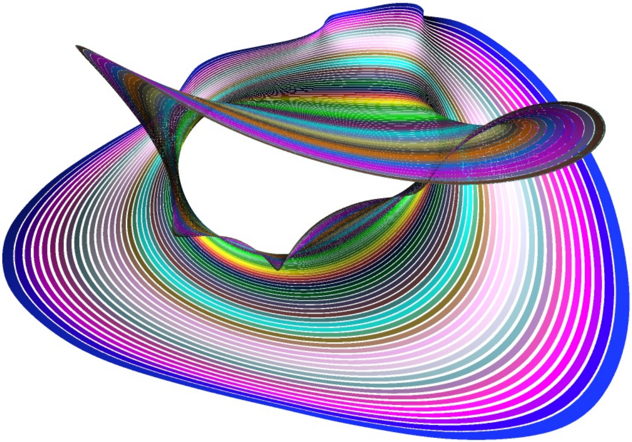
# Herman Rings for Rational Dynamics in $\mathbb{P}^2$



Shigehiro Ushiki Oct. 13, 2023

# Abstract

Some attracting invariant sets are observed numerically for dynamical systems defined by quadratic rational functions on $\mathbb{P}^2$.

These invariant objects seem to be isomorphic to Herman rings.

They are found in families of quadratic rational maps modified from those birational maps inducing surface automorphisms.

Possible bifurcations, which could generate such attractors, are indicated.

# Contents

# 0. Introduction

# Rational maps of $\mathbb{P}^k$

A point $z \in \mathbb{P}^k$ is expressed in homogenious coordinates as $z = [z_0 : z_1 : \cdots : z_k]$.

A rational map of degree $d$ is written as $f = [f_0 : f_1 : \cdots : f_k]$, where $f_j$ are homgeneous polynomials of degree $d$ without common factors.

$f$ is said **dominant** if $\det(f') \not\equiv 0$.

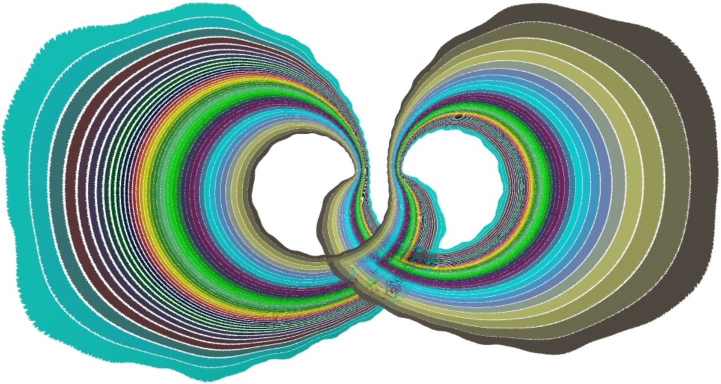
$f$ is said **algebraically stable** if $\deg(f^{\circ n}) = d^n, n \in \mathbb{N}$.

Let $f$ be an algebraically stable dominant rational map of $\mathbb{P}^k$.

DEFINITION A point $p \in \mathbb{P}^k$ is in the **Fatou set** of $f$ if there exists a neighborhood $U$ of $p$ such that the family $f^{\circ n}|_U$ is equicontinuous.

The **Julia set** is the complement of the Fatou set.

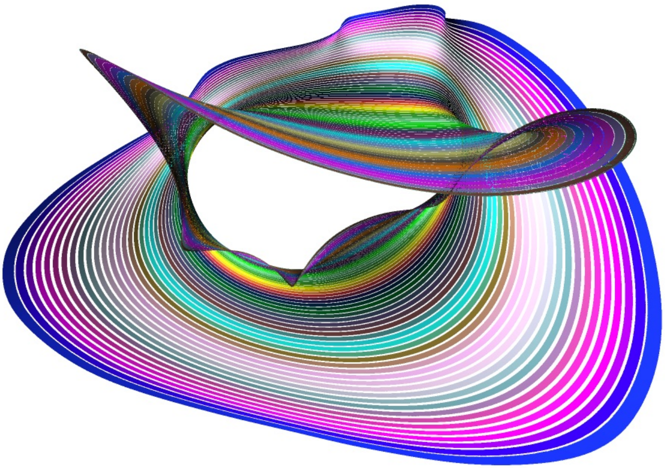
# Apparent Herman ring (LTt514B01bm02)



An attractor of rational map $f = [f_1 : f_2 : f_3] : \mathbb{P}^2 \to \mathbb{P}^2$, with

$$
\begin{aligned}
f_1(x,y,z) &= 0.178023029x^2 - 2xy + 0.286608774xz,\\
f_2(x,y,z) &= 0.042129935x^2 - 0.139011186xy + 1.561721389y^2\\
&\quad + 0.001334534xz - 0.317454834yz - 0.035590774z^2,\\
f_3(x,y,z) &= 0.280638156x^2 - 0.178023029xz + 2yz - 0.567246930z^2.
\end{aligned}
$$

# Apparent Herman ring (LTt154B1b0333)



An attractor of rational map $f = [f_1 : f_2 : f_3] : \mathbb{P}^2 \to \mathbb{P}^2$, with

$$f_1(x,y,z) = -0.139011186x^2 - 2xy - 0.608595974xz,$$

$$f_2(x,y,z) = -0.058307895x^2 + 0.108548371xy + 1.561721389y^2 - 0.025470763xz + 0.247186393yz - 0.011084913z^2,$$

$$f_3(x,y,z) = 0.219139306x^2 + 0.139011186xz + 2yz + 0.389456668z^2.$$

# Herman attractors

In the case of holomorphic mappings $f : \mathbb{P}^2 \to \mathbb{P}^2$, similar attractors are also observed numerically.

