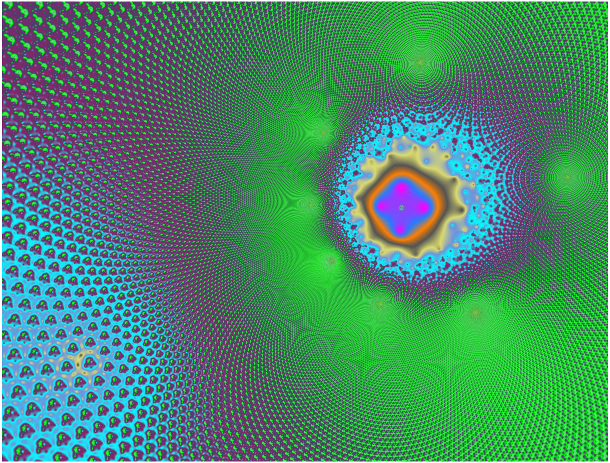
Mordell-Weil Dynamics on Elliptic Surface



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Abstract

Possible structures of Mordell-Weil groups are completely classified by Oguiso and Shioda([OS],1991).

Karayayla([K],2011) gave a list of possible groups of surface automorphisms preserving a section.

In this note, we construct a concrete example of an elliptic surface, whose Mordell-Weil group is isomorphic to the dual lattice of E_7.

The generators of the Mordell-Weil group are specified, in terms of sections of the elliptic fibration, and in terms of automorphisms of invariant cubic curve.

The group of the automorphisms of the elliptic surface is determined.

Contents

0. Introduction

Elliptic surface

Let S be a complex manifold of complex dimension 2. Suppose there is an elliptic fibration onto $\mathbb{P}^1$:

$$\psi : S \to \mathbb{P}^1.$$

And suppose there is a cross section

$$\sigma : \mathbb{P}^1 \to S, \qquad \psi \circ \sigma = id.$$

We want to understand the structure of the group of automorphisms,

$$\mathrm{Aut}(S).$$

Fibration ψ induces homomorphisms (not trivial)

$$0 \to \widetilde{\mathrm{Aut}}(S) \to \mathrm{Aut}(S) \to \mathrm{Aut}(\mathbb{P}^1).$$

$\mathrm{Aut}(S)$

S : elliptic surface. (notations are explained later)
(Gizatullin,[Gi],1980)(Grivaux,[Gr], 2019)

$$0 \to \widetilde{\mathrm{Aut}}(S) \to \mathrm{Aut}(S) \to \mathrm{Aut}(\mathbb{P}^1).$$

$$0 \to \mathrm{Ker}(\mathbf{tr}) \to \mathrm{Pic}(S) \xrightarrow{\mathbf{tr}} \mathrm{Pic}(\mathcal{X})\{\mathbb{C}(t)\} \xrightarrow{\deg} \mathbb{Z}.$$

$$K_S^{\perp}/\mathrm{Ker}(\mathbf{tr}) \cong \mathrm{Pic}^0(\mathcal{X})\{\mathbb{C}(t)\}.$$

(Shioda,[S],1990)

$$MW(S) \cong NS(S)/T, \qquad E(K)/E(K)_{\mathrm{tor}} \cong (E(K)^0)^*.$$

(Karayayla,[K],2011)

$$\mathrm{Aut}(S) = MW(S) \rtimes \mathrm{Aut}_\sigma(S).$$

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}_\sigma(S) \to \mathrm{Aut}_\psi(\mathbb{P}^1) \to 0.$$

We try to find a concrete and explicit example of non-trivial $\mathrm{Aut}(S)$.

An elliptic surface

Consider a surface automorphism with invariant cuspidal cubic curve for orbit data $(1, 3, 5)$, *cyclic*, choosing eigenvalue $\kappa = \exp(2\pi i/7)$.

The configuration of the singular fibers is $\mathrm{III}\ \mathrm{II}\ \mathrm{I}_1^7$.

By Karayayla's table, $\mathrm{Aut}_\sigma(S) \cong \mathbb{Z}/14\mathbb{Z}$.

The type of trivial lattice T is A_1.

The rank of $MW(S)$ is 7. $MW(S)$ does not have torsions.

By Oguiso and Shioda's table, $MW(S) \cong E_7^*$.

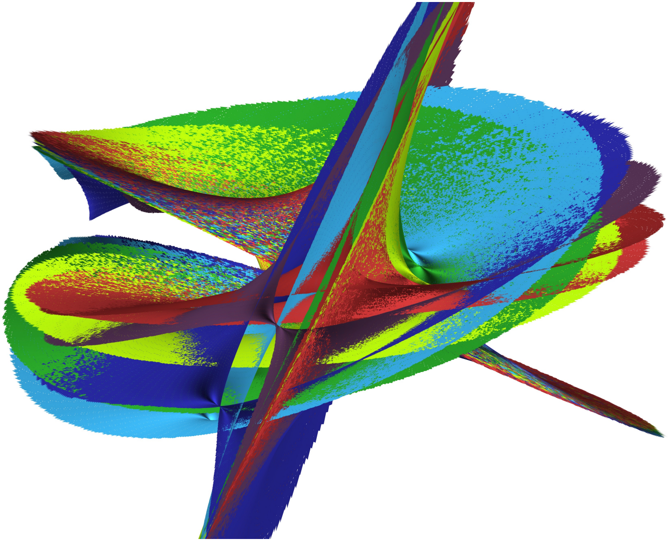
$$\mathrm{Aut}(S) \cong E_7^* \rtimes \mathbb{Z}/14\mathbb{Z}.$$

Find a basis of $MW(S)$ among the exceptional fibers of S.

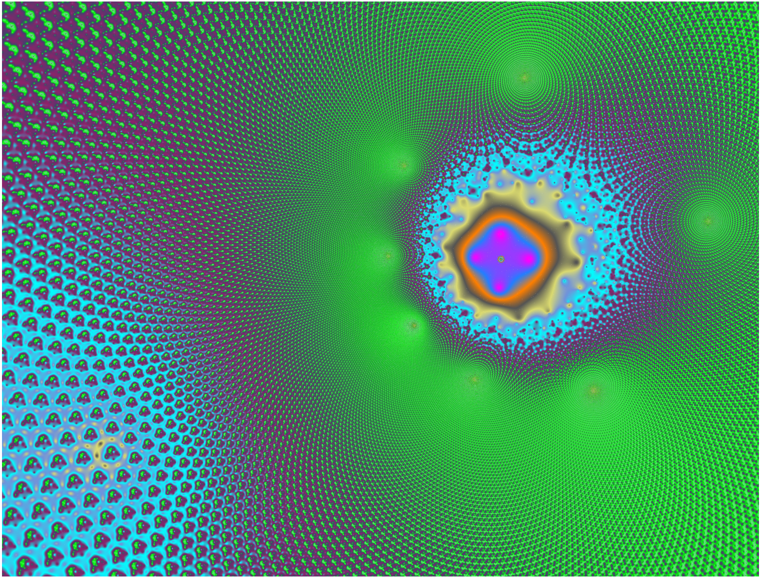
Observe their behavior under homomorphism

$$MW(S) \to \mathrm{Aut}(\mathrm{Pic}(X_{\mathrm{III}})).$$

Elliptic curves of period 7



A section



1. Elliptic surface

Birational map

Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map. Under certain conditions, birational map induces a holomorphic automorphism $F : S \to S$ of rational surface S, which is obtained by successive blowing ups of $\mathbb{P}^2$, with projection $\pi : S \to \mathbb{P}^2$.

$$\begin{array}{ccc} S & \xrightarrow{F} & S \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{P}^2 & \overset{f}{\dashrightarrow} & \mathbb{P}^2. \end{array}$$

Elliptic fibration

A surjective holomorphic map $\psi : S \to \mathbb{P}^1$ is an **elliptic fibration** if almost all fibers, $\psi^{-1}(\xi)$, are smooth curves of genus 1, and no fiber contains an exceptional (-1)-curve.

