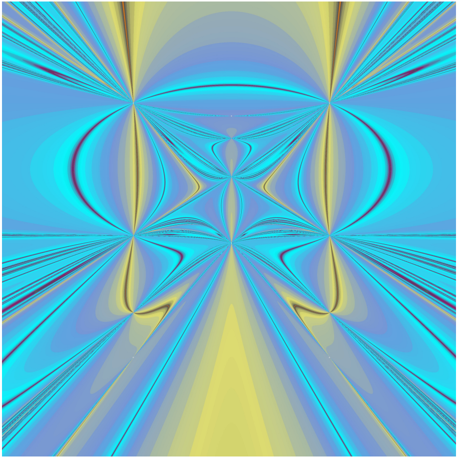
Dynamics in the Julia set of a Surface Automorphism



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Abstract

Automorphisms of complex surfaces can have various dynamics in the Julia set.

In this note, we show an example suggesting the dynamics in the Julia set.

Computer pictures suggest that the dynamics in the Julia set is semi-conjugate to the natural extension of a subshift of finite type.

This dynamics is a kind of Markov chain.

Contents

1. Surface automorphism

Surface automorphism

For $\lambda \in \mathbb{C}^\times$, let us consider a birational map $f : (x, y) \mapsto (X, Y)$, defined by

$$X = \frac{x(y+1)}{y + \frac{3}{2}x^2 - \frac{1}{2}}, \quad Y = \lambda \frac{y^2 - \frac{3}{4}x^2 - \frac{1}{4}}{y + \frac{3}{2}x^2 - \frac{1}{2}}.$$

Its inverse map $f^{-1} : (X, Y) \mapsto (x, y)$ is given by

$$x = \frac{X(\lambda^{-1}Y - 1)}{\lambda^{-1}Y - \frac{3}{2}X^2 + \frac{1}{2}}, \quad y = \frac{(\lambda^{-1}Y)^2 - \frac{3}{4}X^2 - \frac{1}{4}}{\lambda^{-1}Y - \frac{3}{2}X^2 + \frac{1}{2}}.$$

Invariant cubic curve

$$X = \frac{x(y+1)}{y + \frac{3}{2}x^2 - \frac{1}{2}}, \quad Y = \lambda \frac{y^2 - \frac{3}{4}x^2 - \frac{1}{4}}{y + \frac{3}{2}x^2 - \frac{1}{2}}.$$

This biratinal map preserves three lines $\{x = 0\}$, $\{x = 1\}$, $\{x = -1\}$, passing through a point at infinity.

Each of these lines is mapped onto itself.

$$\begin{aligned} (0, y) &\mapsto (0, \lambda(y + \frac{1}{2})), \\ (1, y) &\mapsto (1, \lambda(y - 1)), \\ (-1, y) &\mapsto (-1, \lambda(y - 1)). \end{aligned}$$

Picard parametrization

$$
\begin{aligned}
(0, y) &\mapsto (0, \lambda(y + \frac{1}{2})), \\
(1, y) &\mapsto (1, \lambda(y - 1)), \\
(-1, y) &\mapsto (-1, \lambda(y - 1)).
\end{aligned}
$$

Define a parametrization of cubic curve $x^3 - x = 0$, by

$$
\mathbf{p}_1 : t \mapsto (0, \frac{1}{2}t), \quad \mathbf{p}_2 : t \mapsto (1, -t), \quad \mathbf{p}_3 : t \mapsto (-1, -t).
$$

Through this parametrization, the dynamics in the cubic curve is described as

$$
t \mapsto \lambda(t + 1).
$$

PROPOSITION. For $t_1, t_2, t_3 \in \mathbb{C}$,

$t_1 + t_2 + t_3 = 0$ if and only if $\mathbf{p}_1(t_1), \mathbf{p}_2(t_2), \mathbf{p}_3(t_3)$ are on a line.