# 1. Rotation attractor

# Recurrent Fatou domain

DEFINITION A Fatou component $\Omega$ is said **recurrent** if for some $p_0 \in \Omega$, the $\omega$-limit set of $p_0$ intersects $\Omega$.

DEFINITION A Fatou comonent $\Omega$ is a **Siegel domain** if there exists a subsequence $\{f^{\circ n_i}\}_i$ converging uniformly on compact sets of $\Omega$ to identity.

# Theorem of Fornæss and Sibony

THEOREM(Fornæss-Sibony, 1995). Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a holomorphic rational map of degree $d \geq 2$. Let $U$ be a recurrent Fatou component such that $f(U) = U$. Then one of the following properties is satisfied.

(i) $U$ is the basin of attraction of a fixed point $p \in U$.

(ii) There exists a closed complex submanifold $\Sigma$ of $U$ of complex dimension one such that $f^{\circ n}(K) \to \Sigma$ for every compact subset $K$ of $U$. The Riemann surface $\Sigma$ is biholomorphic to either a disk $D$, a puctured disk or an annulus, and $f|_\Sigma$ is conjugate to an irrational rotation.

(iii) The domain $U$ is a Siegel domain.

# Ueda's construction

In the case of (ii) with invariant annulus, we call it an attracting **Herman ring**.

As Ueda's example of holomorphic rational map shows, there exist attracting Herman rings.

However, his construction essentially constructed from a direct product of one dimensional rational maps with Herman ring.

Let $\Phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ be the two-fold branched covering

$$\Phi([z_0 : z_1], [w_0 : w_1]) = [z_0 w_0 : z_1 w_1 : z_0 w_1 + z_1 w_0].$$

Rational map $h : \mathbb{P}^1 \to \mathbb{P}^1$ induces a rational map $\hat{h} : \mathbb{P}^2 \to \mathbb{P}^2$, with

$$\hat{h} \circ \Phi(z, w) = \Phi(h(z), h(w)).$$

# Rational maps on $\mathbb{P}^2$

The existence of attracting Herman ring is obvious for rational maps on $\mathbb{P}^2$.

However, apart from the constructions essentially by a direct product of one dimensional Herman ring and an attractor, we don't know how to construct such objects.

I encountered apparent attracting Herman rings generated by computer, when I was trying to produce a family of birational maps of $\mathbb{P}^2$.

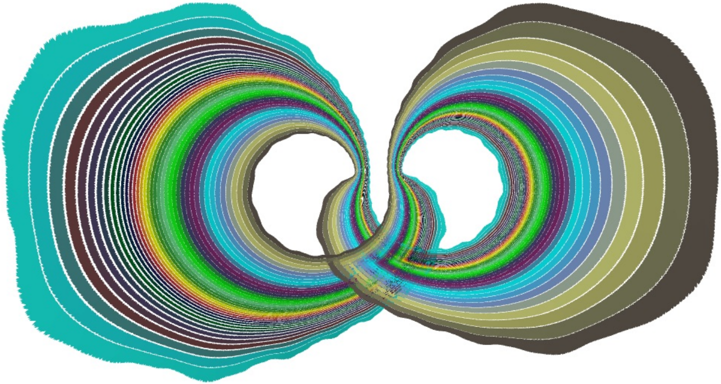
But the computer program had bugs.

The dynamical system which generated these attractors was not a surface automorphism.

I fixed the bugs. When all bugs are removed, I found all the maps I programed give dynamical systems which are conjugate to those already known to me.

# 2. First example

# Apparent Herman ring (LTt514B01bm02)



An attractor of rational map $f = [f_1 : f_2 : f_3] : \mathbb{P}^2 \to \mathbb{P}^2$, with

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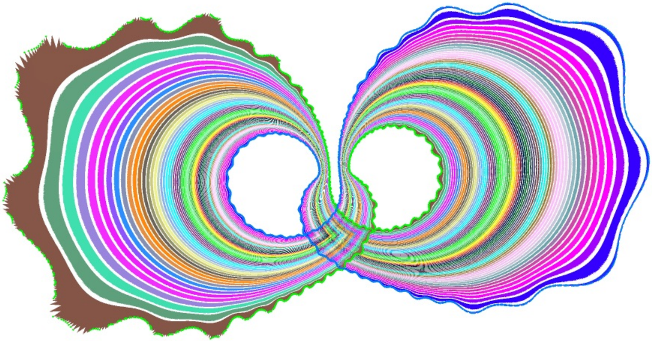
$$f_3(x,y,z) = 0.280638156x^2 - 0.178023029xz + 2yz - 0.567246930z^2.$$

From the formula in the computation, we see the followings. $f$ is a quadratic rational map with all coefficients real. So, the real axis $\mathbb{R}^2$ is mapped to the real axis. The $y$-axis $\{x = 0\}$ is mapped to itself, and line $\{x = z\}$ is mapped to $\{x = -z\}$. Line $\{x = -z\}$ is mapped to $\{x = z\}$.

Point $x = 0$, $y = 0.283623465z$ is indeterminate.

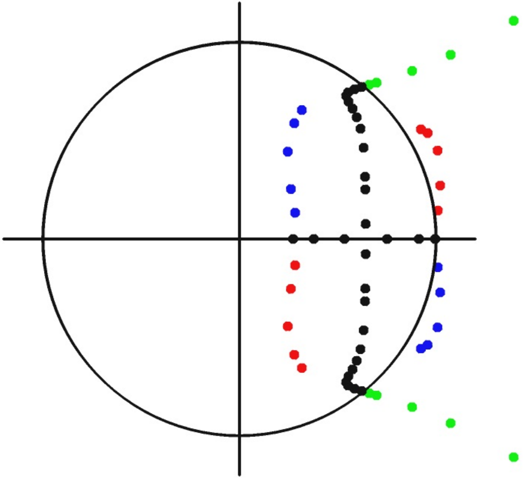
Numerically observed : this map has no other point of indeterminacy.

