily contains two fixed points.
(Our automorphism has four fixed points.)

Following cases are inadequate.

(case LI) three lines passing through a point, each line mapped to itself $\Rightarrow$ antocanonical cubic curve contains four fixed points.

(case LT) three lines passing through a point, two of them are swapped $\Rightarrow$ automorphism cannot have an invariant line intersecting the three lines.

(case QQ) conic and a tangent line, each component mapped itself $\Rightarrow$ automorphism cannot have an invariant line intersecting two components.

Invariant line (sufficient condition)

Suppose that the anticanonical cubic curve of our surface automorphism is one of the followings.

(case C) caspidal cubic curve

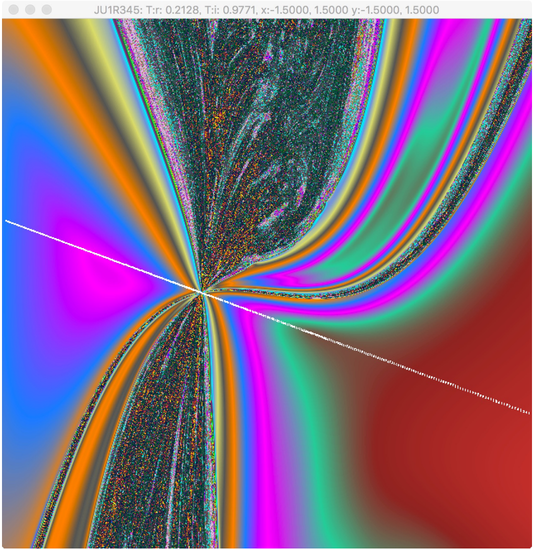
(case LC) three lines passing through a point permuted cyclically

(case QL) conic and a tangent line permuted by the automorphism

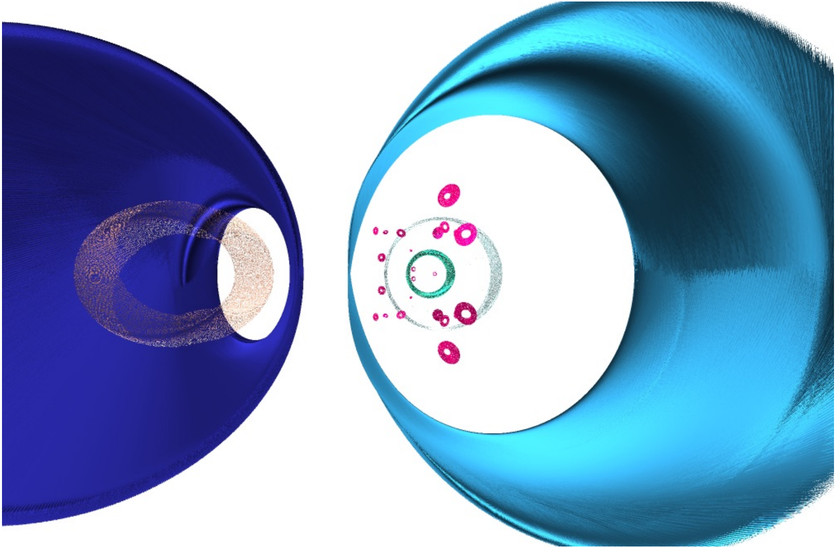
THEOREM. In the case of orbit data $(3, n_2, n_3)$ with $\sigma(1) = 1$, the surface automorphism has an invariant line passing through three blowup points p_1^+, p_1^-, and $f(p_1^-)$.

REM. In this case, the self-intersection of the strict transform of this invariant line is -2.

(Cuspidal) Orbit data (3,4,5),*id*, real slice



Example: super exotic rotation domain (?)



