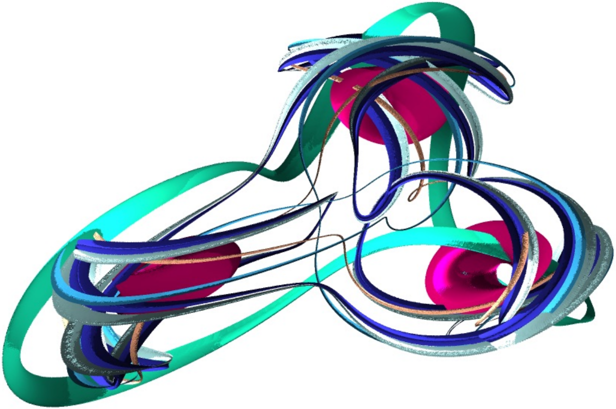
Exotic Rotation Domains in Complex Hénon Dynamics



Shigehiro Ushiki Mar. 27, 2023

Abstract

Abstract : Fatou component of complex dynamical system is called a rotation domain if the dynamics in the set is quasiperiodic. The closure of the orbit of almost any initial point is a circle or a torus. We say a rotation domain is exotic if the domain is not simply connected. In this talk, we explain how to observe such object numerically.

Contents

0. Introduction

Volume preserving complex Hénon map

In this note, we consider complex Hénon map $h : \mathbb{C}^2 \to \mathbb{C}^2$, defined by

$$h(x, y) = (x^2 + c - ay, x).$$

Its differential map is given by

$$dh = \begin{pmatrix} 2x & -a \\ 1 & 0 \end{pmatrix}.$$

The (complex) determinant is given by

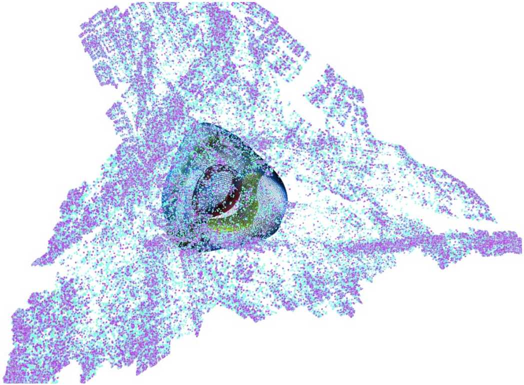
$$\det(dh) = a,$$

And the volume in $\mathbb{C}^2$ is multiplied by $|a|^2$,

$$\mathrm{vol}(h(U)) = |a|^2 \mathrm{vol}(U).$$

Hénon map h is said **volume preserving** if $|a| = 1$.

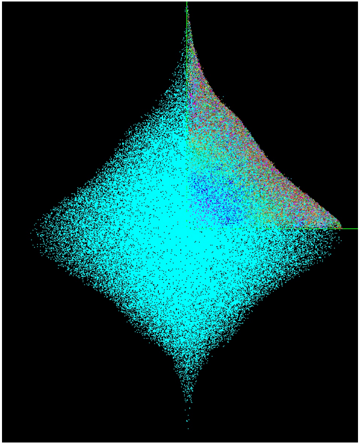
Invariant sets for volume preserving Hénon map



In this picture, many points in the Julia set of a volume preserving complex Hénon map, with several bounded orbits are plotted. These bounded orbits seem to belong to a Siegel disk.

Siegel Reinhardt domain

Siegel disk can be mapped holomorphically to its linear model, a Reinhardt domain.



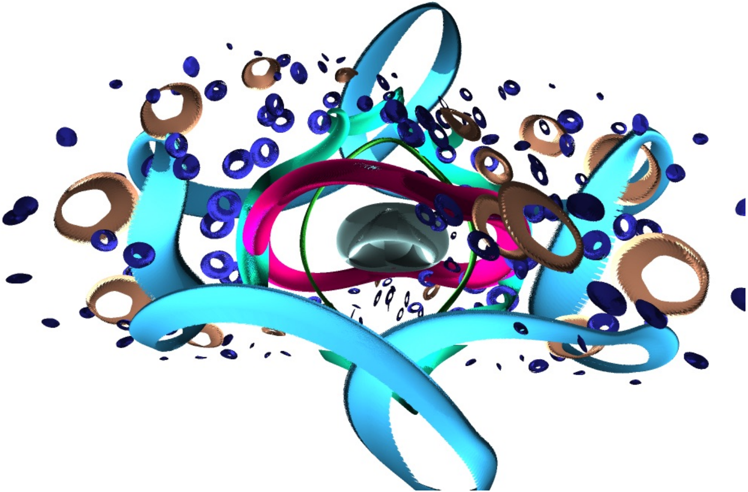
Exotic rotation domain

Siegel disk has a fixed (or periodic) point of the dynamical system.

Rotation domain without periodic point is called an **exotic rotation domain**.

We try to explain how to observe such domains, by means of numerical computation.

Orbits in rotation domains



1. Rotation domain

Fatou set (volume preserving case)

Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a volume preserving complex Hénon map.

A point $p \in \mathbb{C}^2$ is a point of the **forward Fatou set** F_f^+ if there exists an open neighborhood U of p on which the sequence $\{f^n\}_{n \in \mathbb{N}}$ forms a normal family of holomorphic mappings from U to $\mathbb{C}^2$.

Define the **backward Fatou set** F_f^- and the **Fatou set** F_f by

$$F_f^- = F_{f^{-1}}^+, \qquad F_f = F_f^+ \cap F_f^- .$$

REMARK. It is known ([FM], 1989) that $F_f = F_f^+ = F_f^-$.