# Apparent Herman ring (LTt514B01BXm0001bm022)

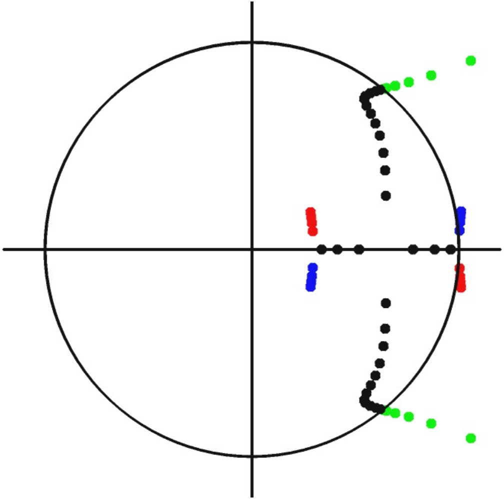


An attractor of holomorphic rational map
$f = [f_1 : f_2 : f_3] : \mathbb{P}^2 \to \mathbb{P}^2$.

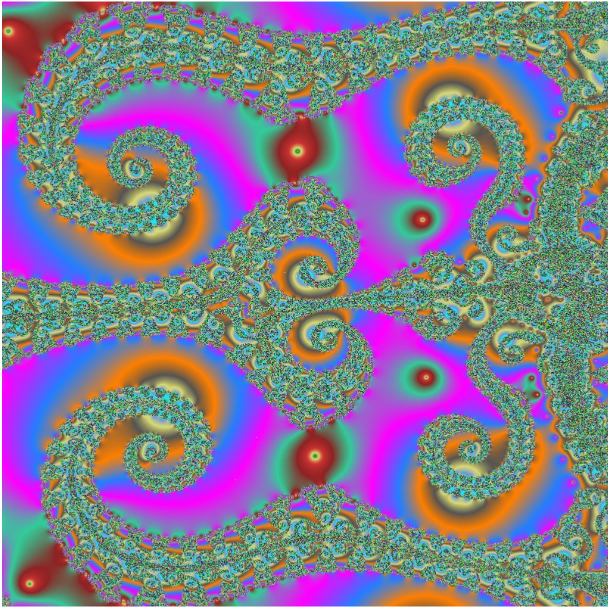
# Eigenvalues of fixed point near bifurcation (LTt514B01)



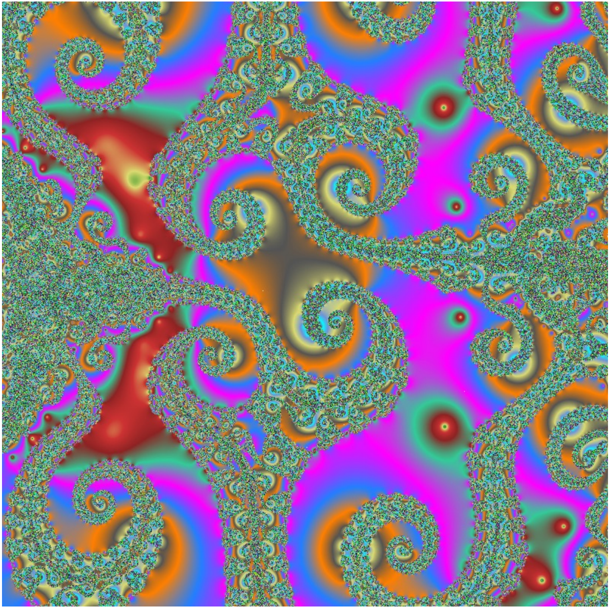
# Eigenvalues of fixed point near bifurcation (LTt514B01BXm0001)



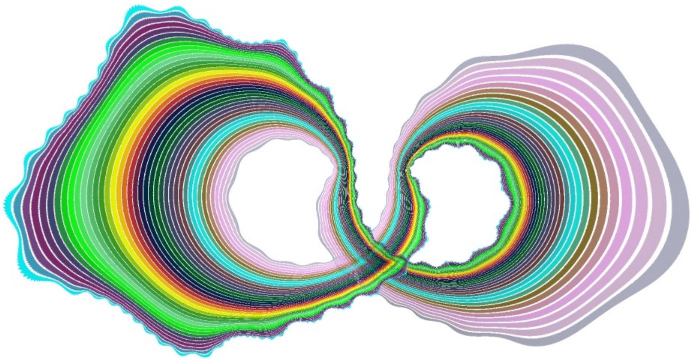
# Horizontal slice near Herman attractor (LTt514B01bm02)



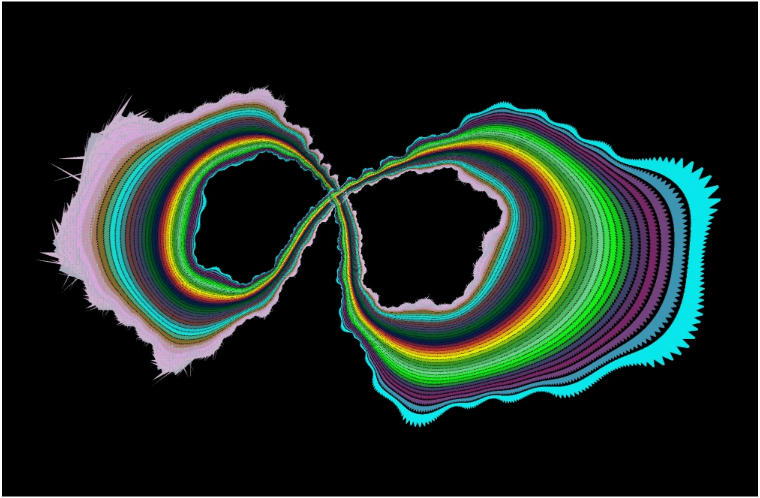
# Vertical slice near Herman attractor (LTt514B01bm02)