An **elliptic surface** S over $\mathbb{P}^1$ is a smooth projective surface with an elliptic fibration over $\mathbb{P}^1$.

For fixed S, fibration $\psi : S \to \mathbb{P}^1$ is unique (up to Möbius transformation).

Kodaira names

Singular fibers are classified by Kodaira. (smooth fiber is indicated by I_0)

$$\mathrm{I}_n,\ n \geq 1,\quad \mathrm{II},\quad \mathrm{III},\quad \mathrm{IV},\quad \mathrm{I}_n^*,\ n \geq 0,\quad \mathrm{IV}^*,\quad \mathrm{III}^*,\quad \mathrm{II}^*.$$

Euler number:

$$e(\mathrm{I}_n) = n, \quad e(\mathrm{II}) = 2, \quad e(\mathrm{III}) = 3, \quad e(\mathrm{IV}) = 4,$$

$$e(\mathrm{I}_n^*) = n+6, \quad e(\mathrm{IV}^*) = 8, \quad e(\mathrm{III}^*) = 9, \quad e(\mathrm{II}^*) = 10.$$

$$\sum_{F_v:\ \text{singular fiber}} e(F_v) = 12.$$

Preservation of elliptic fibration

We say that automorphism $F : S \to S$ **preserves elliptic fibration** $\psi : S \to \mathbb{P}^1$, if commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{F} & S \\ \downarrow \psi & & \downarrow \psi \\ \mathbb{P}^1 & \xrightarrow{\Omega} & \mathbb{P}^1. \end{array}$$

holds for some Möbius transformation $\Omega : \mathbb{P}^1 \to \mathbb{P}^1$.

S can have other automorphisms. Every automorphism of S preserves the fibration.

Let $\mathrm{Aut}(S)$ denote the group of automorphisms of S.

Let $\mathrm{Aut}_\psi(\mathbb{P}^1)$ denote the group of Möbius transformations induced by fibration ψ.

$$\mathrm{Aut}_\psi(\mathbb{P}^1) \; = \; \left\{ \Omega : \mathbb{P}^1 \to \mathbb{P}^1 \mid \Omega \circ \psi = \psi \circ F, \text{ for some } F \in \mathrm{Aut(S)} \right\}.$$

Section

Suppose fibration $\psi : S \to \mathbb{P}^1$ has a section $\sigma : \mathbb{P}^1 \to S$, *i.e.*, $\psi \circ \sigma = id.$, and let

$$\mathrm{Aut}_\sigma(S) \; = \; \left\{ F \in \mathrm{Aut}(S) \mid F(\sigma(\mathbb{P}^1)) = \sigma(\mathbb{P}^1) \right\},$$

Karayayla([K],2011) showed a short exact sequence of groups

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}_\sigma(S) \to \mathrm{Aut}_\psi(\mathbb{P}^1) \to 0.$$

He gave a list of all the possible configurations of singular fibers for each type of the group of Möbius transformations.

Rem. **There are rational elliptic surfaces which do not admit sections.**

Mordell-Weil group

By specifying a section $\sigma : \mathbb{P}^1 \to S$, the set of sections of fibration $\psi : S \to \mathbb{P}^1$ form an additive group, regarding the specified section σ as the origin of each fiber. The addition of sections is defined by the group law in each smooth fiber as elliptic curve, and by taking the closure for a section. This group is called the Mordell-Weil group, $MW(S)$, of S.

Karayayla([K], 2011) proved for rational surface with a section :

THEOREM.

$$\mathrm{Aut}(S) = MW(S) \rtimes \mathrm{Aut}_\sigma(S).$$

$$(t_{\zeta_1} \circ \alpha_1)(t_{\zeta_2} \circ \alpha_2) = (t_{\zeta_1 + \alpha_1(\zeta_2)} \circ (\alpha_1 \circ \alpha_2)).$$

(t_ζ denotes the translation induced by $\sigma \to \zeta$.)

Mordell-Weil rank

It is known ([Gi], 1980) that in the case of rational surface,

$$\mathrm{rank}(MW(S)) = 8 - \sum_{v \in R} (m_v - 1).$$

Where, R is the set of points $v \in \mathbb{P}^1$, such that $F_v = \psi^{-1}(v)$ is not smooth, and m_v is the number of irreducible components of the singular fiber.

REM. (Grivaux,[Gr], 2019).
(Gizatullin,[Gi],1980)

$$0 \to \widetilde{\mathrm{Aut}}(S) \to \mathrm{Aut}(S) \to \mathrm{Aut}(\mathbb{P}^1).$$

$$0 \to \mathrm{Ker}(\mathbf{tr}) \to \mathrm{Pic}(S) \xrightarrow{\mathbf{tr}} \mathrm{Pic}(\mathcal{X})\{\mathbb{C}(t)\} \xrightarrow{\deg} \mathbb{Z}.$$

$$K_S^{\perp}/\mathrm{Ker}(\mathbf{tr}) \cong \mathrm{Pic}^0(\mathcal{X})\{\mathbb{C}(t)\} \hookrightarrow \widetilde{\mathrm{Aut}}(S).$$

2. Construction of elliptic surface

Cuspidal cubic curve

Let C denote the cubic curve $\{y = x^3\}$ in $\mathbb{P}^2$.

This curve has a parametrization

$$p : \mathbb{C} \to C, \quad p(t) = (t, t^3).$$

We want to find birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, which map C onto itself.

$$f(C) = C.$$

f has indeterminacy points $I(f)$. The equality should be understood "modulo exceptional points".

$$f(C) = \overline{f(C \setminus I(f))}.$$

f induces an automorphism of the cubic curve C, which can be described by an affine map $t \mapsto \lambda(t + \mu)$ for some constants $\lambda \in \mathbb{C}^\times,\ \mu \in \mathbb{C}$.

Explicit formula for invariant cuspidal cubic curve case

PROPOSITION. For $\lambda \in \mathbb{C}^\times$ and $a_1, a_2, a_3 \in \mathbb{C}$ with $a_1 + a_2 + a_3 \neq 0$, there exists a quadratic birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, such that

$$f(C) = C, \quad I(f) = \{p(a_1), p(a_2), p(a_3)\},$$

inducing $t \mapsto \lambda(t + \frac{\nu_1}{3})$, with $\nu_1 = a_1 + a_2 + a_3$.