Rotation domain

Suppose Ω is a connected component of Fatou set F_f with $f(\Omega) = \Omega$. Define the set of all limits of convergent subsequences $\mathcal{G}$ by

$$\mathcal{G} = \left\{ g = \lim_{n_j \to \infty} f^{n_j} : \Omega \to \overline{\Omega} \right\}.$$

If $g = \lim_{n_j \to \infty} f^{n_j}$ is such a limit, then g must preserve volume, and thus it is locally invertible. It follows that $g : \Omega \to \Omega$.

It is known that $\mathcal{G}$ is a compact Lie group, by a theorem of H. Cartan. The connected component $\mathcal{G}_0$ of the identity must be a (real) torus.

Rank of a rotation domain

In the volume preserving Hénon map case,

THEOREM. (Bedford-Smilie, 1991).
$\mathcal{G}_0$ is isomorphic to $\mathbb{T}^\rho$ with $\rho = 1$ or 2.

Such a domain is called a **rotation domain**, and we refer to ρ as the **rank** of the rotation domain.

Reinhardt domain

Let $R \subset \mathbb{C}^2$ be a connected open set. We say that R is a **Reinhardt domain** if $(e^{i\theta}z, e^{i\phi}w) \in R$ for all $(z, w) \in R$ and all $\theta, \phi \in \mathbb{R}$.

If Ω is a rank 2 rotation domain, then the $\mathcal{G}$-action on Ω may be conjugated to the standard linear action on $\mathbb{C}^2$.

THEOREM. (Barret-Bedford-Dadok, 1989) There are a Reinhardt domain $R \subset \mathbb{C}^2$, a linear map $L : (x, y) \mapsto (\alpha x, \beta y)$, $|\alpha| = |\beta| = 1$, and a biholomorphic map $\psi : \Omega \to R$ such that $\psi \circ f = L \circ \psi$.

Siegel's theorem (n-dimensional case)

$(\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$ is said to satisfy a **multiplicative diophantian condition** if there are positive constants C and ν, such that

$$|\lambda_1^{k_1} \cdots \lambda_n^{k_n} - \lambda_s| \geq C(k_1 + \cdots k_n)^{-\nu}$$

for $s = 1, \cdots, n$, and $k_1, \cdots, k_n \geq 0$, with $k_1 + \cdots + k_n \geq 2$.

Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be holomorphic near a fixed point $O \in \mathbb{C}^n$, and let $\lambda_1, \cdots, \lambda_n$ denote the eigenvalues of df_O.

THEOREM. (Siegel, 1942)
If these eigenvalues satisfy a multiplicative diophantian condition, then f is holomorphically linearizable near the fixed point.

REMARK. Siegel's theorem was for $n = 1$. According to V. Arnold, higher dimensional cases was a "folklore theorem" for thirty years when he published a proof in textbook ([A2],1978). A proof of this theorem was also included in Sternberg ([St], 1961). There are sharper linearizabilty conditions. See Bryuno ([B],1965), Yoccoz ([Y],1984).

Siegel disk

Suppose $|\lambda_s| = 1, \quad s = 1, \cdots, n$, and a multiplicative diophantian condition or the Bryuno condition holds.

The maximal linearizable neighborhood of the fixed point is called a **Siegel disk**.

The dynamics in the Siegel disk is holomorphically conjugate to the linear part of f at the fixed point.

The image, by the conjugacy, of the Siegel disk is invariant under the linear map df_O.

This linearising map is called a **Siegel linearizer**.

Siegel uniformizer

Open neighborhood of the origin invariant under diagonal linear map of eigenvalues $\lambda_s,\ |\lambda_s| = 1,\ s = 1, \cdots, n$ is a Reinhardt domain.

The inverse map from the image domain to Siegel disk is holomorphic.

Our Reinhardt domain must be a maximal domain of holomorphy of this inverse map.

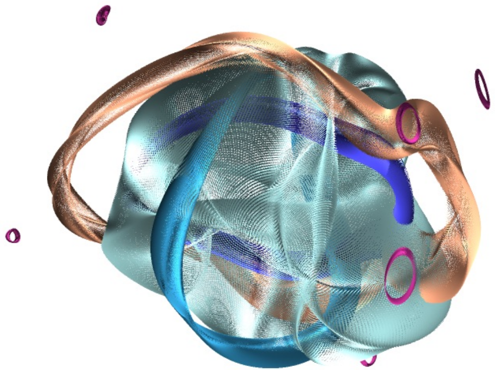
It is a logarithmically convex complete Reinhardt domain.

Let Ω be a Siegel disk, and let $\psi : \Omega \to R$ be the Siegel linearizer onto Reihardt domain R.

Its inverse map $\varphi = \psi^{-1} : R \to \Omega$ is called a **Siegel uniformizer**.

Some orbits in a Siegel disk

Some orbits in a Siegel disk



Complete Reinhardt domain (2-dimensional case)

A Reinhardt domain $R \subset \mathbb{C}^2$ is said **complete** if $(x, y) \in R$ for all $(z, w) \in R$ and all $|x| < |z|, |y| < |w|$.

The first quadrant part B of the real slice $R \cap \mathbb{R}^2$ is called the **base** of R.

Siegel disk has a fixed (or periodic) point.

The Reinhardt domain isomorphic to Siegel disk is a complete and logarythmically convex Reinhardt domain.

A rotation domain without periodic point will be called an **exotic rotation domain**.

Reinhardt domain for exotic rotation domain is not complete.

Exotic rotation domain ?

2. Reversible dynamics

Reversible maps

Let $\tau : \mathbb{C}^2 \to \mathbb{C}^2$ be the involution defined by $\tau(x, y) = (\bar{y}, \bar{x})$, called **swap conjugacy**.

We say that a map f is τ-**reversible** if $\tau \circ f \circ \tau = f^{-1}$.