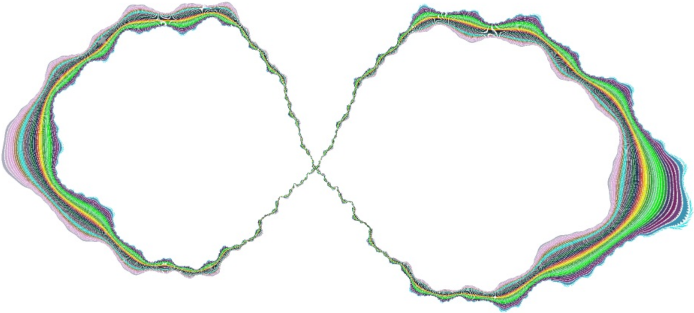
# Similar attractor (LTt514B01bm021)



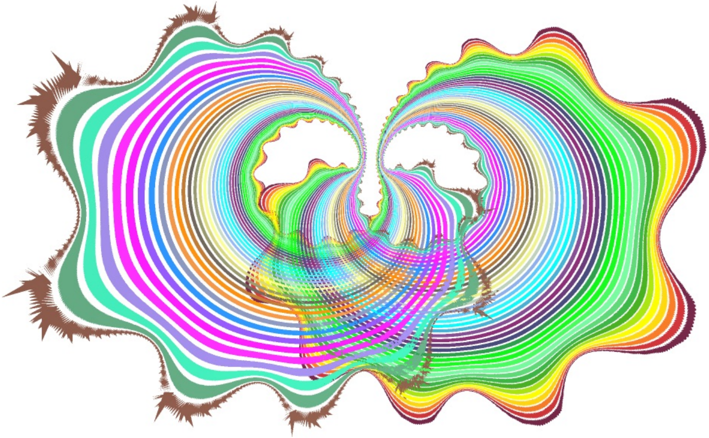
# Similar attractor (LTt514B01bm025)



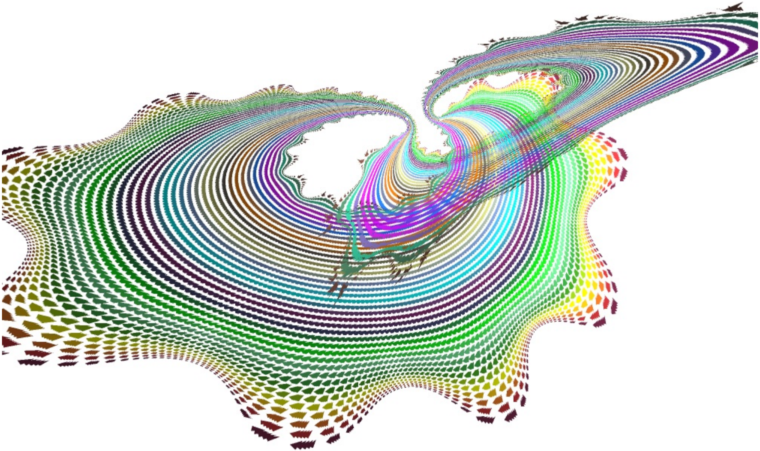
# Thin attractor (LTt514B01bm028)



# Still another attractor(LTt154B12b0143)

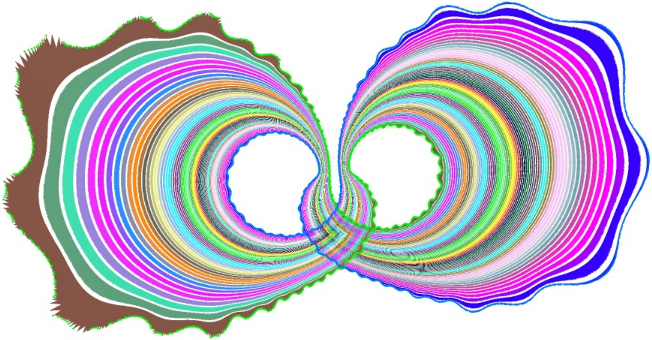


# Different projection (LTt154B12b0143)



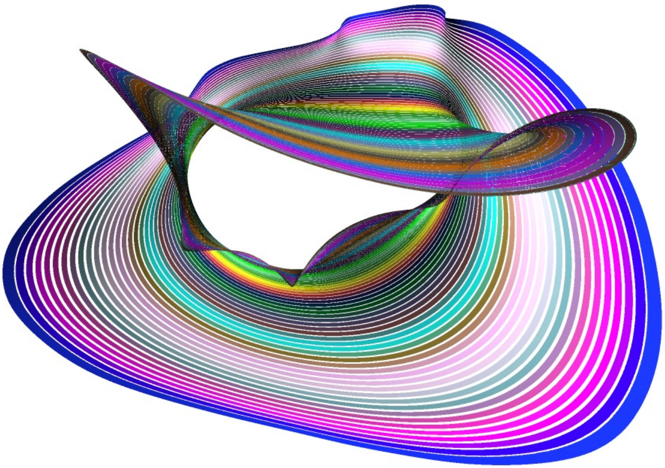
# Holomorphic map case (LTt514B01BXm0001m022)

In a family of holomorphic mappings, similar bifurcation is observed numerically.