PROOF. Let $p_1^+ = (a_1, a_1^3)$, $p_1^- = (b_1, b_1^3)$, and $f(p_1^-) = (c_1, c_1^3)$. Then,

$$a_1 = -\frac{d^2(d-1)}{d^3-1} + \frac{1}{3}, \quad b_1 = -\frac{d-1}{d^3-1} + \frac{1}{3}, \quad c_1 = -\frac{d(d-1)}{d^3-1} + \frac{1}{3}.$$

Immediately we see that $a_1 + b_1 + c_1 = 0$. Hence three points $p_1^+, p_1^-, f(p_1^-)$ are on a line. Let L denote this line. As L passes through the indeterminate point p_1^+, its image $f(L)$ is a line. Since $f(L)$ passes through $p_1^+ = f^2(p_1^-)$ and $f(p_1^-)$, it coincides with L.

In our case, L is disjoint from the invariant cubic curve.

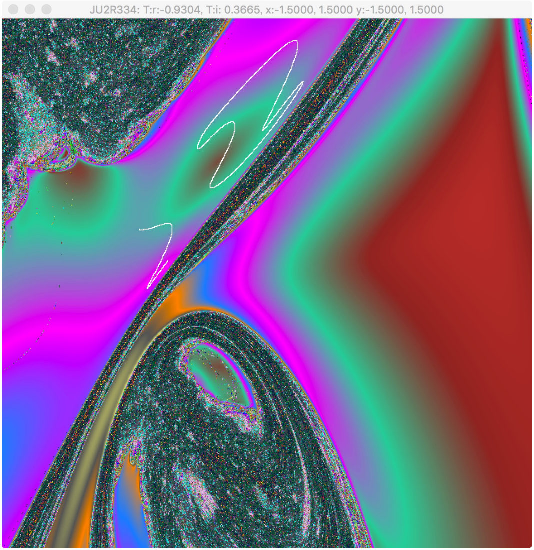
Invariant line intersecting the cubic curve will be treated later.

Line of periodic points

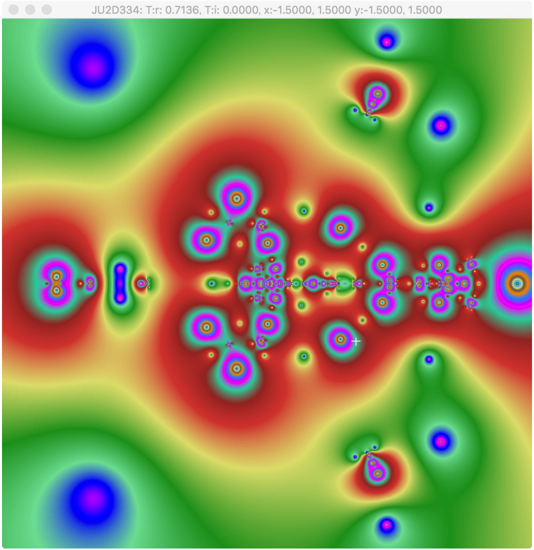
THEOREM. In the case of orbit data $(3,3,n)$, $\sigma = id.$ or $\sigma = (1,2)$, with $n \geq 4$, the surface automorphism f has an invariant line of period-three periodic points.

PROOF. Similarly as in the case of $(3, n_2, n_3)$, $\sigma(1) = 1$, f has an invariant line, say L, passing through points p_1^+, p_1^-, and $f(p_1^-)$. In this case we have $p_2^+ = p_1^+$ and $p_2^- = p_1^-$. The image $f(p_1^+)$ is the line passing through p_2^- and p_3^-. The point in the strict transform of L must be mapped to a point in the same line. So p_1^+ is mapped to p_2^-. This shows that the Möbius transformation $f|_L$ has a periodic point of period 3. Consequently, all the points of L, except for two fixed points, are periodic points of period 3.

Orbit data (3,3,4), id, $\mathbb{P} \times \mathbb{D}(?)$, t_3, rank 1.