PROPOSITION. The quadratic birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ in the previous proposition is given by

$$X = \lambda \left(x + \frac{\nu_1}{3} + \frac{\nu_1(y - x^3)}{\nu_1 x^2 - \nu_2 x + \nu_3 - y} \right),$$

$$Y = \lambda^3 \left((x + \frac{\nu_1}{3})^3 + (y - x^3)(1 + \frac{\nu_1^2 x + \frac{\nu_1^3}{3} - \nu_1 \nu_2}{\nu_1 x^2 - \nu_2 x + \nu_3 - y}) \right).$$

Where $\nu_2 = a_1 a_2 + a_2 a_3 + a_3 a_1$, and $\nu_3 = a_1 a_2 a_3$.

Exceptional lines

A quadratic birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ always acts by blowing up three indeterminacy points in $\mathbb{P}^2$ and blowing down the three exceptional lines joining them.

The inverse map f^{-1} is also quadratic and the images of three exceptional lines of f are the indeterminacy points of f^{-1}.
The indeterminacy points $I(f^{-1}) = \{p(b_1), p(b_2), p(b_3)\}$ are given by

$$b_i = \lambda(a_i - \frac{2\nu_1}{3}), \qquad i = 1, 2, 3.$$

Rem.

$$-a_j - a_k = a_i - \nu_1.$$

Surface automorphism

We have

$$I(f) = \{p(a_1), p(a_2), p(a_3)\}$$

and

$$I(f^{-1}) = \{p(b_1), p(b_2), p(b_3)\}$$

If, for some positive integers n_1, n_2, n_3, and permutation $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$,

$$p(a_{\sigma(i)}) = f^{\circ(n_i - 1)} p(b_i), \qquad i = 1, 2, 3,$$

holds, then f lifts to a surface automorphism by blowing up $(n_1 + n_2 + n_3)$ points (provided they are all distinct)

$$p(b_i), f(p(b_i)), \cdots, f^{\circ(n_i - 1)}(p(b_i)), \qquad i = 1, 2, 3.$$

Orbit data

Positive integers (n_1, n_2, n_3) with permutation σ is said an **orbit data**.

Following Diller([D], 2011), we look for determinant λ and a quadratic birational transformation f, which maps C onto itself and realizes the prescribed orbit data.

In order to simplify our calculations, we suppose the fixed point of $t \mapsto \lambda(t + \frac{\nu_1}{3})$ is $\frac{1}{3}$, so that

$$\nu_1 = \frac{1}{\lambda} - 1.$$

Rem. The case without fixed point, $\lambda = 1$, and the case with fixed point $= 0$, are not treated.

Conditions

In terms of inner dynamics, the conditions are as follows.

$$a_{\sigma(i)} = \lambda^{n_i - 1}(b_i - \frac{1}{3}) + \frac{1}{3}, \quad i = 1, 2, 3,$$

$$b_i = \lambda a_i + \frac{2}{3}(\lambda - 1), \qquad i = 1, 2, 3,$$

$$a_1 + a_2 + a_3 = \frac{1}{\lambda} - 1.$$

Eliminate $a_i, b_i, i = 1, 2, 3$, to obtain an equation in λ, which is a necessary condition.

Polynomial equation and Picard coordinates(cyclic case)

Necessary condition $P(\lambda) = 0$ for cyclic permutation case $(\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1)$:

$$P(\lambda) = (\lambda - 2)\lambda^{n_1+n_2+n_3} + (\lambda - 1)(\lambda^{n_1+n_2} + \lambda^{n_2+n_3} + \lambda^{n_3+n_1})$$
$$+(\lambda - 1)(\lambda^{n_1} + \lambda^{n_2} + \lambda^{n_3}) + 2\lambda - 1.$$

If $\lambda^{n_i+n_j+n_k} \neq 1$, we have :

$$a_i = -\frac{\lambda^{n_k-1}(\lambda^{n_j}(\lambda^{n_i} + 1) + 1)(\lambda - 1)}{\lambda^{n_i+n_j+n_k} - 1} + \frac{1}{3}$$
$$((i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)).$$

Characteristic polynomial

From orbit data and a choice of a root of $P(\lambda)$, surface automorphism, $F : S \to S$, is induced by bowing-up the birational map $f : \mathbb{P}^2 \to \mathbb{P}^2$.

The cohomology group $H^2(S, \mathbb{Z})$ is spanned by the class of a generic line and the classes of the exceptional fibers. It is isomorphic to a Minkowski lattice $\mathbb{Z}^{1,n}$ with inner product

$$(x \cdot y) = x_0 y_0 - x_1 y_1 - x_2 y_2 - \cdots - x_n y_n$$

defined by the intersection product.

Orbit data determines the characteristic polynomial $P(\lambda)$ of $F^* : H^2(S, \mathbb{Z}) \to H^2(S, \mathbb{Z})$.

The polynomial $P(\lambda)$ obtained as a necessary condition and the characteristic polynomial $P(\lambda)$ coincide (not by chance).

Orbit data (1,3,5)*cyclic*

We choose orbit data $(1, 3, 5)$,*cyclic*.

In our case of orbit data $(1, 3, 5)$, *cyclic*, the characteristic polynomial is as follows.

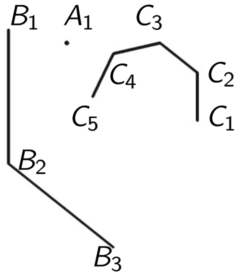
$$P(\lambda) = (\lambda - 1)(\lambda^2 - 1)(\lambda^7 - 1).$$

Among the orbit data with $n_1 + n_2 + n_3 = 9$, this is the only one case with factor $(\lambda^7 - 1)$.

In the case of a root of unity for a candidate of the choice of eigenvalue, we must be careful about the conflict and zero-division in the process.

We choose a primitive 7th root of unity $\kappa = \exp(2\pi i/7)$.
Picard coordinates of base points are computed by the formulas above.