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{f} & \mathbb{C}^2 \\ \updownarrow \tau & & \updownarrow \tau \\ \mathbb{C}^2 & \xleftarrow{f} & \mathbb{C}^2. \end{array}$$

Conjugate diagonal $\Delta' = \{(x, \bar{x}) | x \in \mathbb{C}\}$ is the set of fixed points of involution τ.

Reversible Hénon map

THEOREM. A (quadratic) Hénon map is τ-reversible if and only if it has the form

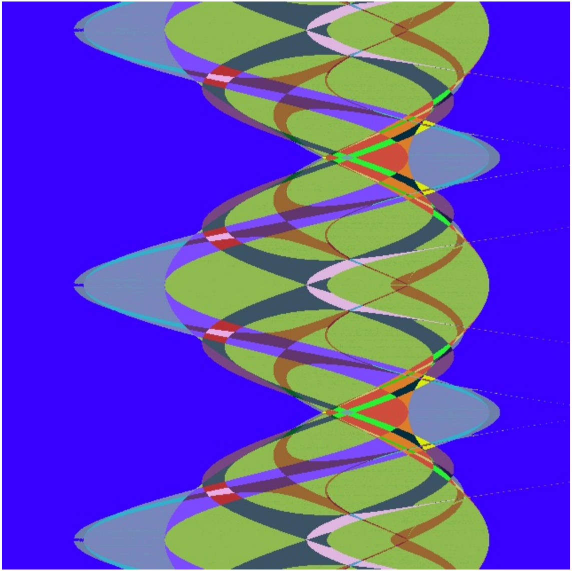
$$f(x,y) = (\beta(x^2+\alpha) - \beta^2 y, x)$$

where $\alpha \in \mathbb{R}$ and $|\beta| = 1$.

In fact, set $X = \beta(x^2+\alpha) - \beta^2 y$, $Y = x$, then we have $x = Y$, $y = \bar{\beta}(Y^2+\alpha) - \bar{\beta}^2 X$.

Hénon map $h(z,w) = (z^2 + c - aw, z)$ is conjugate to f by change of coordinates $(z,w) = (\beta x, \beta y)$ with $a = \beta^2, c = \alpha\beta^2$.

Parameter space of τ-reversible Hénon map, $(\alpha, \arg(\beta))$



Swap conjugacy τ and complex conjugacy σ

Let $\sigma : \mathbb{C}^2 \to \mathbb{C}^2$ be the anti-holomorphic involution defined by the complex conjugation $\sigma(p, q) = (\bar{p}, \bar{q})$.

Let $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ be a holomorphic isomorphism defined by

$$\phi(p, q) = (p + iq, p - iq).$$

PROPOSITION. Involutions τ and σ are conjugate by ϕ.

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{\phi} & \mathbb{C}^2 \\ \updownarrow \sigma & & \updownarrow \tau \\ \mathbb{C}^2 & \xrightarrow{\phi} & \mathbb{C}^2. \end{array}$$

PROPOSITION. ϕ maps the real axis $\mathbb{R}^2$ onto the conjugate diagonal Δ'.

τ-reversibility and σ-reversibility

PROPOSITION. If f is τ-reversible, then $g = \phi^{-1} \circ f \circ \phi$ is σ-reversible.

$$\begin{array}{ccc}
\mathbb{C}^2 & \xleftarrow{f} & \mathbb{C}^2 \\
\updownarrow \tau & & \updownarrow \tau \\
\mathbb{C}^2 & \xrightarrow{f} & \mathbb{C}^2 \\
\uparrow \phi & & \uparrow \phi \\
\mathbb{C}^2 & \xrightarrow{g} & \mathbb{C}^2 \\
\updownarrow \sigma & & \updownarrow \sigma \\
\mathbb{C}^2 & \xleftarrow{g} & \mathbb{C}^2 .
\end{array}$$

Reversible linear map

PROPOSITION. The eigenvalues of σ-reversible 2×2 matrix can be written as

$$\delta\gamma \quad \text{and} \quad \delta\gamma^{-1},$$

with $|\delta| = 1$, and $|\gamma| = 1$ or $\gamma \in \mathbb{R}$.

PROOF. Let L be a σ-reversible 2×2 matrix. As $\bar{L} = L^{-1}$, $|\det L| = 1$. Let $\det(L) = \delta^2$, and let $A = \delta^{-1}L$. Then $\det(A) = 1$, and $\bar{A} = A^{-1}$. Hence it follows that $\operatorname{trace}(A) \in \mathbb{R}$.

REMARK. Same result holds for τ-reversible matrix.

Involution and matrix

	A	A^{-1}	$-A$
$\bar{A}$	real	*reversible*	pure imaginary
${}^tA$	symmetric	orthogonal	skew symmetric
A^*	Hermite	unitary	skew Hermite

Real eigenvector

PROPOSITION. Simple eigenvalue λ, with $|\lambda| = 1$, of σ-reversible 2×2 matrix L has a real eigenvector $v \in \mathbb{R}^2$.

PROOF. Without loss of generality, we can assume $v = (1, q)$, with $Lv = \lambda v$. By σ-reversibility of L, we have

$$L\bar{v} = \overline{\bar{L}v} = \overline{L^{-1}v} = \overline{\lambda^{-1}v} = \overline{\bar{\lambda}v} = \lambda\bar{v}.$$

Since the eigenspace of λ is spanned by v, we conclude $v = \bar{v} \in \mathbb{R}^2$.

REMARK. Same result holds for τ-reversible matrix, with $\mathbb{R}^2$ replaced by Δ'.

Reversible dynamics

Let $G : \mathbb{C}^2 \to \mathbb{C}^2$ be a σ-reversible biholomorphic diffeomorphism with fixed point $P \in \mathbb{R}^2$, $G(P) = P$.

Let $L = dG_P : T_P \to T_P$ be the differential map at the fixed point.

Then L is σ-reversible, too. (Use real eigenvectors as the basis of T_P.)