Orbit data (3,3,4), id, $\mathbb{P}$, t_r, attractor, diagonal slice



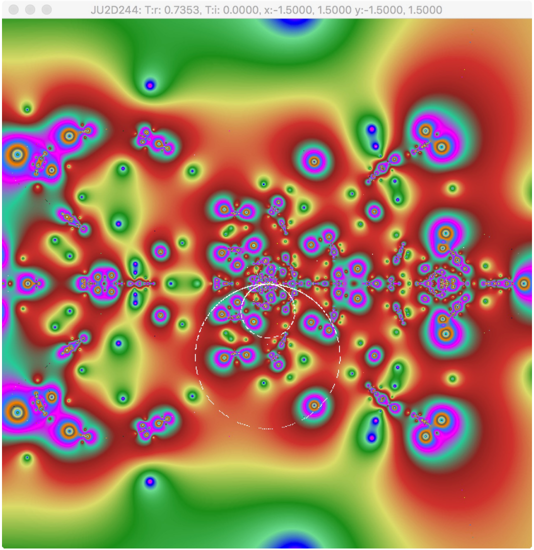
4. Invariant conic

Attracting quadratic curve

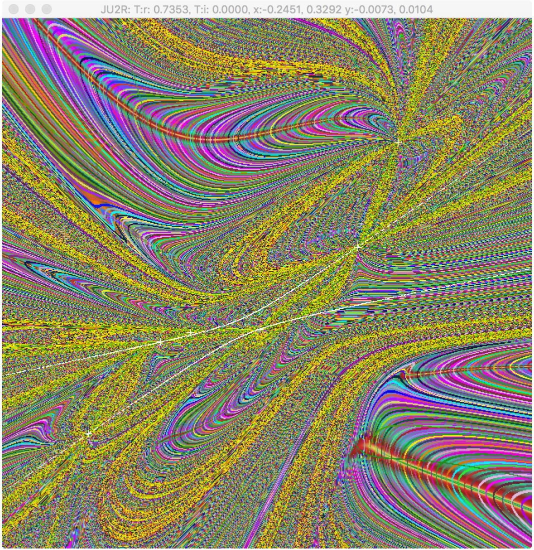
There are cases where the attractor is an invariant quadratic curve, disjoint from the cubic curve.

Following pictures are in the case of caspidal cubic curve.
Invariant quadratic curves exist in other cases, too.

Orbit data (2,4,4), transposition (1,2), diagonal slice



Attracting quadratic curve with irrational(?) rotation, real slice



Invariant conic (necessary condition)

If there is an invariant quadratic curve, disjoint from the anticanonical cubic curve, it must pass trough 6 points to be blown up, two of which are indeterminate points of the base birational map.

The sum of the Picard coordinates of these 6 blowup points vanish.

The number of blowup points in each component of the anticanonical curve must be 2 times the degree of the component.

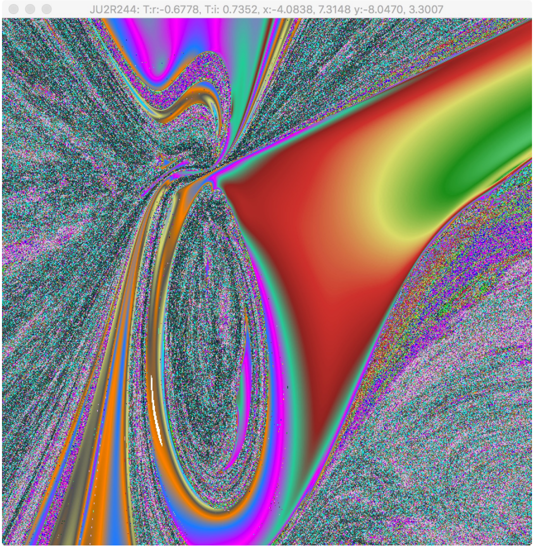
The invariant quadratic curve contains two fixed points.