# 3. Second example

# Apparent Herman ring (LTt154B1b0333)



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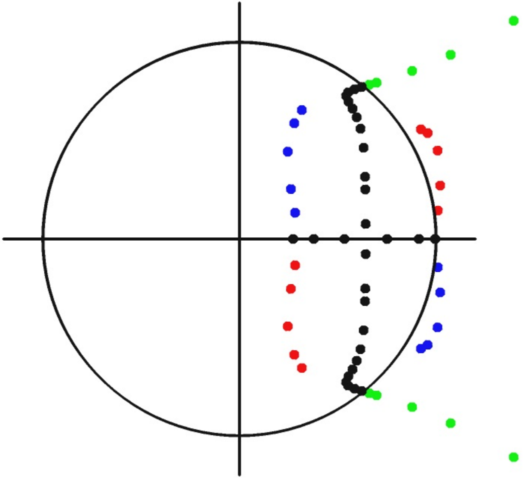
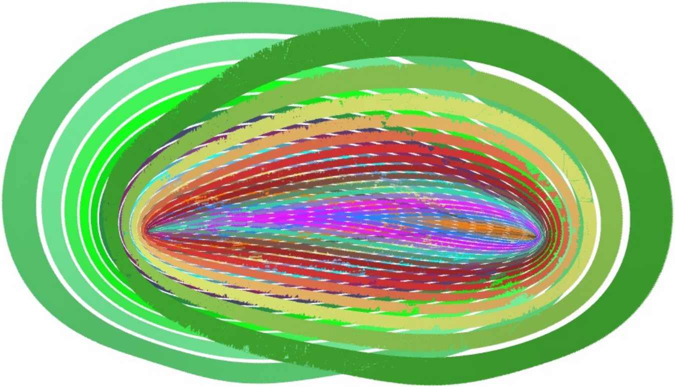
This mapping does not appear in the family of mappings in the following.

(Apparent) Herman rings appear from "Hopf bifurcation", that is, when a pair of complex conjugate eigenvalues of a fixed point cross the unit circle.

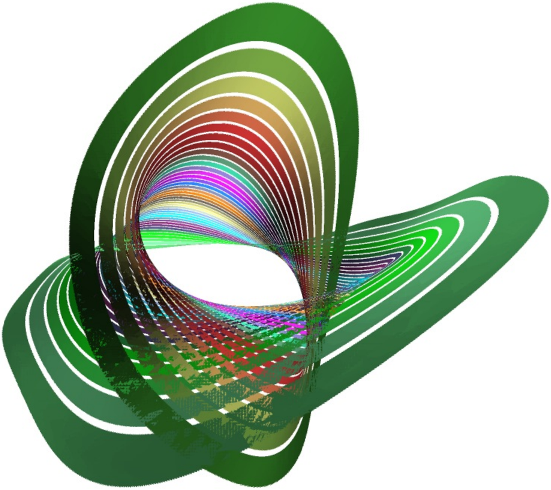
# Eigenvalues of fixed point near bifurcation (LTt514B01)



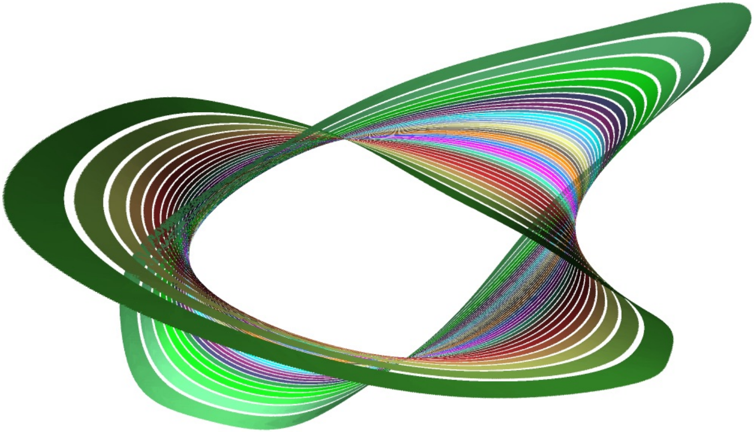
# Apparent Herman ring (LTt514B01b0165)



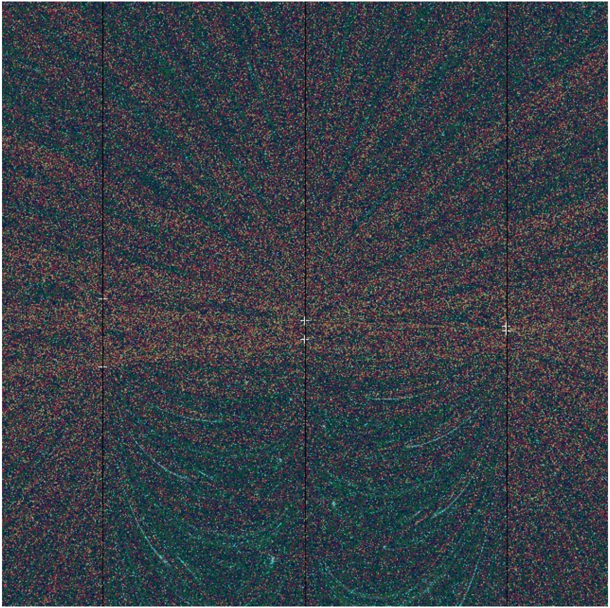
# Apparent Herman ring (LTt514B01b0165)



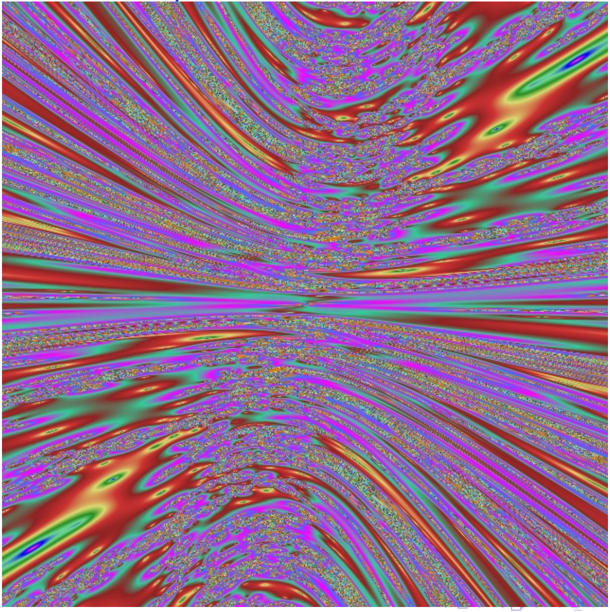
# Apparent Herman ring (LTt514B01b0165)