Suppose the fixed point P is a center of Siegel disk Ω and

$$\varphi : R \to \Omega$$

is a Siegel uniformizer satisfying

$$G \circ \varphi = \varphi \circ L \quad \text{and} \quad d\varphi_{(0,0)}(\mathbb{R}^2) = \mathbb{R}^2 \subset T_P.$$

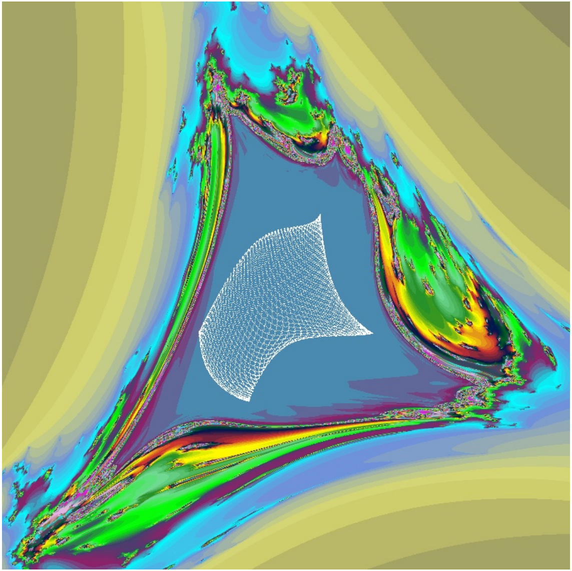
Real Sigel uniformizer

THEOREM. Siegel uniformizer φ induces a real analytic diffeomorphism of $R \cap \mathbb{R}^2$ onto $\Omega \cap \mathbb{R}^2$.

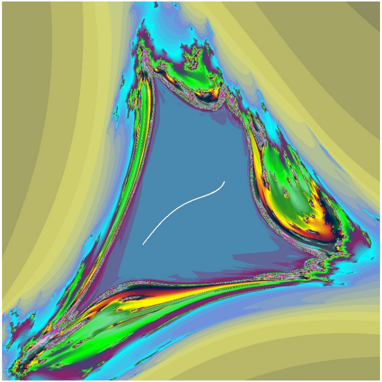
PROOF. We denote $\bar{G} = \sigma \circ G \circ \sigma$, $\bar{L} = \sigma \circ L \circ \sigma$, and $\bar{\varphi} = \sigma \circ \varphi \circ \sigma$. (These are holomorphic.) From $G \circ \varphi = \varphi \circ L$, we have $\varphi \circ L^{-1} = G^{-1} \circ \varphi$. By σ-reversibility, we have $L^{-1} = \bar{L}$ and $G^{-1} = \bar{G}$. And by taking the σ-conjugacy, we get $\bar{\varphi} \circ L = G \circ \bar{\varphi}$. Finally, $d\bar{\varphi}_{(0,0)}|_{\mathbb{R}^2} = d\varphi_{(0,0)}|_{\mathbb{R}^2}$ imply $\bar{\varphi} = \varphi$ by the uniqueness of Siegel uniformizer.

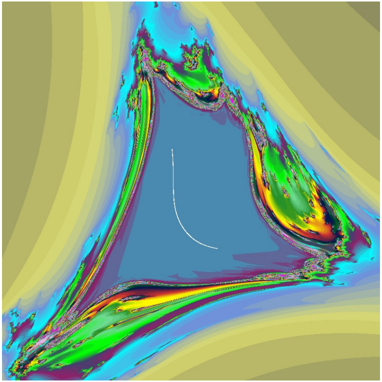
COROLLARY. Siegel uniformizer of a fixed point in Δ' for τ-reversible map induces a real analytic diffeomorphism of $R \cap \mathbb{R}^2$ onto $\Omega \cap \Delta'$.