Invariant quadratic curve

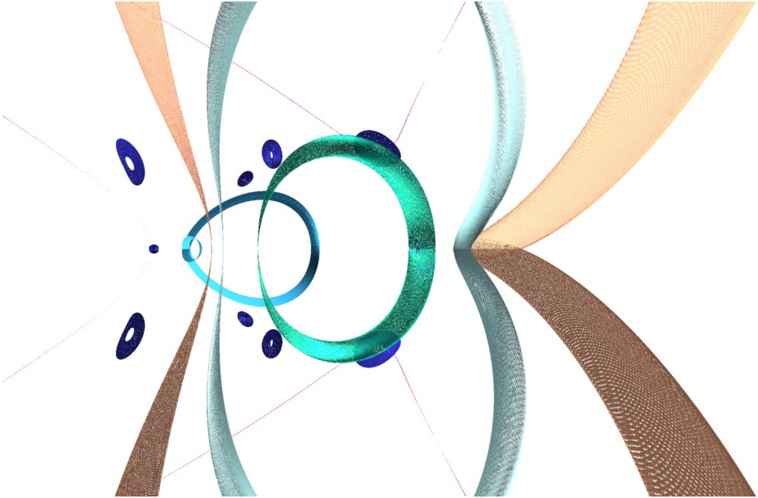
Suppose the anticanonical curbic curve of surface automorphism is a caspidal cubic curve.

THEOREM. In the case of orbit data $(2, 4, n)$ with transposition (1,2), the surface automorphism has an invariant quadratic curve passing through six blowup points $p_1^+, p_1^-, p_2^+, p_2^-, f(p_2^-), f^2(p_2^-)$.

Caspidal, orbit data (2,4,4),tr, t3, real slice



Quadratic curve in rotation of rank 2(?)



PROOF. Quadratic curve is mapped to a quadratic curve by Cremona transformation if the quadratic curve passes through exactly two indeterminate points. If there exists a quadratic curve passing through these 6 points, its image by f is a quadratic curve, since p_1^+ and p_2^+ are indeterminate points. Points $p_1^+ = f(p_1^-)$, $p_2^+ = f^3(p_2^-)$, $f(p_2^-)$, $f^2(p_2^-)$ are in the image quadratic curve. The line passing through p_1^+ and p_3^+, which contains another point in the quadratic curve, is mapped to p_2^-. Hence p_2^- is in the image of the quadratic curve. Similarly, p_1^- is in the image, too. The image quadratic curve must be the same quadratic curve, since 6 points determine the quadratic curve.

So, we only need to prove the existence of a quadratic curve passing through the 6 points.

Let

$$a_1 = -\frac{d(d^4+1)(d-1)}{d^6-1} + \frac{1}{3}, \quad a_2 = -\frac{d^3(d^2+1)(d-1)}{d^6-1} + \frac{1}{3},$$

$$b_1 = -\frac{(d^2+1)(d-1)}{d^6-1} + \frac{1}{3}, \quad b_2 = -\frac{(d^4+1)(d-1)}{d^6-1} + \frac{1}{3},$$

$$c_1 = -\frac{d(d^2+1)(d-1)}{d^6-1} + \frac{1}{3}, \quad c_2 = -\frac{d^2(d^2+1)(d-1)}{d^6-1} + \frac{1}{3}.$$

These are the x-coordinates of the blowup points.

$$\begin{aligned} p_1^+ &= (a_1, a_1^3), \quad & p_1^- &= (b_1, b_1^3), \\ p_2^+ &= (a_2, a_2^3), \quad & p_2^- &= (b_2, b_2^3), \\ f(p_2^-) &= (c_1, c_1^3), \quad & f^2(p_2^-) &= (c_2, c_2^3). \end{aligned}$$

Immediately, we see that

$$a_1 + a_2 + b_1 + b_2 + c_1 + c_2 = 0.$$

Consider polynomial of degree 6 :

$$P(z) = (z - a_1)(z - a_2)(z - b_1)(z - b_2)(z - c_1)(z - c_2)$$
$$= z^6 + A_4 z^4 + A_3 z^3 + A_2 z^2 + A_1 z + A_0.$$

Let $Q(x, y)$ be a quadratic polynomial defined by

$$Q(x, y) = y^2 + A_4 xy + A_3 y + A_2 x^2 + A_1 x + A_0.$$

The 6 points $p_1^+, p_1^-, p_2^+, p_2^-, f(p_2^-), f^2(p_2^-)$ satisfy $Q(x, y) = 0$. Hence the quadratic curve $Q(x, y) = 0$ passes through these 6 points.