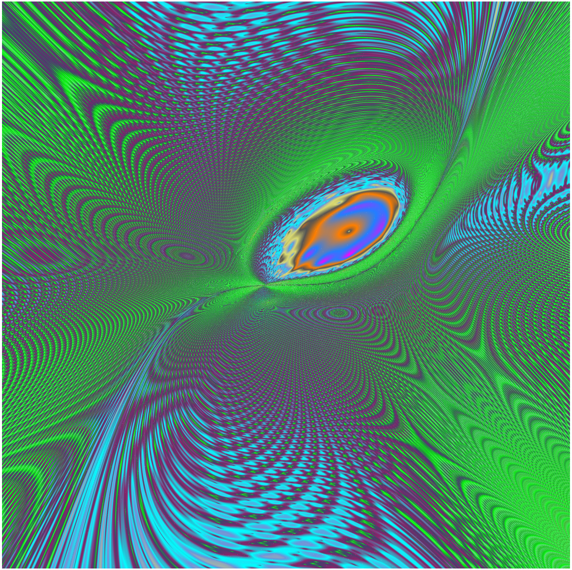
Picard coordinates for eigenvalue $\exp(\frac{2\pi i}{7})$ in $\mathrm{Pic}_0(X_{\mathrm{II}})$



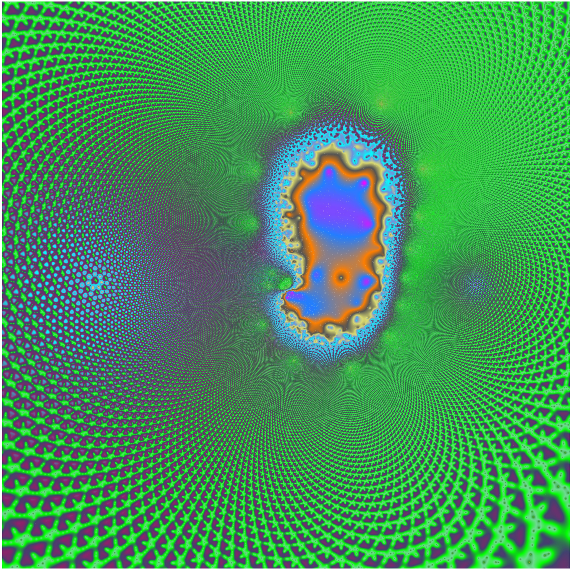
Surface automorphism $F : S \to S$.

PROPOSITION. There exists a surface automorphism $F : S \to S$, with invariant cuspidal cubic curve, realizing orbit data $(1, 3, 5)$*cyclic*, and the determinant κ (along the cuspidal cubic curve) which is a primitive 7th root of unity.

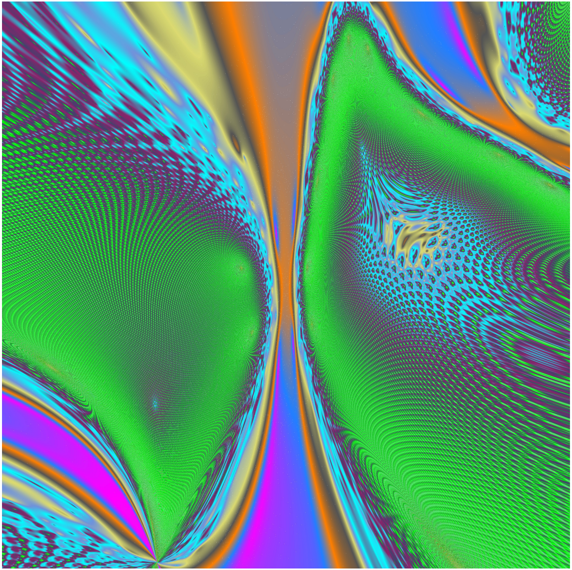
$F : S \to S$, CSPc135t1R, real slice



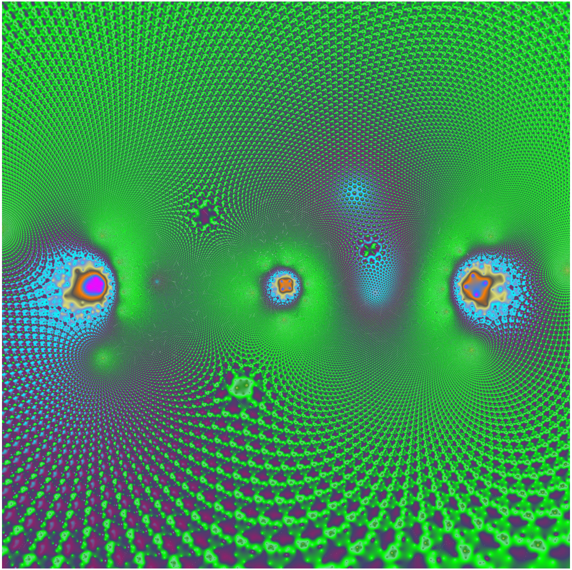
$F : S \to S$, CSPc135t1D, diagonal slice



$F : S \to S$, QLc135t2R, real slice



$F : S \to S$, QLc135t2D, diagonal slice



3. Elliptic fibration

Elliptic fibration of F

Surface automorphism F is as in the previous section.

Proposition. Surface automorphism, $F : S \to S$, preserves an elliptic fibration $\psi : S \to \mathbb{P}^1$, whose configuration of singular fibers is $\mathrm{III}\ \mathrm{II}\ \mathrm{I}_1^7$.

Proposition.

$$\mathrm{Aut}_\psi(\mathbb{P}^1) \cong \mathbb{Z}/7\mathbb{Z}.$$

Proposition.

$$\mathrm{Aut}_\sigma(S) \cong \mathbb{Z}/14\mathbb{Z}.$$

$F^* : H^2(S,\mathbb{Z}) \to H^2(S,\mathbb{Z})$

Now, let $A_1 \in H^2(S,\mathbb{Z})$ denote the cohomology class of the exceptional fiber $[\pi^{-1}(p(b_1))]$. Let $B_i = [\pi^{-1}(f^{i-1}(p(b_2)))]$, $i = 1,2,3$, and $C_i = [\pi^{-1}(f^{i-1}(p(b_3))]$, $i = 1,2,3,4,5$.

Let $H \in H^2(S,\mathbb{Z})$ denote the class of a generic line $[\pi^{-1}(L)]$.

A basis of $H^2(S,\mathbb{Z})$ is given by classes

$$H, A_1, B_1, B_2, B_3, C_1, C_2, C_3, C_4, C_5.$$

Automorphism $F^* : H^2(S,\mathbb{Z}) \to H^2(S,\mathbb{Z})$ acts as follows.

$$H \mapsto 2H - A_1 - B_3 - C_5,$$

$$A_1 \mapsto H - A_1 - B_3,$$

$$B_3 \mapsto B_2 \mapsto B_1 \mapsto H - B_3 - C_5,$$

$$C_5 \mapsto C_4 \mapsto C_3 \mapsto C_2 \mapsto C_1 \mapsto H - A_1 - C_5.$$

Anticanonical class

The characteristic polynomial of $F^* : H^2(S, \mathbb{Z}) \to H^2(S, \mathbb{Z})$ is

$$P(\lambda) = (\lambda - 1)(\lambda^2 - 1)(\lambda^7 - 1).$$

Eigenvalue $\lambda = 1$ has multiplicity 3.

The eigenspace of F^* for eigenvalue 1 is one-dimensional and spanned by the anti-canonical class.

$$\mathcal{F} = -K_S = 3H - A_1 - B_1 - B_2 - B_3 - C_1 - C_2 - C_3 - C_4 - C_5.$$

This is the class of the invariant cuspidal cubic curve.
This vector sits at the top of the Jordan block of size 3.
And it is the class of fibers of the elliptic fibration below.