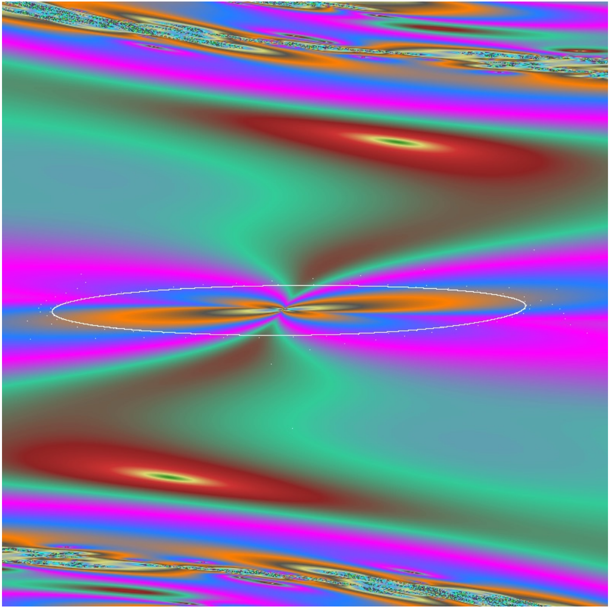
# Real slice (LTt514B01b0165R)



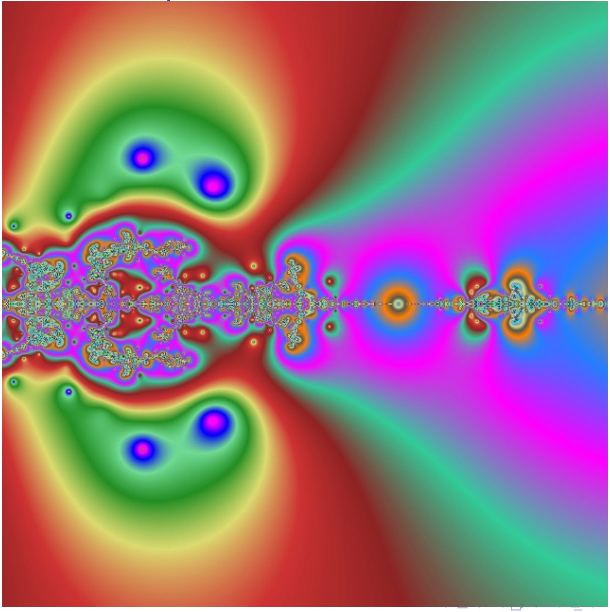
# Imaginary slice passing the fixed point (LTt514B01b0165RN)



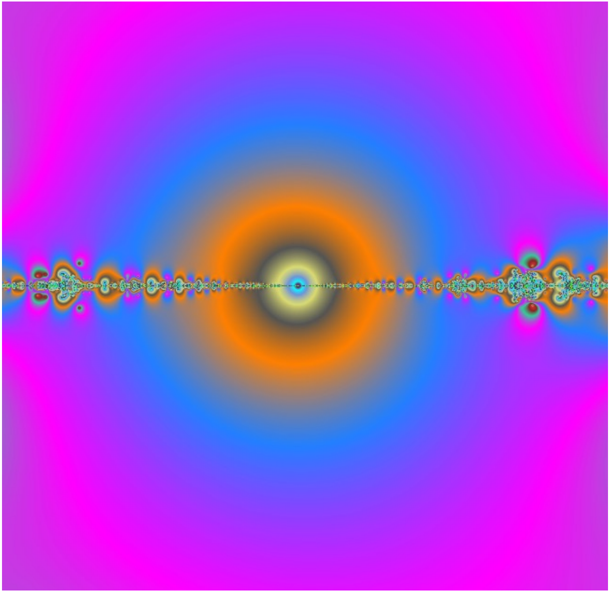
# Imaginary slice near the fixed point (LTt514B01b0165RNa)



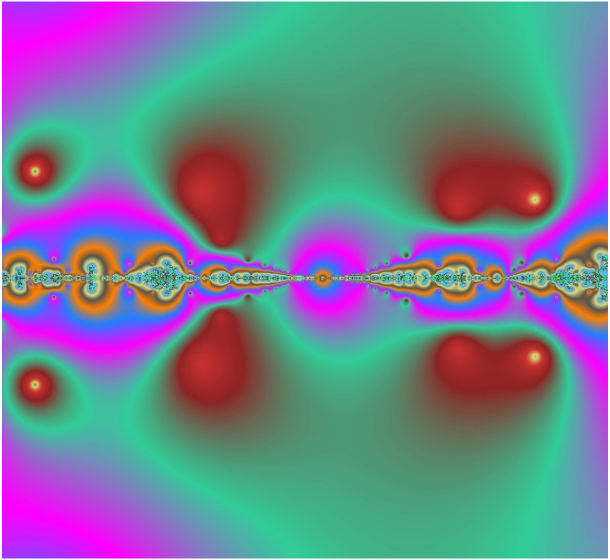
# Horizontal slice passing the fixed point (LTt514B01b0165Xx)