We conclude that quadratic curve $\{Q(x, y) = 0\}$ is invariant under f.

REM. The strict transform of this quadratic curve has self-intersection -2.

Other cases of invariant conic

There are other cases.

THEOREM In the case of three lines passing through a point, permuted by transposition $\eta = (1,2)$, with orbit data $(4, n, 2)$,$\sigma = id.$, the surface automorphism has an invariant quadratic curve passing through six blowup points $p_1^-, f(p_1^-), f^2(p_1^-), p_1^+, p_3^-, p_3^+$.

THEOREM In the case of conic and a tangent line, permuted by transposition $\eta = (1,2)$, with orbit data $(1, 5, n)$,$\sigma = (1,2)$., the surface automorphism has an invariant quadratic curve passing through six blowup points $p_1^+, p_2^-, f(p_2^-), f^2(p_2^-), f^3(p_2^-), p_2^+$.

Still other cases of invariant conic

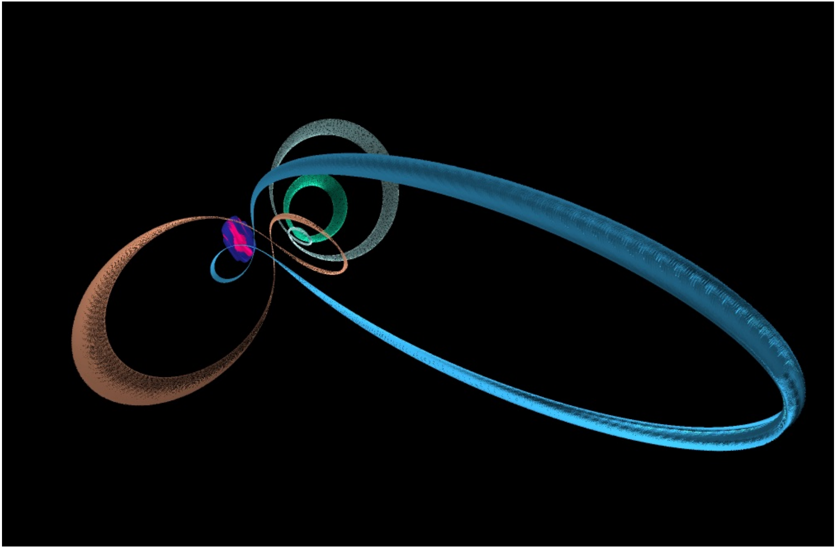
THEOREM In the case of caspidal anticanonical curve, with orbit data $(4, 2, n)$,$\sigma = id.$, the surface automorphism has an invariant quadratic curve passing through six blowup points $p_1^-, f(p_1^-), f^2(p_1^-), p_1^+, p_2^-, p_2^+$.

In this case the quadratic curve consists of two lines passing through a fixed point and permuted by the automorphism.

THEOREM In the case of caspidal cubic curve, with orbit data $(1, 5, n)$,$\sigma = (1, 2).$, the surface automorphism has an invariant quadratic curve passing through six blowup points $p_1^+, p_2^-, f(p_2^-), f^2(p_2^-), f^3(p_2^-), p_2^+$.

In this case the quadratic curve consists of two lines passing through a fixed point and permuted by the automorphism.

Quadratic curve in a rotation domain of rank 2 (?)