Dynamics in the Poicard coordinate

Assume $\lambda \neq 1$.

$$t \mapsto \lambda(t+1).$$

Fixed point is : $t = \frac{\lambda}{1-\lambda}$. Its k-th iteration:

$$t \mapsto \lambda^k t + \lambda^k + \lambda^{k-1} + \cdots + \lambda.$$

Three fixed points of f are:

$$(0, \frac{1}{2}\frac{\lambda}{1-\lambda}), \quad (1, -\frac{\lambda}{1-\lambda}), \quad (-1, -\frac{\lambda}{1-\lambda}).$$

These are (non-flip) saddles with eigenvalues λ and λ^{-4}.

(The fixed point at infinity is a source (or a sink, or a center of a rotation domain of rank 1) with eigenvalues λ^{-1} and λ^{-1}.)

Indeterminacy points

$f(x, y) = (X, Y);$

$$X = \frac{x(y+1)}{y + \frac{3}{2}x^2 - \frac{1}{2}}, \quad Y = \lambda \frac{y^2 - \frac{3}{4}x^2 - \frac{1}{4}}{y + \frac{3}{2}x^2 - \frac{1}{2}}.$$

Indeterminacy points of f are as follows.

$$(0, \frac{1}{2}), \ \ (1, -1), \ \ (-1, -1).$$

Their Picard coordinates are all $t = 1$.

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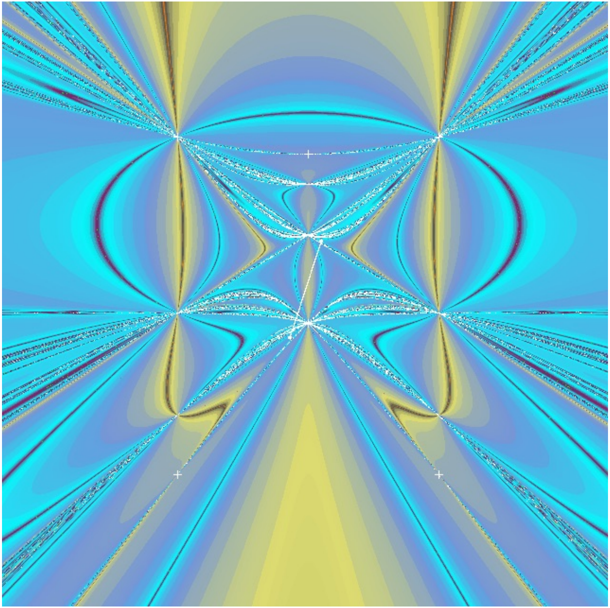


Indeterminay points of the inverse map

$f^{-1}(X, Y) = (x, y);$

$$x = \frac{X(\lambda^{-1}Y - 1)}{\lambda^{-1}Y - \frac{3}{2}X^2 + \frac{1}{2}}, \quad y = \frac{(\lambda^{-1}Y)^2 - \frac{3}{4}X^2 - \frac{1}{4}}{\lambda^{-1}Y - \frac{3}{2}X^2 + \frac{1}{2}}.$$

The indeterminacy points of the inverse map f^{-1} are as follows.

$$(0, -\frac{1}{2}\lambda), \quad (1, \lambda), \quad (-1, \lambda).$$

In Picard coordintes, $t = -\lambda$, for all.

Surface automorphism

To construct a surface automorphism from a quadratic birational map, we require that the indeterminacy points of the inverse map, which are exceptional values of f, is mapped to the indeterminacy points of f by some iterate of f.

In our case, dynamics and indeterminacy points in three lines are the same in the Picard coordinates.

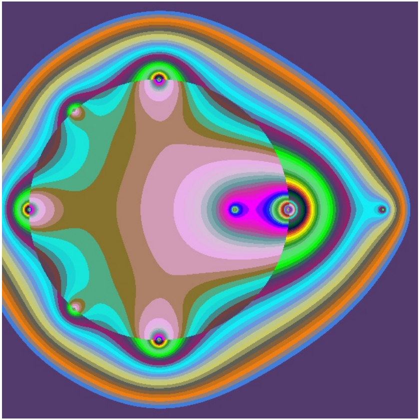
In this note, we try to construct a surface automorphism with orbit data $(4, 4, 4)$, *id*., *i.e.*,