Periodic curve of period 2

Periodic roots of period two are as follows. They are mapped to each other by F^*.

$$\mathcal{Q} = 2H - A_1 - B_1 - B_3 - C_1 - C_3 - C_5,$$

$$\mathcal{L} = H - B_2 - C_2 - C_4.$$

Especially, the vectors sum up to $\mathcal{F}$, as they form a singular fiber of the elliptic fibration (nodality is verified later).

Proposition. Classes $\mathcal{Q}$ and $\mathcal{L}$ are nodal roots and they consist a singular fiber of type III.

Nodal root

For Rational surface, following commutative diagram holds.

$$
\begin{array}{ccccccc}
 & 0 \longrightarrow & \mathrm{Pic}(S) & \xrightarrow{c_1} & H^2(S,\mathbb{Z}) & \longrightarrow 0, & \\
 & & \downarrow r & & \downarrow \iota^* & & \\
0 \to \mathrm{Pic}_0(X) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{\deg} & H^2(X,\mathbb{Z}) & \longrightarrow 0. &
\end{array}
$$

X : cuspidal cubic, three lines through a point, quadric with a tangent line

$\mathrm{Pic}_0(X) \simeq \mathbb{C}$,

X : nodal cubic (one, two, or three nodes)

$\mathrm{Pic}_0(X) \simeq \mathbb{C}/\mathbb{Z}$,

X : elliptic cubic

$\mathrm{Pic}_0(X) \simeq \mathbb{C}/\Lambda$.

Genus formula

If $\mathcal{R} \in H^2(S, \mathbb{Z})$ is a cohomology class of an irreducible component of a reducible singular fiber of the fibration, then

$$\mathcal{R}^2 = -2, \quad \text{and} \quad r \circ c_1^{-1}(\mathcal{R}) = 0.$$

The condition $r \circ c_1^{-1}(\mathcal{R}) = 0$ implies $\mathcal{R}$ is nodal, *i.e.* it represents the class of a curve.

And $\mathcal{R}^2 = -2$ implies the curve is isomorphic to a Riemann sphere.

The **arithmetic genus** of a curve C representing class $\mathcal{R}$ is

$$g(C) = \frac{1}{2}\mathcal{R}.(\mathcal{R} + K_S) + 1.$$

Singular fiber of type III

Periodic roots $\mathcal{Q}$ and $\mathcal{L}$ represent a cycle of periodic curves of period two.

$$\mathcal{Q}^2 = \mathcal{L}^2 = -2, \quad \mathcal{Q} \cdot \mathcal{L} = 2,$$

$$\mathcal{Q} + \mathcal{L} = -K_S, \quad K_S \cdot K_S = 0.$$

$$\mathcal{Q} \cdot K_S = \mathcal{L} \cdot K_S = 0.$$

The sum of Picard coordinates of components of $\mathcal{Q}$ and $\mathcal{L}$ vanish, so that

$$r \circ c_1^{-1}(\mathcal{Q}) = r \circ c_1^{-1}(\mathcal{L}) = 0 \in \mathrm{Pic}(X).$$

Next, we show that $\mathcal{Q}$ and $\mathcal{L}$ form a singular fiber of type III, we use the Lefschetz formula and the Atyah-Bott formula.

Lefschetz formula and Atyah-Bott formula

Suppose $F : S \to S$ satisfy $\det(DF - I) \neq 0$ at all fixed points.

Lefschetz formula :

$$\sum_{F(p)=p} \mathrm{sign}(\det(DF_p - I)) \;=\; \sum_{k=0}^{4} (-1)^k \mathrm{Tr}(F_*|_{H_k(S,\mathbb{R})}).$$

Atyah-Bott formula : for $r = 0, 1, 2$,

$$\sum_{F(z)=z} \frac{\mathrm{Tr} \wedge^r DF_z}{\det(I - DF_z)} = \sum_{s=0}^{4} (-1)^s \mathrm{Tr}(F^*|_{H^{r,s}(S)}).$$

Singular fiber of type III

From the characteristic polynomial

$$P(\lambda) = (\lambda - 1)(\lambda^2 - 1)(\lambda^7 - 1)$$

of $F^* : H^2(S, \mathbb{Z}) \to H^2(S, \mathbb{Z})$, and the Lefschetz formula, we see that $F : S \to S$ has three fixed points, a cycle of period two, (and a cycle of period 7).

Two fixed points are in the cuspidal cubic curve.

$\mathcal{Q}$ and $\mathcal{L}$ has one fixed point and a cycle of period two. So, they form a singular fiber of type III.

Eigenvalues of periodic points

Eigenvalues of the cuspidal fixed point in the invariant cuspidal cubic curve are κ^{-2} and κ^{-3}.

Eigenvalues at the fixed point $(\frac{1}{3}, \frac{1}{27})$ are κ and κ^{-6}.

Eigenvalues of the third fixed point, $-\kappa^3$ and $-\kappa^5$, can be computed from the determinant κ at the fixed point and the Atyah-Bott formula applied to F with $r = 0$.

Eigenvalues of the 2-cycle, κ^4 and κ^5, can be computed from the determinant κ^2 of the cycle and the Atyah-Bott formula applied to F^2 with $r = 0$.

Quadic and a tangent line

This shows that the 2-cycle is not hyperbolic.

$\mathcal{Q}$ and $\mathcal{L}$ form a singular fiber of type III, *i.e.*, quadric and a tangent line.

The eigenvalues of the 2-cycle and of the third fixed point imply the multiplier in the singular fiber, of type III, is κ^2.

REM. Surface automorphism derived from invariant cubic curve consisting of a quadric and a tangent line for orbit data $(1, 3, 5)$*cyclic* and eigenvalue κ^2, also has an invariant cuspidal cubic curve. This automorphism is conjugate to our automorphism.

Irreducible singular fibers I_1^7

Periodic roots of period 7 are

$$\mapsto A_1 - B_3 \mapsto B_1 - C_1 \mapsto B_2 - C_2 \mapsto B_3 - C_3 \mapsto$$

$$H - A_1 - B_1 - C_4 \mapsto H - B_1 - B_2 - C_5 \mapsto H - A_1 - B_2 - B_3 \mapsto,$$

and

$$\mapsto A_1 - C_3 \mapsto H - A_1 - C_1 - C_4 \mapsto H - B_1 - C_2 - C_5 \mapsto$$

$$H - A_1 - B_2 - C_3 \mapsto H - B_1 - B_3 - C_4 \mapsto H - B_2 - C_1 - C_5 \mapsto$$