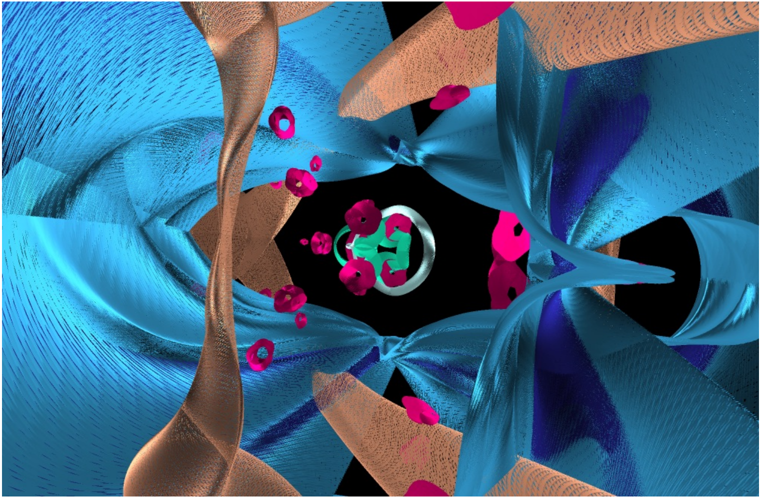


Invariant curves of self intersectin -1

There are many cases of surface automorphisms having an invariant line, an invariant quadratic curve or both of them with self intersection -1, which can be blown down.

There is a case where three disjoint lines permuted cyclically, disjoint from the anticanonical cubic curve in the case of a conic with a tangent line, each component mapped to itself, with orbit data $(2, 3, 7), \sigma = id$.

Thank you !



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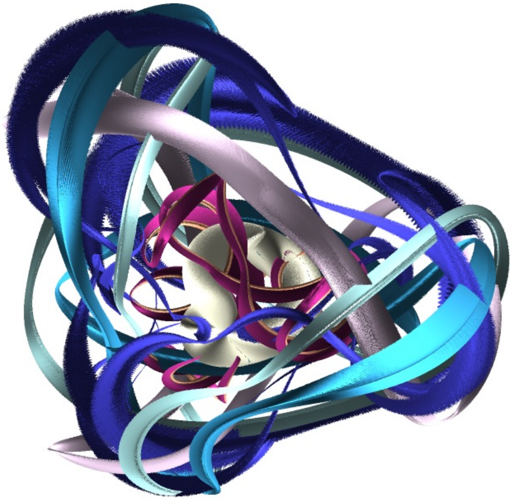
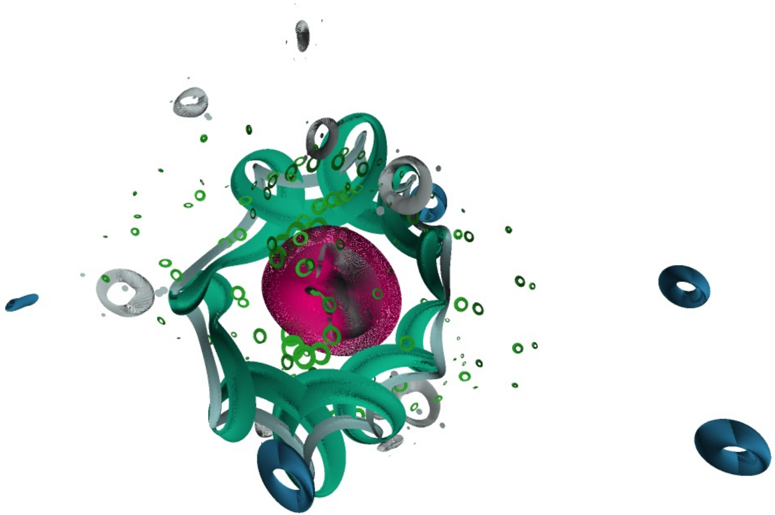
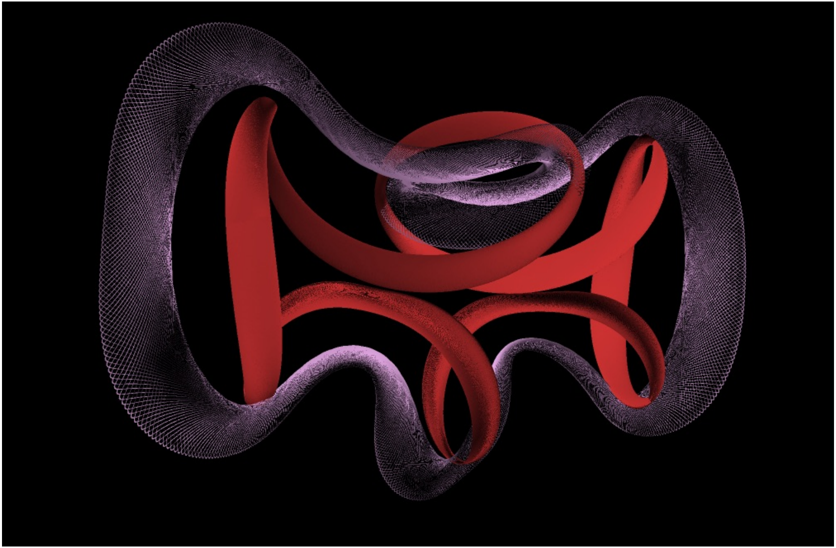
Example: (case 3) exotic rotation domain (?)