$$1 = \lambda^3(-\lambda) + \lambda^3 + \lambda^2 + \lambda,$$

or

$$\lambda^4 - \lambda^3 - \lambda^2 - \lambda + 1 = 0.$$

Quadratic birational transformation



Let $\lambda = 0.580691832...$ be the smallest positive real root of equation

$$z^4 - z^3 - z^2 - z + 1 = 0.$$

Blow-up base

Let

$$\begin{aligned} t_1 &= -\lambda, \\ t_2 &= -\lambda^2 + \lambda, \\ t_3 &= -\lambda^3 + \lambda^2 + \lambda, \\ t_4 &= -\lambda^4 + \lambda^3 + \lambda^2 + \lambda, \end{aligned}$$

$$\mathbf{a}_i = \mathbf{p}_1(t_i), \quad \mathbf{b}_i = \mathbf{p}_2(t_i), \quad \mathbf{c}_i = \mathbf{p}_3(t_i), \qquad i = 1, 2, 3, 4.$$

By blowing-up these twelve points, f defines a surface automorphism.

Real slice

Let $X_{\mathbb{C}}$ denote the complex surface obtained by these blow-ups, and let

$$F_{\mathbb{C}} : X_{\mathbb{C}} \to X_{\mathbb{C}}$$

denote the induced surface automorphism.

As we took $\lambda \in \mathbb{R}$, the real axis $\mathbb{RP}^2 \subset \mathbb{CP}^2$ is invariant under f, the subset $X_{\mathbb{R}}$, obtained by blowing-up the twelve points in the real way, is invariant under $F_{\mathbb{C}}$,

$$X_{\mathbb{R}} \subset X_{\mathbb{C}}$$

and $F_{\mathbb{C}}$ induces a real analytic automrphism, $F_{\mathbb{R}} = F_{\mathbb{C}} \mid_{X_{\mathbb{R}}}$,

$$F_{\mathbb{R}} : X_{\mathbb{R}} \to X_{\mathbb{R}}.$$

2. Homology

Our complex surface $X_{\mathbb{C}}$ is obtained by blowing-up the complex projective space $\mathbb{CP}^2$ in twelve points.

Let $\mathcal{L}$ denote the homology class of a generic complex line in $\mathbb{CP}^2$.

The homology classes of the exceptional fibers at base points $\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i,\ i = 1, 2, 3, 4$, will be denoted as $\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i,\ i = 1, 2, 3, 4$.

Homology group $H_2(X_{\mathbb{C}}, \mathbb{Z}) \simeq \mathbb{Z}^{1,12}$ is generated by

$$\mathcal{L}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4.$$

Homomorphism induced by automorphism $F_{\mathbb{C}}$ acts as follows.

$$\mathcal{L} \mapsto 2\mathcal{L} - \mathcal{A}_1 - \mathcal{B}_1 - \mathcal{C}_1,$$

$$\mathcal{A}_1 \mapsto \mathcal{A}_2, \quad \mathcal{A}_2 \mapsto \mathcal{A}_3, \quad \mathcal{A}_3 \mapsto \mathcal{A}_4, \quad \mathcal{A}_4 \mapsto \mathcal{L} - \mathcal{B}_1 - \mathcal{C}_1,$$

$$\mathcal{B}_1 \mapsto \mathcal{B}_2, \quad \mathcal{B}_2 \mapsto \mathcal{B}_3, \quad \mathcal{B}_3 \mapsto \mathcal{B}_4, \quad \mathcal{B}_4 \mapsto \mathcal{L} - \mathcal{A}_1 - \mathcal{C}_1,$$

$$\mathcal{C}_1 \mapsto \mathcal{C}_2, \quad \mathcal{C}_2 \mapsto \mathcal{C}_3, \quad \mathcal{C}_3 \mapsto \mathcal{C}_4, \quad \mathcal{C}_4 \mapsto \mathcal{L} - \mathcal{A}_1 - \mathcal{B}_1.$$

Characteristic polynomial

These data describe the linear isomorphism $F_{\mathbb{C}*} : H_2(X_{\mathbb{C}}, \mathbb{Z}) \to H_2(X_{\mathbb{C}}, \mathbb{Z})$, whose characteristic polynomial, $\chi_{\mathbb{C}}(z)$, is as follows.