Δ'-slice of a Siegel disk

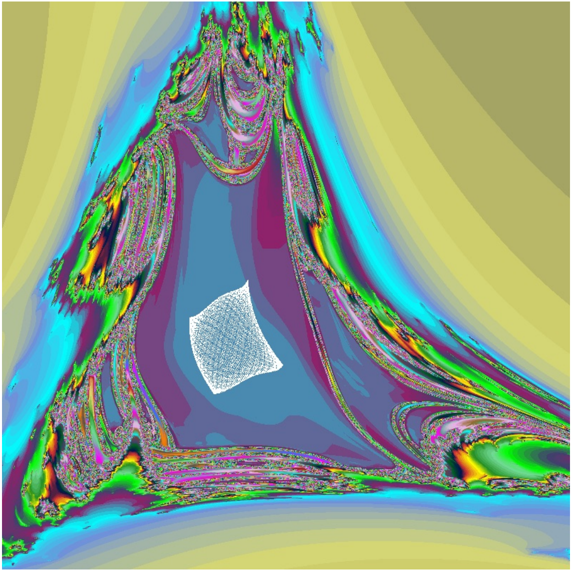


Δ'-slice of a Siegel disk



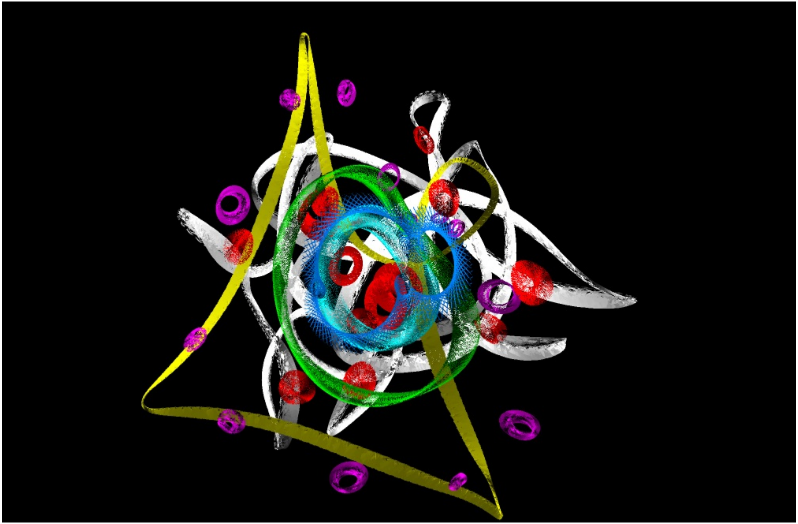


Slice of another Siegel disk



$\alpha = -0.2101699, \beta = \exp(0.7694274i)$

Some orbits near the Siegel disk



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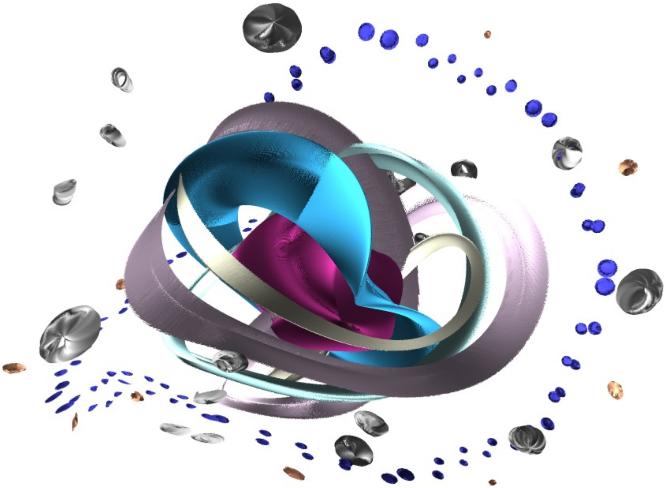
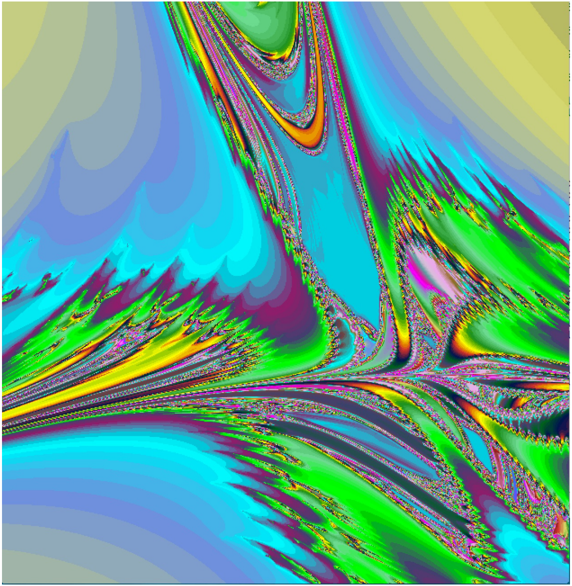
Some orbits near a Siegel disk