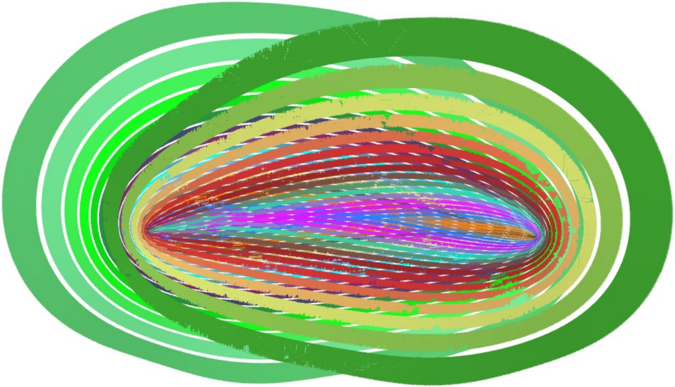
# Zoom of the previous (LTt514B01b0165Xxa)



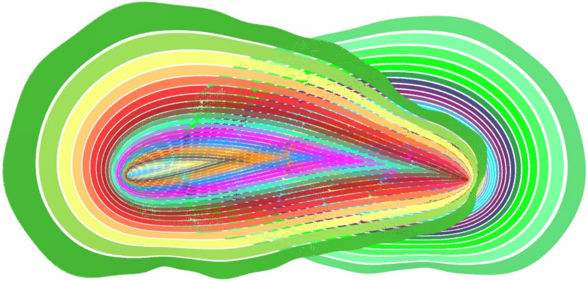
# Vertical slice passing the fixed point (LTt514B01b0165Xya)



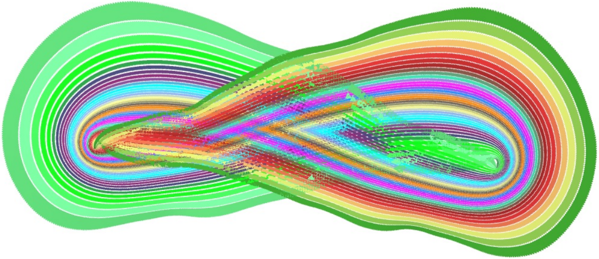
# Apparent Herman ring (LTt514B01b0165)



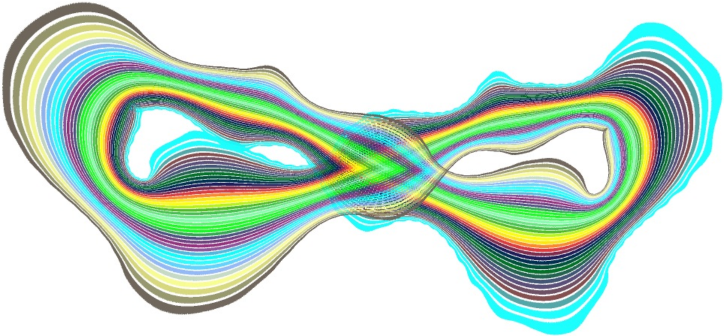
# Apparent Herman ring (LTt514B01b017)



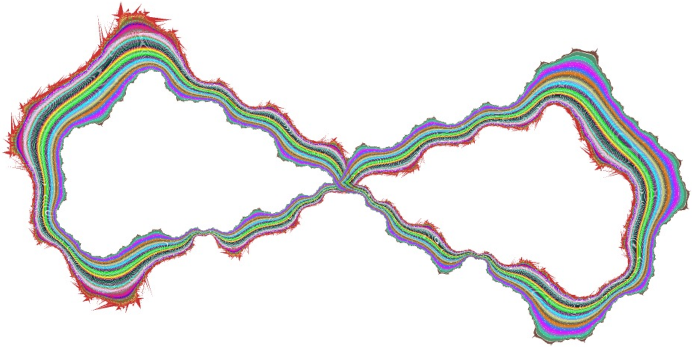
# Apparent Herman ring (LTt514B01b0175)



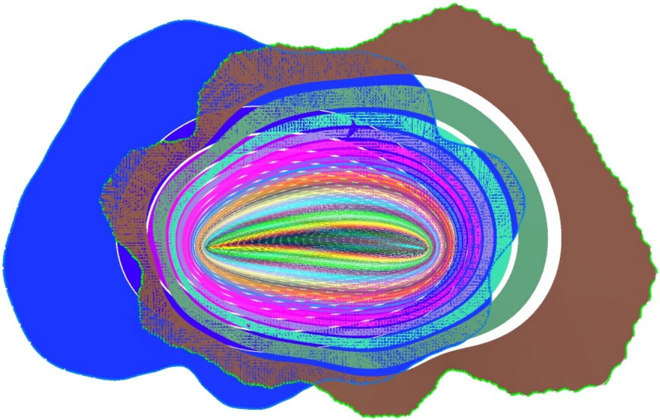
# Another Herman ring (LTt514B01b018)



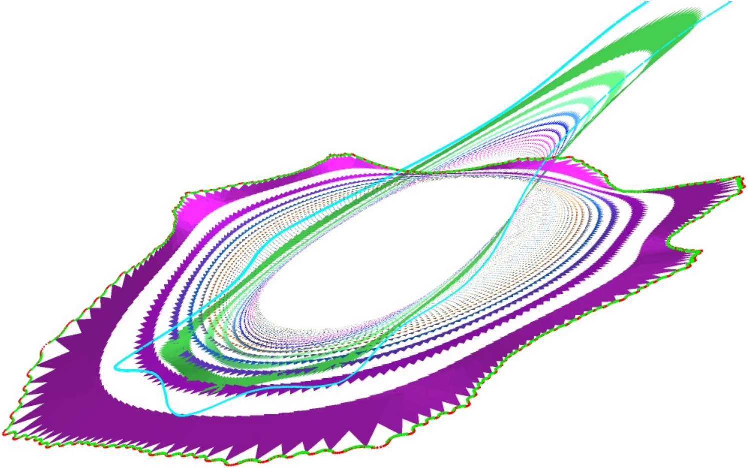
# Another Herman ring (LTt514B01b0188)