Example: (case 3) exotic rotation domain (?)



Example: (case 3) exotic rotation domain (?)



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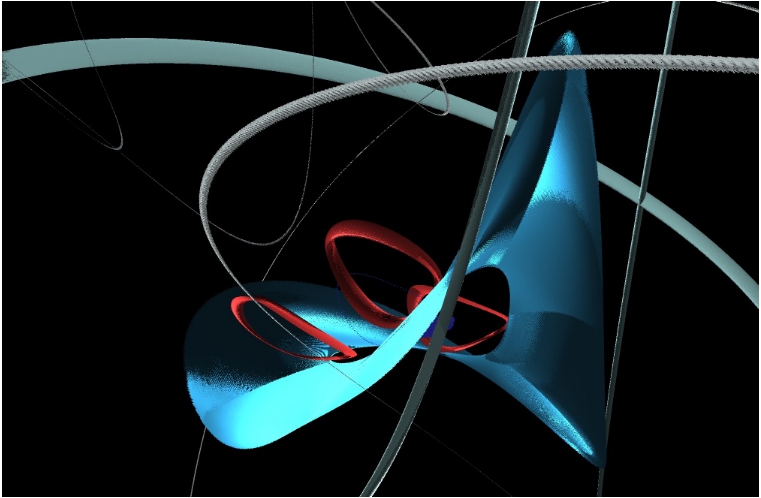
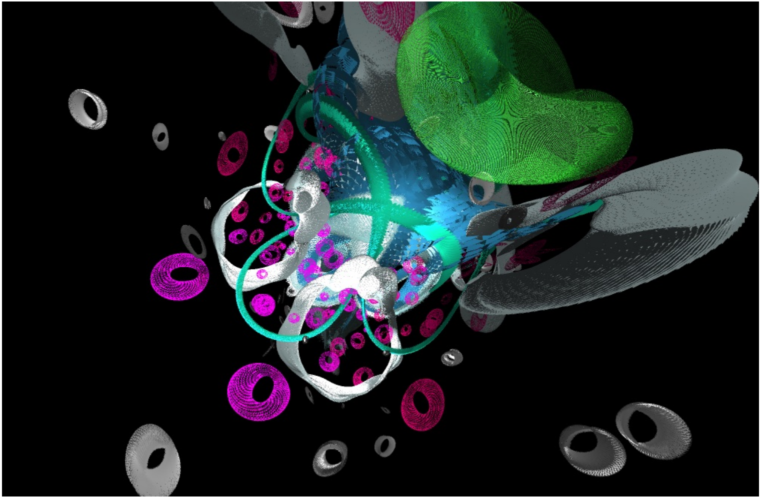


CSPIt237t2



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