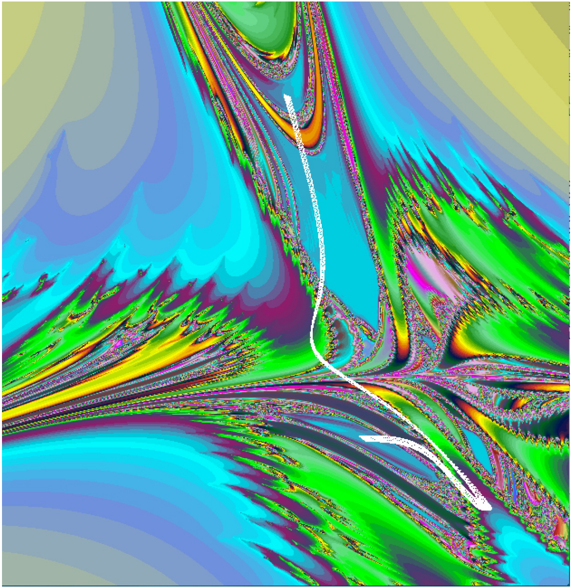


3. Exotic rotation domain

Conjugate diagonal slice for $\alpha = 0.269423, \beta = e^{1.02773i}$



An Orbit in a rotation domain

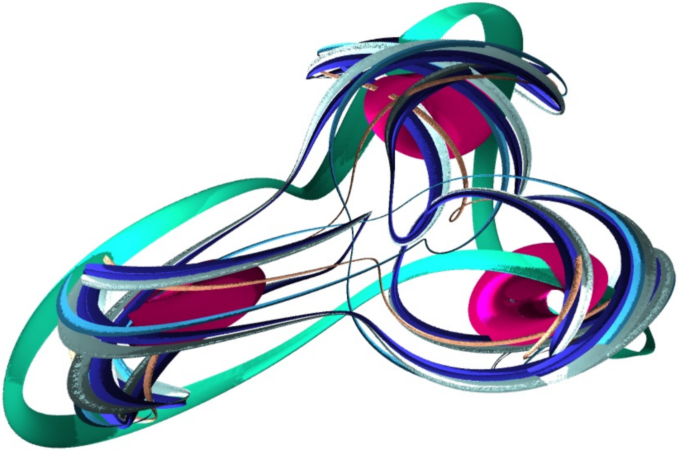


Multiply connected Reinhardt domain

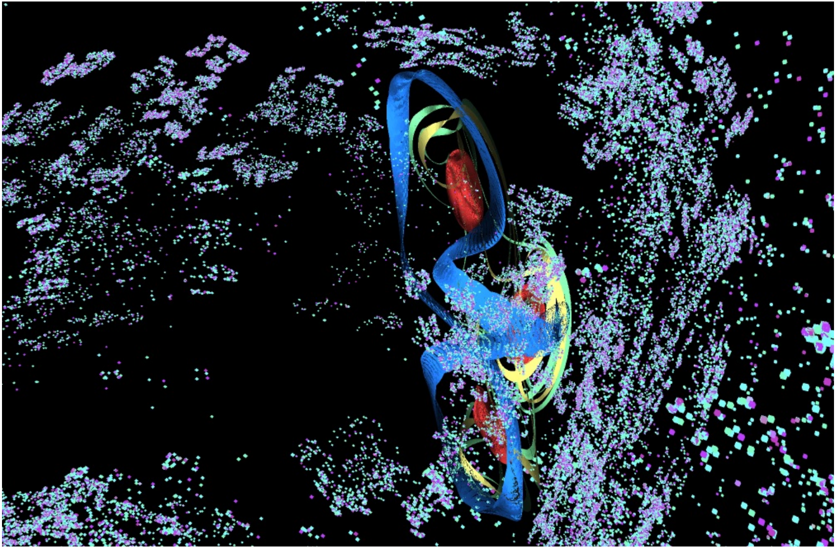
Multiply connected Reinhardt domain could be a model of rotation domain of rank 2.

Theorem(Bedford). Let f be reversible by τ, and let $\Omega = f(\Omega)$ be a rank 2 Fatou component with $\Omega \cap \Delta' \neq \emptyset$. If Ω contains a fixed point, then $\Omega \cap \Delta'$ is connected; otherwise it has exactly two connected components.

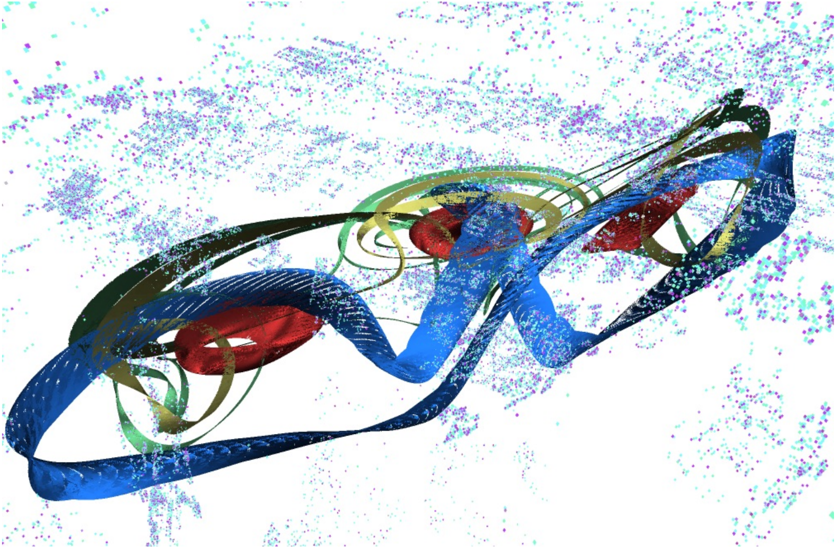
Orbits in rotation domains



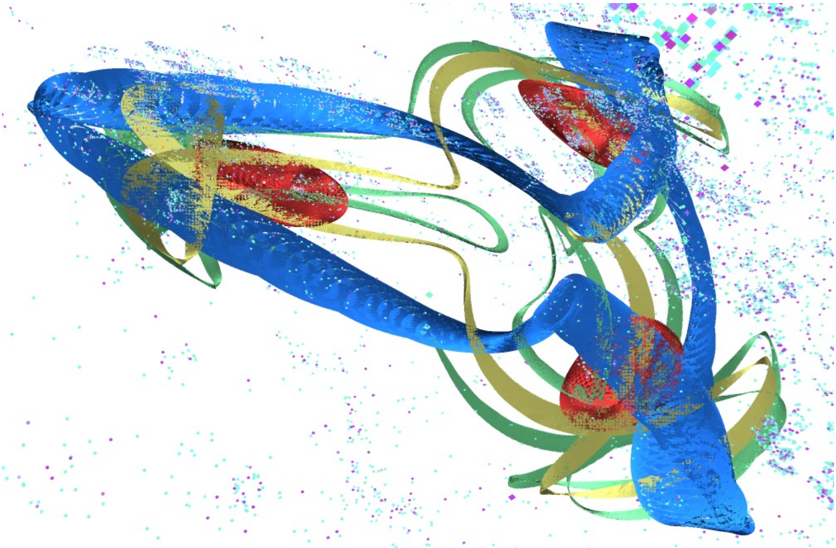
Orbits in rotation domains and points in Julia set



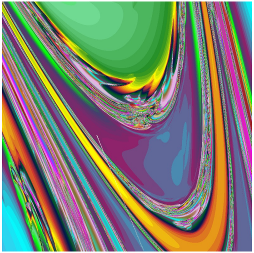
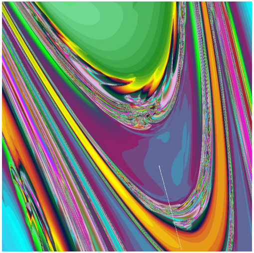
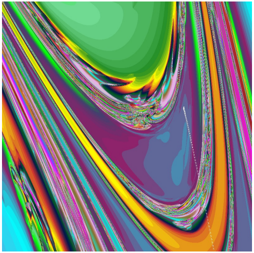
Same as previous