$$\chi_{\mathbb{C}}(z) = z^{13} - 2z^{12} + 3z^8 - 3z^5 + 2z - 1,$$

or

$$\chi_{\mathbb{C}}(z) = (z-1)(z^4-1)^2(z^4 - z^3 - z^2 - z + 1).$$

We denote by $C_{\mathbb{C}}$, the cyclotomic factor and by $S_{\mathbb{C}}$, the Salem factor.

$$C_{\mathbb{C}}(z) = (z-1)(z^4-1)^2, \quad S_{\mathbb{C}}(z) = z^4 - z^3 - z^2 - z + 1.$$

Eigenvalues and traces

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of $n \times n$-matrix L, with characteristic polynomial

$$\det(zI - L) = P(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n.$$

Let

$$\tau_k = \sum_{i=1}^{n} \lambda_i^k = \mathrm{trace}(L^k),$$

$$\sigma_k = \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k} = (-1)^k a_k.$$

Then

$$\begin{aligned}
\tau_\ell \sigma_k &= (\sum_{i_0} \lambda_{i_0}^\ell)(\sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}) \\
&= \sum_{i_1 < i_2 < \cdots < i_k} \quad \sum_{i_0 \in \{i_1, \cdots, i_k\}} \lambda_{i_0}^\ell \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k} \\
&+ \sum_{i_1 < i_2 < \cdots < i_k} \quad \sum_{i_0 \notin \{i_1, \cdots, i_k\}} \lambda_{i_0}^\ell \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}
\end{aligned}$$

Traces

Traces τ_k can be computed from the coefficients of the characteristic polynomial, inductively.

$$\sigma_k = (-1)^k a_k,$$

$$\tau_k = \tau_{k-1}\sigma_1 - \tau_{k-2}\sigma_2 + \cdots + (-1)^k \tau_1 \sigma_{k-1} + (-1)^{k+1} k \sigma_k.$$

$$\tau_1 = -a_1,$$

$$\tau_2 = a_1^2 - 2a_2,$$

$$\tau_3 = -a_1^3 + 3a_1 a_2 - 3a_3,$$

$$\tau_4 = a_1^4 - 4a_1^2 a_2 + 2a_2^2 + 4a_1 a_3 - 4a_4.$$

We denote as $\tau_k(P)$ to indicate the trace of linear map with characteristic polynomial $P(z)$.

3. Lefschetz formula

Lefschetz number

Let $|K|$ be a finite polyhedra, and let $f : |K| \to |K|$ be a continuous map.

Let $T_i(|K|)$ denote the torsion subgroup of the homology group $H_i(|K|, \mathbb{Z})$.

Let $B_i(|K|) = H_i(|K|, \mathbb{Z}) / T_i(|K|)$.

f induces a homomorphism $f_*|_{B_i(|K|)} : B_i(|K|) \to B_i(|K|)$.

Lefschetz number $\Lambda(f)$ of f is defined by

$$\Lambda(f) = \sum_{i=0}^{\dim K} (-1)^i \operatorname{trace}(f_*|_{B_i(|K|)}).$$

Lefschetz formula

Suppose M is a compact smooth manifold without boundary.
And suppose $f : M \to M$ is a differentiable map satisfying $\det(Df - I) \neq 0$ at all fixed points.

The **Lefschetz index** of fixed point p of f is defined as

$$\mathrm{Ind}(f; p) \;=\; \mathrm{sign}(\det(Df_p - I)).$$

The **Lefschetz formula** is

$$\sum_{f(p)=p} \mathrm{Ind}(f; p) \;=\; \sum_{k=0}^{\dim M} (-1)^k \mathrm{trace}(f_*|_{H_k(M,\mathbb{R})}).$$

Surface automorphism $F_{\mathbb{C}}$

Let us consider the case of $F_{\mathbb{C}} : X_{\mathbb{C}} \to X_{\mathbb{C}}$.

$$\begin{aligned} H_0(X_{\mathbb{C}}, \mathbb{Z}) &\simeq \mathbb{Z}, & \mathrm{trace}(F_*|_{H_0}) &= 1, \\ H_2(X_{\mathbb{C}}, \mathbb{Z}) &\simeq \mathbb{Z}^{1,12}, & \mathrm{trace}(F_*|_{H_2}) &= 2, \\ H_4(X_{\mathbb{C}}, \mathbb{Z}) &\simeq \mathbb{Z}, & \mathrm{trace}(F_*|_{H_4}) &= 1. \end{aligned}$$

By the Lefschetz formula, we conclude that $F_{\mathbb{C}}$ has four fixed points, because in the complex dynamical system case, Lefschetz index is always 1.