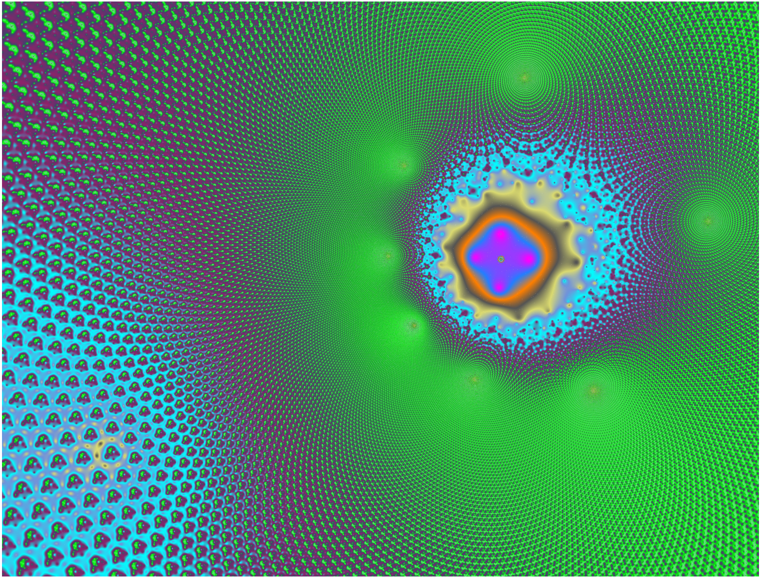
$$H - A_1 - B_3 - C_2 \mapsto .$$

These roots are not nodal, *i.e.*, the Picard sum of these roots are not zero.

However, the Lefschetz formula implies the existence of a cycle of period 7.

By counting the number of branches of J-map, I_7 is not possible.

Section



Configuration of singular fibers

Configuration of singular fibers for our automorphism

$F : S \to S$ is

$$\mathrm{III}\ \mathrm{II}\ \mathrm{I}_1^7.$$

$$P(\lambda) = (\lambda - 1)(\lambda^2 - 1)(\lambda^7 - 1).$$

Fiber	J	d	r	e	G	Pic_0	δ	p
II	0	7	6	2	0	$\mathbb{C}$	κ	21
III	1	7	6	3	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{C}$	κ^2	$\mathbf{1} + \mathbf{2}$
I_1^7	∞	7×1	0	7×1	$\mathbb{Z}/7\mathbb{Z}$	$\mathbb{C}^\times$	?	$\mathbf{7}$

Persson's list of configurations

In the list of configurations of singular fibers given by Persson([P],1990), those containing I_7 or I_1^7 are :

$$\mathrm{III}\ \mathrm{II}\ \mathrm{I}_1^7,\quad \mathrm{II}^2\ \mathrm{I}_7\ \mathrm{I}_1,\quad \mathrm{III}\ \mathrm{I}_7\ \mathrm{I}_1^2,\quad \mathrm{II}\ \mathrm{I}_7\ \mathrm{I}_2\ \mathrm{I}_1,\quad \mathrm{II}\ \mathrm{I}_7\ \mathrm{I}_1^3,$$

$$\mathrm{II}\ \mathrm{I}_3\ \mathrm{I}_1^7,\quad \mathrm{I}_7\ \mathrm{I}_2\ \mathrm{I}_1^3,\quad \mathrm{I}_7\ \mathrm{I}_1^5,\quad \mathrm{I}_3\ \mathrm{I}_2\ \mathrm{I}_1^7.$$

4. Mordell-Weil group

Mordell-Weil group

By specifying a section $\sigma : \mathbb{P}^1 \to S$, the set of sections of fibration $\psi : S \to \mathbb{P}^1$ form an additive group, regarding the specified section σ as the origin of each fiber. The addition of sections is defined by the group law in each smooth fiber as elliptic curve, and by taking the closure for a section. This group is called the Mordell-Weil group, $MW(S)$, of S.

Let T denote the subgroup of $NS(S) \cong H^2(S, \mathbb{Z})$ generated by the class of the specified section $O = [\sigma(\mathbb{P}^1)]$ and all the classes of irreducible components of fibers of the fibration.

Theorem(Shioda,[S],1990).

$$MW(S) \cong NS(S)/T. \qquad \text{(as group)}$$

Orthogonal projection

Let F_v denote reducible fiber with $F_v = \psi^{-1}(v)$, $v \in R \subset \mathbb{P}^1$,

$$F_v = \Theta_{v,0} + \cdots + \Theta_{v,m_v-1}, \qquad (O \cdot \Theta_i) = \delta_{0,i}$$

$$T_v = \Theta_{v,1} + \cdots + \Theta_{v,m_v-1},$$

$$T = < O > \oplus < \mathcal{F} > \oplus \bigoplus_{v \in R} T_v.$$

Let $L = T^\perp$ be the orthogonal complement of T in $NS(S)$ with respect to the intersection product. Orthogonal projection $\varphi : MW(S) \to L \otimes \mathbb{Q}$ can be expressed as follows.

$$\varphi(P) = (P) - (O) - ((P \cdot O) + \chi)\mathcal{F} +$$

$$+ \sum_{v \in R} (\Theta_{v,1}, \cdots, \Theta_{v,m_v-1})(-A_v^{-1}) \begin{pmatrix} (P \cdot \Theta_{v,1}) \\ \vdots \\ (P \cdot \Theta_{v,m_v-1}) \end{pmatrix}.$$

where $A_v = \{(\Theta_{v,i} \cdot \Theta_{v,j})\}_{1 \le i,j \le m_v-1}$, and $\chi = \chi(S) = -(O \cdot O) = 1$.

Root system

In our case with configuration of singular fibers $\mathrm{III}\ \mathrm{II}\ \mathrm{I}_1^7$, singular fiber of type III is reducible.

Fiber	I_m	I_m^*	II^*	III^*	IV^*	IV	III
T_v	A_{m-1}	D_{m+4}	E_8	E_7	E_6	A_2	A_1

Let $\mathrm{T} = \bigoplus_{v \in R} T_v$. The table of Oguiso and Shioda([OS] 1991) tells us:

No.	r	rkT	T	$\det \mathrm{T}$	n	$E(K)^0$	$E(K)$
2	7	1	A_1	2	1	E_7	E_7^*

.

$$
\begin{array}{ccccccccccc}
\circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ \\
 & & & & | & & & & & & \\
 & & & & \circ & & & & & &
\end{array}
$$

Rem. $\quad E(K)^0 \cong L = T^{\perp}, \quad E(K) \cong L^* \cong MW(S). \qquad$ (as group)

Group of the automorphisms

Together with the Karayayla's table, the group of the automorphisms of our surface S is:

$$\mathrm{Aut}(S) \cong E_7^* \rtimes \mathbb{Z}/14\mathbb{Z}.$$

We want to have a more concrete idea of the group.