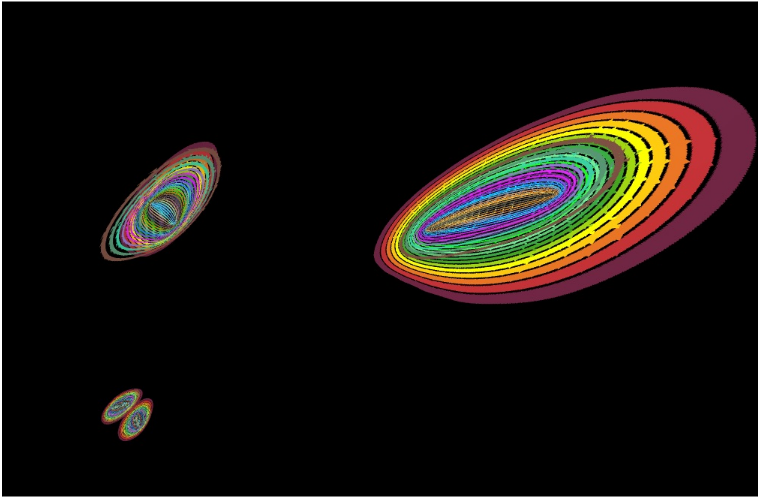
# Holomorphic map case (LTt514BXm001)



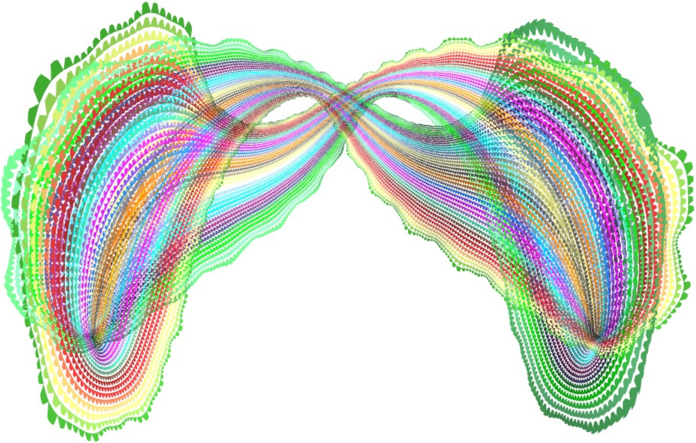
# Different projection (LTt514BXm001)



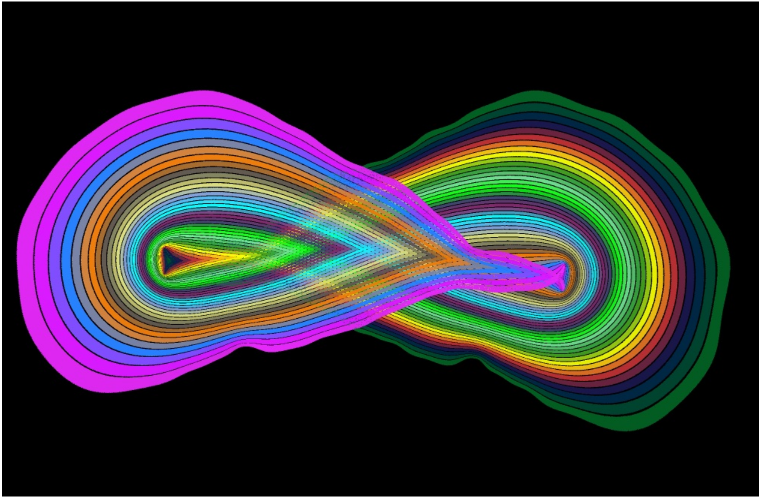
# Another Herman ring (LTt514B0b04)



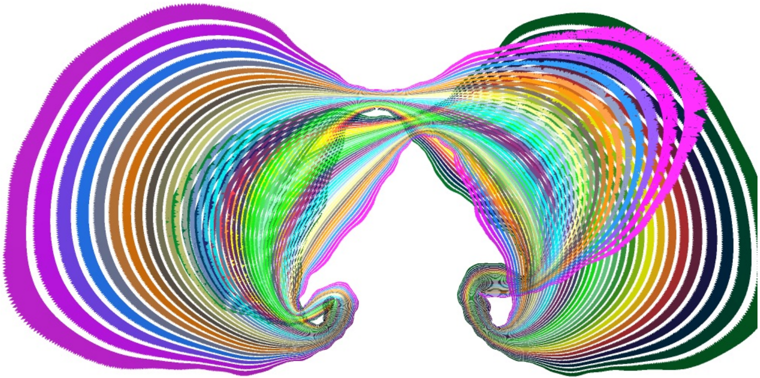
# Still another Herman ring (LTt514B0bm044)



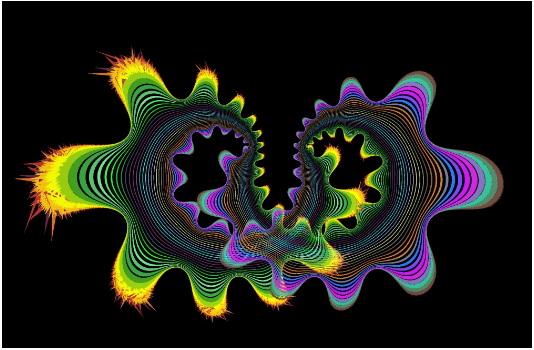
# Still another Herman ring (LTt514BX), holomorphic case



# Still another Herman ring (LTt514BX), holomorphic case



# Thank you ! (LTt514BB2b0145)



Thank you !

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