The characteristic polynomial of $F_{\mathbb{C}*}|_{H_2}$ is :

$$\chi_{\mathbb{C}}(z) = z^{13} - 2z^{12} + 3z^8 - 3z^5 + 2z - 1,$$

or

$$\chi_{\mathbb{C}}(z) = (z-1)(z^4-1)^2(z^4 - z^3 - z^2 - z + 1).$$

Then

$$\Lambda(F_{\mathbb{C}}^k) = 2 + \mathrm{trace}(F_{\mathbb{C}*}^k|_{H_2}).$$

gives the number of fixed points of $F_{\mathbb{C}}^k$.

Trace of $F_{\mathbb{C}*}^k|_{H_2}$ can be computed from the coefficients of the characteristic polynomial.

Partial trace

In the followings, we shall examine the contribution of each eigenvalue to periodic orbits.

Our characteristic polynomial factorizes into cyclotomic factors, $C_{\mathbb{C}}(z)$, and a Salem polynomial, $S_{\mathbb{C}}(z)$.

$$C_{\mathbb{C}}(z) = (z-1)(z^4-1)^2, \qquad S_{\mathbb{C}}(z) = z^4 - z^3 - z^2 - z + 1.$$

To describe periodic cycles in terms of Lefschetz index, for $m \in \mathbb{N}$ and $k \in \mathbb{Z}$, let

$$\mathbf{m}(k) = \begin{cases} m & k \equiv 0 \pmod{m} \\ 0 & \text{otherwise} \end{cases}.$$

Then

$$\Lambda(F_{\mathbb{C}}^k) = 3 \cdot \mathbf{1}(k) + 2 \cdot \mathbf{4}(k) + \tau_k(S_{\mathbb{C}}).$$

$$\tau_1(S_{\mathbb{C}}) = 1, \quad \tau_2(S_{\mathbb{C}}) = 3, \quad \tau_3(S_{\mathbb{C}}) = 7, \quad \tau_4(S_{\mathbb{C}}) = 7.$$

4. Real slice

Real surface

Consider the real slice $F_{\mathbb{R}} : X_{\mathbb{R}} \to X_{\mathbb{R}}$.

$X_{\mathbb{R}}$ is a non-orientable real 2-dimensional analytic manifold without boundary.

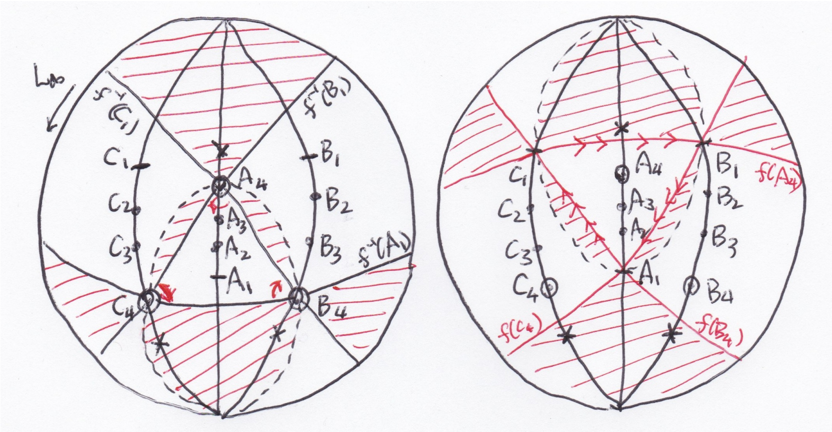
In this case, $X_{\mathbb{R}}$ is invariant under $F_{\mathbb{C}} : X_{\mathbb{C}} \to X_{\mathbb{C}}$.

$$
\begin{aligned}
&H_0(X_{\mathbb{R}}, \mathbb{Z}) \simeq \mathbb{Z}, \\
&H_1(X_{\mathbb{R}}, \mathbb{Z}) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}^{12}, \qquad B_1(X_{\mathbb{R}}) \simeq \mathbb{Z}^{12}, \\
&H_2(X_{\mathbb{R}}, \mathbb{Z}) \simeq 0.
\end{aligned}
$$

Let A_i, B_i, C_i denote the 1-dimensional homology class representing the real slice of $\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i$, respectively, with the clockwise orientation, for $i = 1, 2, 3, 4$.