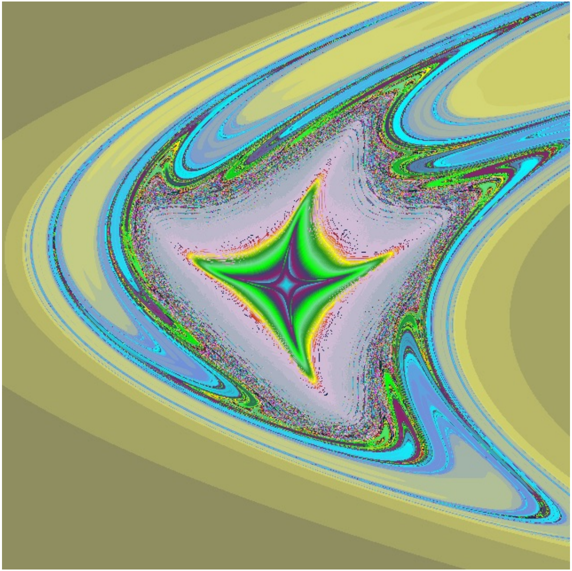
Same as previous, another projection



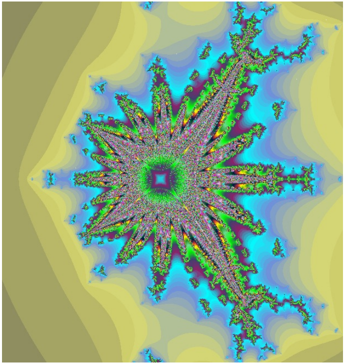
Δ'-slice

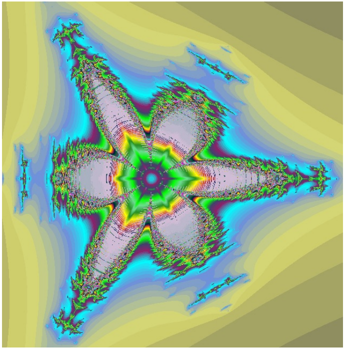
  

Real volume preserving Hénon map, $\alpha = 0, \beta = 1$.

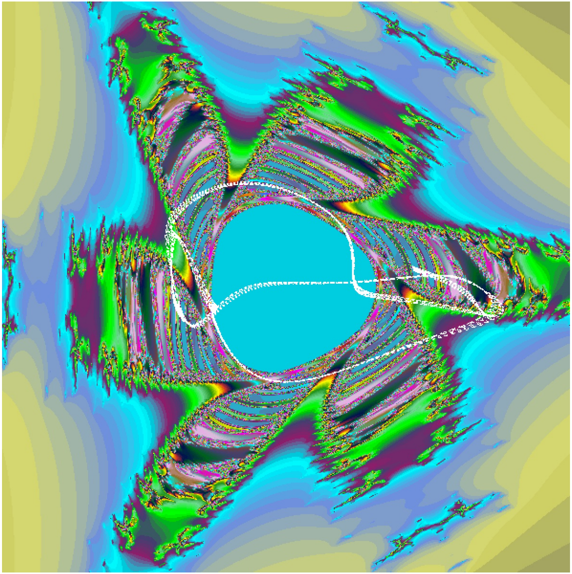


Real volume preserving Hénon map, Δslice and Δ'slice

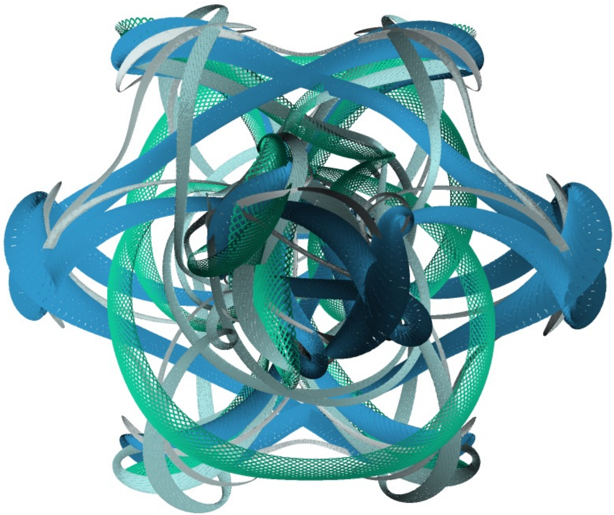




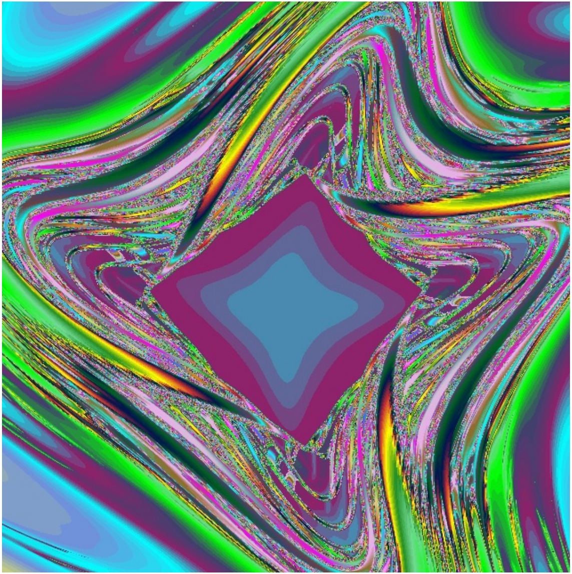
Conjugate diagonal slice for $\alpha = 0, \beta = e^{0.005i}$