Root system E_7

We choose section C_3 as the origin of $MW(S)$.
Trivial lattice T is spanned by C_3, $\mathcal{F}$, and $\mathcal{L}$, with

$$\mathcal{F} = 3H - A_1 - B_1 - B_2 - B_3 - C_1 - C_2 - C_3 - C_4 - C_5,$$

$$\mathcal{L} = H - B_2 - C_2 - C_4.$$

The orhogonal complement $L = T^{\perp}$ in $NS(S) \cong H^2(S, \mathbb{Z})$ is spanned by

$$s_0 = B_1 - A_1, \quad s_1 = B_3 - C_1, \quad s_2 = C_1 - C_5, \quad s_3 = C_5 - B_1,$$

$$s_4 = A_1 + B_1 + C_4 - H, \quad s_5 = C_2 - C_4, \quad s_6 = B_2 - C_2,$$

which form a basis of root system E_7.

$$\begin{array}{ccccccccccc} s_1 & - & s_2 & - & s_3 & - & s_4 & - & s_5 & - & s_6 \\ & & & & | & & & & & & \\ & & & & s_0 & & & & & & \end{array}$$

PROPOSITION([SS]) For a rational elliptic surface,

$$E(K) = \ < P; \bar{P} \cdot \bar{O} = 0 >$$

We have at least 8 sections, exceptional fibers of $\pi : S \to \mathbb{P}^2$, representing classes

$$A_1, B_1, B_2, B_3, C_1, C_2, C_4, C_5.$$

Their images $\varphi(A_1), \cdots, \varphi(C_5)$ span L^*, the dual of L.

In our case, $\varphi : MW(S) \to L^*$ can be expressed as follows.

$$\varphi(P) = (P) - (O) - ((P \cdot O) + \chi)\mathcal{F} + \frac{1}{2}(P \cdot \mathcal{L})\mathcal{L}.$$

where $O = C_3$ and $\chi = \chi(S) = -(O \cdot O) = 1$.

$\varphi : MW(S) \to L^*$

$$s_0 = B_1 - A_1, \quad s_1 = B_3 - C_1, \quad s_2 = C_1 - C_5, \quad s_3 = C_5 - B_1,$$
$$s_4 = A_1 + B_1 + C_4 - H, \quad s_5 = C_2 - C_4, \quad s_6 = B_2 - C_2.$$

$$\begin{aligned}
\varphi(A_1) = A_1 - C_3 - \mathcal{F} &= -3H + 2A_1 + B_1 + B_2 + B_3 + C_1 + C_2 + C_4 + C_5 \\
&= s_0 + s_1 + 2s_2 + 3s_3 + 3s_4 + 2s_5 + s_6, \\
\varphi(B_1) &= B_1 - C_3 - \mathcal{F} = \varphi(A_1) + s_0, \\
\varphi(B_3) &= \varphi(A_1) + s_0 + s_1 + s_2 + s_3, \\
\varphi(C_1) &= \varphi(A_1) + s_0 + s_2 + s_3, \\
\varphi(C_5) &= \varphi(A_1) + s_0 + s_3,
\end{aligned}$$

$\varphi : MW(S) \to L^*$

$$\begin{aligned}
\varphi(B_2) &= B_2 - C_3 - \mathcal{F} + \frac{1}{2}\mathcal{L} \\
&= B_2 - A_1 + \varphi(A_1) + \frac{1}{2}(H - B_2 - C_2 - C_4) \\
&= \varphi(A_1) + \frac{1}{2}(-(A_1 + B_1 + C_4 - H) + (B_1 - A_1) + (B_2 - C_2)) \\
&= \varphi(A_1) - \frac{1}{2}(s_4 - s_0 - s_6), \\
\varphi(C_2) &= \varphi(B_2) - s_6, \\
\varphi(C_4) &= \varphi(B_2) - s_5 - s_6.
\end{aligned}$$

The dual lattice L^* is generated by

$$s_0, s_1, s_2, s_3, s_5, s_6, \quad \text{and} \quad s_* = \frac{1}{2}(s_4 - s_0 - s_6).$$

Generators in $MW(S)$

These elements are expressed as

$$s_0 = \varphi(B_1) - \varphi(A_1), \quad s_1 = \varphi(B_3) - \varphi(C_1),$$

$$s_2 = \varphi(C_1) - \varphi(C_5), \quad s_3 = \varphi(C_5) - \varphi(B_1),$$

$$s_5 = \varphi(C_2) - \varphi(C_4), \quad s_6 = \varphi(B_2) - \varphi(C_2),$$

$$s_* = \varphi(A_1) - \varphi(B_2).$$

We see that 7 elements in $MW(S)$:

$$B_1 - A_1, \quad B_3 - C_1, \quad C_1 - C_5, \quad C_5 - B_1,$$

$$C_2 - C_4, \quad B_2 - C_2, \quad A_1 - B_2$$

are "linearly independent" and form a basis (of type A_7).

These elements specify translations in each fiber of the elliptic fibration.

5. Homomorphism $MW(S) \to \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$

Nodal root

For Rational surface, following commutative diagram holds.

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & \mathrm{Pic}(S) & \xrightarrow{c_1} & H^2(S,\mathbb{Z}) & \longrightarrow & 0, \\ & & & & \downarrow r & & \downarrow \iota^* & & \\ 0 & \to & \mathrm{Pic}_0(X) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{\deg} & H^2(X,\mathbb{Z}) & \longrightarrow & 0. \end{array}$$

X : cuspidal cubic, three lines through a point, quadric with a tangent line : $\mathrm{Pic}_0(X) \simeq \mathbb{C}$,
X : nodal cubic (one, two, or three nodes) : $\mathrm{Pic}_0(X) \simeq \mathbb{C}/\mathbb{Z}$,
X : elliptic cubic : $\mathrm{Pic}_0(X) \simeq \mathbb{C}/\Lambda$.

REM. For rational elliptic surface S,

$$NS(S) \cong \mathrm{Pic}(S) \cong H^2(S,\mathbb{Z}).$$

Homomorphism $MW(S) \to \mathrm{Aut}(F_v)$

Since element $P \in MW(S)$, regarded as an automorphism of S preserving the fibration and sending the specified section O to section P, P induces an automorphism of each fiber F_v.

We consider $P - O$ in place of P, to be more compatible with the group structure as automorphisms.

Element $A - B \in MW(S)$ should be considered as $(A - O) - (B - O)$.

We denote $\mathcal{MW}(S) = \{A - B \mid A, B \in MW(S)\}$.

Especially, we have homomorphisms

$$\mathbf{r}_v : \mathcal{MW}(S) \to \mathrm{Aut}(\mathrm{Pic}(F_v)),$$

for each fiber F_v.