And let L_∞ denote the homology class of the line at infinity with counter-clockwise orientation.

$F_{\mathbb{R}} : X_{\mathbb{R}} \to X_{\mathbb{R}}$



Homology group

Let Q denote the homology class of the invariant cubic curve.

$$Q = L_\infty + A_1 + A_2 + A_3 + A_4 + B_1 + B_2 + B_3 + B_4 + C_1 + C_2 + C_3 + C_4.$$

Immediately, we see that $2Q = 0$ in $H_1(X_\mathbb{R})$. The torsion subgroup $T_1(X_\mathbb{R})$ is generated by Q.

The quotient group $B_1(X_\mathbb{R})$ is spanned by $A_i, B_i, C_i, \ i = 1, 2, 3, 4.$, by eliminating L_∞ and setting Q to be 0.

In the followings, we abuse these homology classes with those in $B_1(X_\mathbb{R})$.

That is, homology classes are regarded as that modulo Q.

Local index

In our case, $F_{\mathbb{R}}$ has an eigenform $\eta = \frac{dx \wedge dy}{x^3 - x}$ satisfying $F_{\mathbb{R}}^* \eta = \lambda^{-1} \eta$, in the complement of the invariant cubic curve $\{x^3 - x = 0\}$.

$\lambda > 0$ is a reciprocal of a Salem number.

If $p \in X_{\mathbb{R}} \setminus Q$ is an isolated fixed point of $F_{\mathbb{R}}^m$, then $\det DF_{\mathbb{R}}^m|_p = \lambda^m > 0$.

If p is a sink or a source of period m, then $\mathrm{Ind}(f^m, p) = 1$, since

$$\mathrm{Ind}(f^m, p) = \mathrm{sign}(\det(Df_p^m - I)).$$

If p is a saddle of period m, with eigenvalues μ_1, μ_2, then

$$\mathrm{Ind}(f^m, p) = 1, \quad \text{if} \quad \mu_1 < -1 < \mu_2 < 0. \quad \text{(bi-flip saddle)}$$

$$\mathrm{Ind}(f^m, p) = -1, \quad \text{if} \quad 0 < \mu_1 < 1 < \mu_2. \quad \text{(non-flip saddle)}$$

Lefschetz index of saddles

If p is a bi-flip saddle of period m, with $\mathrm{Ind}(f^m, p) = 1$, then

$$\mathrm{Ind}(f^{jm}, p) = -(-1)^j, \qquad j = 1, 2, \cdots .$$

If p is a non-flip saddle of period m, with $\mathrm{Ind}(f^m, p) = -1$, then

$$\mathrm{Ind}(f^{jm}, p) = -1, \qquad\qquad j = 1, 2, \cdots .$$

Using notation $\quad \mathbf{m}(k) = \begin{cases} m & k \equiv 0 \pmod{m} \\ 0 & \text{otherwise} \end{cases}$,

$$\mathrm{Ind}(F_{\mathbb{R}}^k, p) = \mathbf{m}(k), \qquad \text{for source or sink of period } m,$$

$$\mathrm{Ind}(F_{\mathbb{R}}^k, p) = \mathbf{m}(k) - \mathbf{2m}(k), \qquad \text{for bi-flip saddle of period } m,$$

$$\mathrm{Ind}(F_{\mathbb{R}}^k, p) = -\mathbf{m}(k), \qquad \text{for non-flip saddle of period } m.$$