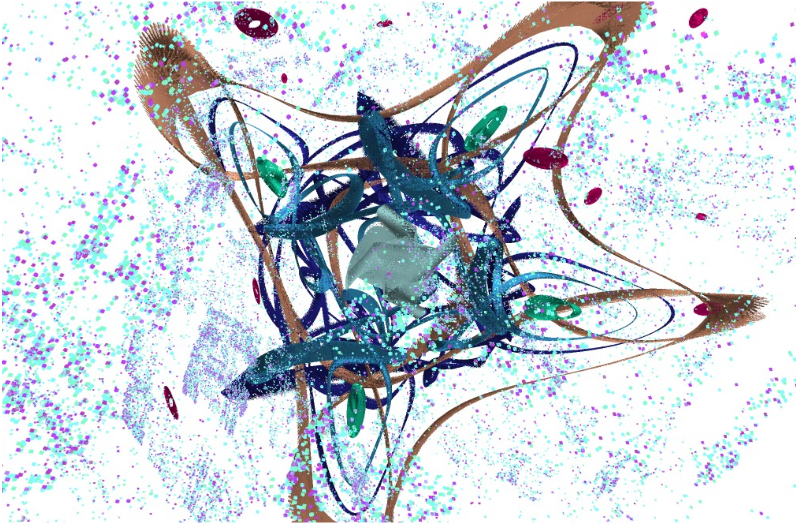
Orbits in a rotation domain



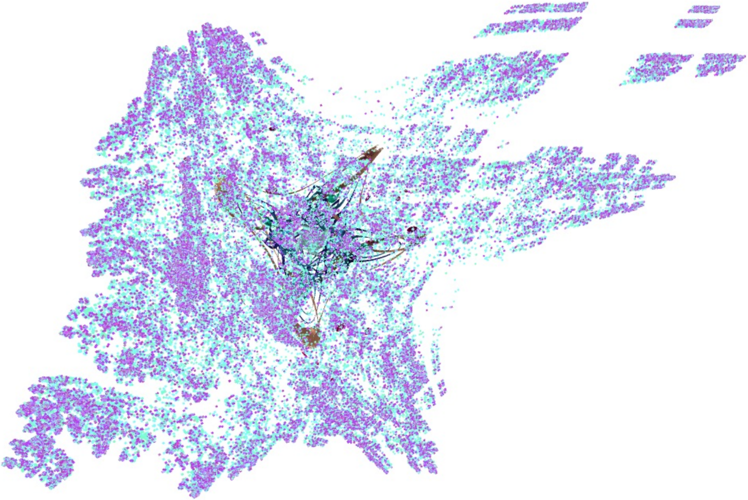
Real slice



Rotaion domains and Julia set

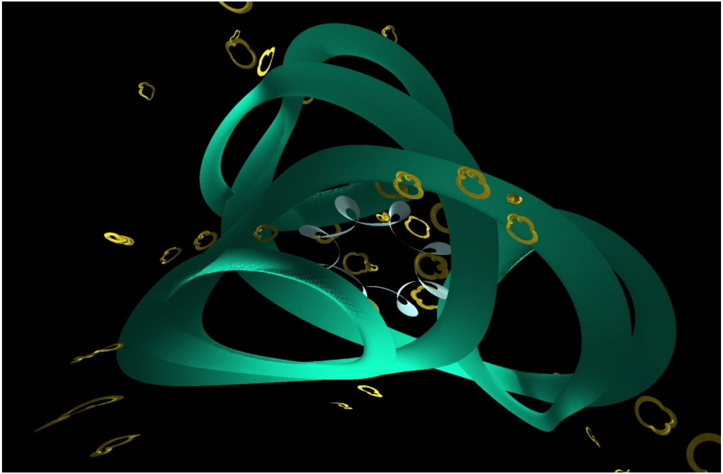


Same as above

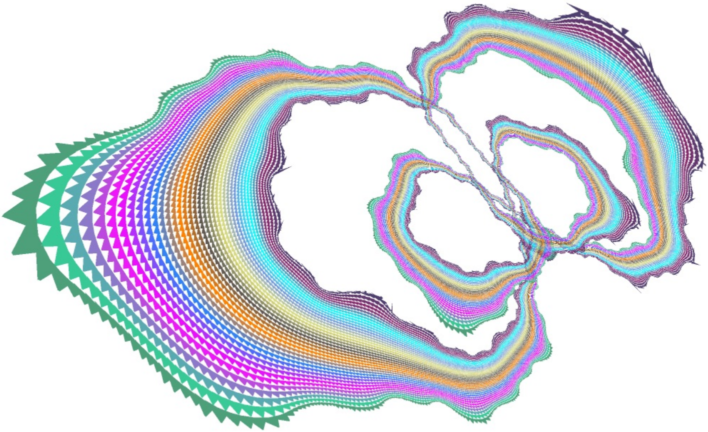


4. Surface automorphism

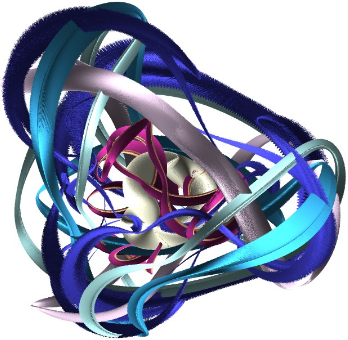
Orbits in a rotation domain of surface automorphism



Attracting Herman ring

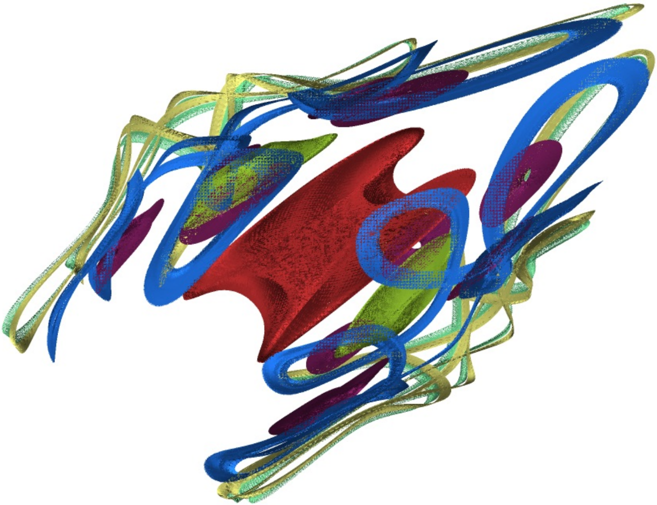


Thank you !



Thank you !

Periodic rotation domain



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