In our case

In our case, invariant cubic curve X_{III} consists of quadric $\mathcal{Q}$ and a tangent line $\mathcal{L}$.

$$
\begin{array}{ccccccccc}
 & 0 & \longrightarrow & \mathrm{Pic}(S) & \xrightarrow{c_1} & H^2(S,\mathbb{Z}) & \longrightarrow & 0, \\
 & & & \downarrow r_{\mathrm{III}} & & \downarrow \iota^* & & \\
0 \to & \mathbb{C} & \longrightarrow & \mathrm{Pic}(X_{\mathrm{III}}) & \xrightarrow{\deg} & \mathbb{Z}^2 & \longrightarrow & 0.
\end{array}
$$

Group structure of singular fiber

Let $\mathbb{G}_m \cong \mathbb{C}/\mathbb{Z}, \quad \mathbb{G}_a \cong \mathbb{C}.$

PROPOSITION(Shioda [SS]). The singular fibers of elliptic surfaces admit the follwing group structure :

$$\begin{aligned}
\mathbb{G}_m \times G(F_v) &: G(\mathrm{I}_n) \cong \mathbb{Z}/n\mathbb{Z}, \\
\mathbb{G}_a \times G(F_v) &: G(\mathrm{I}^*_{2m}) \cong (\mathbb{Z}/2\mathbb{Z})^2, \\
& \quad G(\mathrm{I}^*_{2m+1}) \cong \mathbb{Z}/4\mathbb{Z}, \\
& \quad G(\mathrm{II}) \cong G(\mathrm{II}^*) \cong \{0\}, \\
& \quad G(\mathrm{III}) \cong G(\mathrm{III}^*) \cong \mathbb{Z}/2\mathbb{Z}, \\
& \quad G(\mathrm{IV}) \cong G(\mathrm{IV}^*) \cong \mathbb{Z}/3\mathbb{Z}.
\end{aligned}$$

Picard automorphism

For $F_v = X_{\mathrm{III}}$, we have $\mathbb{G}_a \cong \mathbb{C}$, $G(\mathrm{III}) \cong \mathbb{Z}/2\mathbb{Z}$.

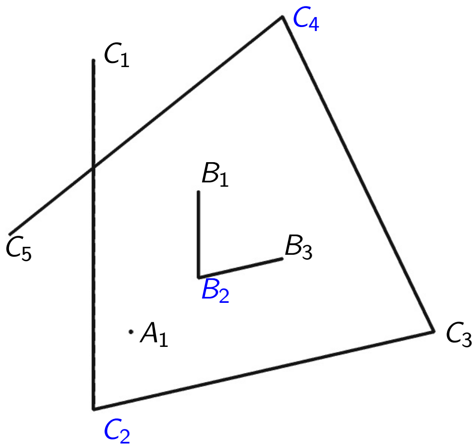
Consider the map $\mathbf{r}_{\mathrm{III}} : \mathcal{MW}(S) \to \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$.

For $A - B \in \mathcal{MW}(S)$,

$$\mathbf{r}_{\mathrm{III}}(A - B) = ((r(A) - r(B)), ((A \cdot \mathcal{L}) - (B \cdot \mathcal{L})) \bmod 2).$$

The Picard coordinates in X_{III} is not canonically defined, but the group structure of translations in X_{III} is well defined.

Picard coordinates for eigenvalue $\exp(\frac{4\pi i}{7})$ in $\mathrm{Pic}_0(X_{\mathrm{III}})$



Picard coordinates of sections

Let r_0, r_1, r_2, r_3, r_5, r_6 and r_* denote the Picard coordinates of 7 elements in $\mathcal{MW}(S)$:

$$B_1 - A_1, \quad B_3 - C_1, \quad C_1 - C_5, \quad C_5 - B_1,$$

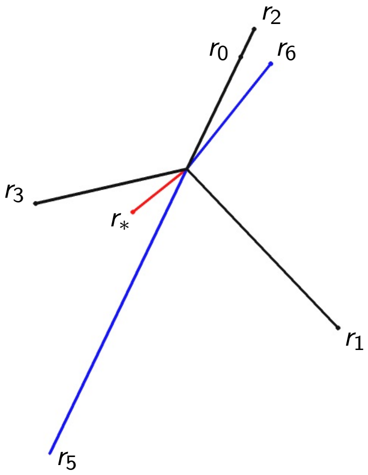
$$C_2 - C_4, \quad B_2 - C_2, \quad A_1 - B_2,$$

which are plotted in the next slide.

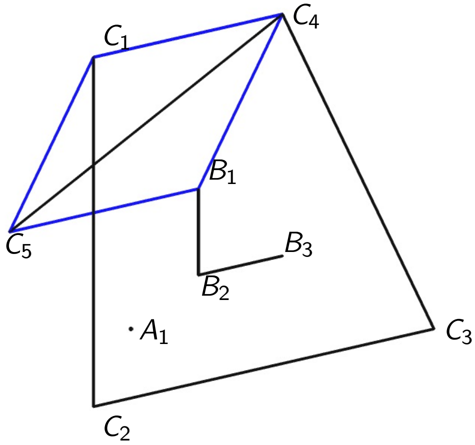
r_0, r_2 and r_5 are on a line.
These vectors span a $\mathbb{Z}$-module of rank 6 with relation

$$r_0 + r_2 + r_5 + r_6 + r_* = 0.$$

Picard coordinates for eigenvalue $\exp(\frac{4\pi i}{7})$ in $\mathbb{G}_a$



Picard coordinates for eigenvalue $\exp(\frac{4\pi i}{7})$ in $\mathrm{Pic}_0(X_{\mathrm{III}})$



$\mathbf{r}_{\mathrm{III}} : \mathcal{MW}(S) \to \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$

Homomorphism

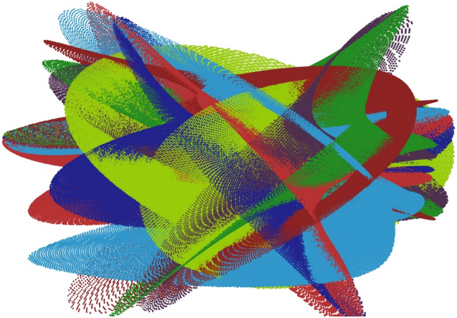
$$\mathbf{r}_{\mathrm{III}} : \mathcal{MW}(S) \to \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$$

is given by

$$\begin{aligned}
\mathbf{r}_{\mathrm{III}}(B_1 - A_1) &= (r_0, 0),\\
\mathbf{r}_{\mathrm{III}}(B_3 - C_1) &= (r_1, 0),\\
\mathbf{r}_{\mathrm{III}}(C_1 - C_5) &= (r_2, 0),\\
\mathbf{r}_{\mathrm{III}}(C_5 - B_1) &= (r_3, 0),\\
\mathbf{r}_{\mathrm{III}}(C_2 - C_4) &= (r_5, 0),\\
\mathbf{r}_{\mathrm{III}}(B_2 - C_2) &= (r_6, 0),\\
\mathbf{r}_{\mathrm{III}}(A_1 - B_2) &= (r_*, 1),
\end{aligned}$$

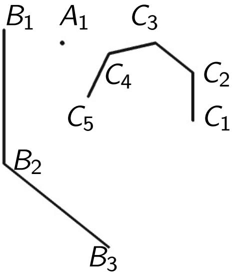
These span a lattice of “rank 7” in $\mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$.

Thank you.



Thank you.

Picard coordinates for eigenvalue $\exp(\frac{2\pi i}{7})$ in $\mathrm{Pic}_0(X_{\mathrm{II}})$



Generalized eigenvector

Generalized eigenvector for F^* is:

$$H - 2B_1 - B_2 - 2C_1 - C_2 + C_4 + 2C_5$$

$$= H - A_1 - B_1 - C_4 - (B_1 - A_1) - 2(C_1 - C_5) - 2(C_2 - C_4) - (B_2 - C_2)$$

$$= -s_0 - 2s_2 - s_4 - 2s_5 - s_6.$$

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