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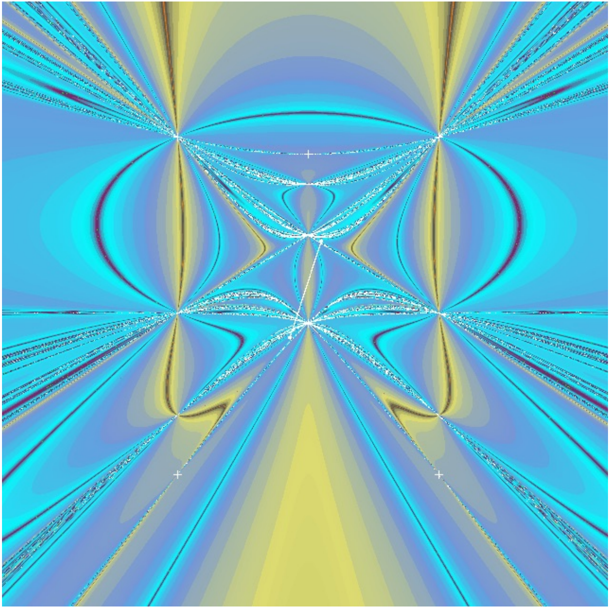


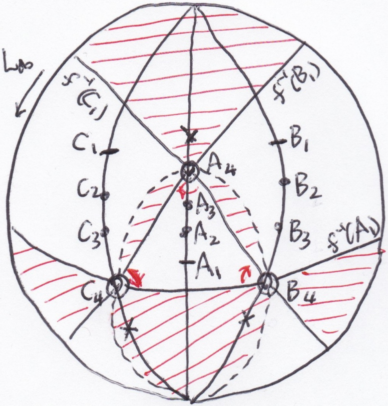
Jacobian of $F_{\mathbb{R}}$

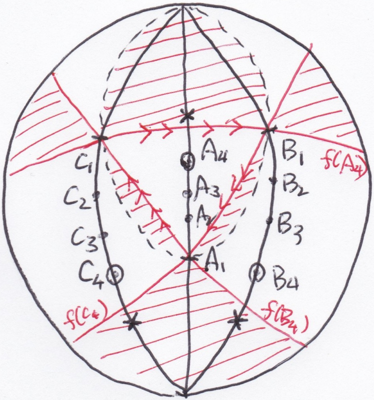
The determinant of $F_{\mathbb{R}}$ is as follows.

$$\det(DF_{\mathbb{R}}) = \frac{\lambda(y+1)(y-\frac{1}{2}+\frac{3}{2}x)(y-\frac{1}{2}-\frac{3}{2}x)}{(y-\frac{1}{2}+\frac{3}{2}x^2)^3}.$$

The determinant of $F_{\mathbb{R}}$, with respect to the real coordinates (x, y), changes sign across lines of critical points, curve of poles, and across the line at infinity.

$F_{\mathbb{R}} : X_{\mathbb{R}} \to X_{\mathbb{R}}$





Homomorphism $F_{\mathbb{R}*}$

Indeterminacy points A_4, B_4, C_4 are mapped to lines through two of the blown down points A_1, B_1, C_1.

By $F_{\mathbb{R}*}$, generators of $H_1(X_{\mathbb{R}}, \mathbb{Z})$ are mapped as follows.

$$A_1 \mapsto -A_2, \quad A_2 \mapsto -A_3, \quad A_3 \mapsto -A_4,$$
$$A_4 \mapsto Q + A_1 + A_2 + A_3 + A_4 + B_2 + B_3 + B_4 + C_2 + C_3 + C_4,$$
$$B_1 \mapsto -B_2, \quad B_2 \mapsto -B_3, \quad B_3 \mapsto -B_4,$$
$$B_4 \mapsto Q + A_2 + A_3 + A_4 + B_1 + B_2 + B_3 + B_4 - C_2 - C_3 - C_4,$$
$$C_1 \mapsto -C_2, \quad C_2 \mapsto -C_3, \quad C_3 \mapsto -C_4,$$
$$C_4 \mapsto Q + A_2 + A_3 + A_4 - B_2 - B_3 - B_4 + C_1 + C_2 + C_3 + C_4,$$
$$Q \mapsto Q.$$

Linear action on $H_1(X_{\mathbb{R}}, \mathbb{Z})/\mathbb{Z}_2 Q$

The characteristic polynomial is given by the determinant of the following matrix.

$$\begin{array}{cc}
 & \begin{array}{cccccccccccc} A_1 & A_2 & A_3 & A_4 & B_1 & B_2 & B_3 & B_4 & C_1 & C_2 & C_3 & C_4 \end{array} \\
\begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} &
\left(\begin{array}{cccccccccccc}
-t & & & 1 & & & & & & & & \\
-1 & -t & & 1 & & & & 1 & & & & 1 \\
 & -1 & -t & 1 & & & & 1 & & & & 1 \\
 & & -1 & 1-t & & & & 1 & & & & 1 \\
 & & & & -t & & & 1 & & & & \\
 & & & 1 & -1 & -t & & 1 & & & & -1 \\
 & & & 1 & & -1 & -t & 1 & & & & -1 \\
 & & & 1 & & & -1 & 1-t & & & & -1 \\
 & & & & & & & & -t & & & 1 \\
 & & & 1 & & & & -1 & -1 & -t & & 1 \\
 & & & 1 & & & & -1 & & -1 & -t & 1 \\
 & & & 1 & & & & -1 & & & -1 & 1-t
\end{array}\right)
\end{array}.$$

Characteristic polynomial

The determinant of this matrix turns out to be as follows.

$$\chi_{\mathbb{R}}(t) = (t-1)^4(t^2+1)^2(t^4+t^3-t^2+t+1)$$

$$= t^{12}-3t^{11}+3t^{10}+t^9-9t^8+18t^7-22t^6+18t^5-9t^4+t^3+3t^2-3t+1.$$

Decompose it to cyclotomic factor, $C_{\mathbb{R}}(t)$, and Salem factor, $S_{\mathbb{R}}(t)$.

$$C_{\mathbb{R}}(t) = (t-1)^4(t^2+1)^2, \qquad S_{\mathbb{R}}(t) = t^4+t^3-t^2+t+1.$$

As $C_{\mathbb{R}}(t) = (t-1)^4(t^4-1)^2/(t^2-1)^2$,

$$\Lambda(F_{\mathbb{R}}^k) = \mathbf{1}(k) - (4\cdot\mathbf{1}(k) + 2\cdot\mathbf{4}(k) - 2\cdot\mathbf{2}(k) + \tau_k(S_{\mathbb{R}})).$$

And since $S_{\mathbb{R}}(-z) = S_{\mathbb{C}}(z)$,

$$\tau_k(S_{\mathbb{R}}) = (-1)^k\tau_k(S_{\mathbb{C}}).$$

Lefschetz numbers

Compare this with that of complex version.

$$C_{\mathbb{C}}(z) = (z-1)(z^4-1)^2, \qquad S_{\mathbb{C}}(z) = z^4 - z^3 - z^2 - z + 1.$$

$$\Lambda(F_{\mathbb{C}}^k) = 3 \cdot \mathbf{1}(k) + 2 \cdot \mathbf{4}(k) + \tau_k(S_{\mathbb{C}}).$$

$$\tau_1(S_{\mathbb{C}}) = 1, \quad \tau_2(S_{\mathbb{C}}) = 3, \quad \tau_3(S_{\mathbb{C}}) = 7, \quad \tau_4(S_{\mathbb{C}}) = 7.$$

More precisely,

$$\Lambda(F_{\mathbb{C}}^k) = 4 \cdot \mathbf{1}(k) + \mathbf{2}(k) + 2 \cdot \mathbf{4}(k) + (\tau_k(S_{\mathbb{C}}) - \mathbf{1}(k) - \mathbf{2}(k)).$$

And

$$\Lambda(F_{\mathbb{R}}^k) = \mathbf{1}(k) - (4 \cdot \mathbf{1}(k) + 2 \cdot \mathbf{4}(k) - 2 \cdot \mathbf{2}(k) + \tau_k(S_{\mathbb{R}}))$$

$$= \mathbf{1}(k) - 3 \cdot \mathbf{1}(k) - 2 \cdot \mathbf{4}(k) - (-1)^k(\tau_k(S_{\mathbb{C}}) - \mathbf{1}(k) - \mathbf{2}(k)).$$

Here, we used $\tau_k(S_{\mathbb{R}}) = (-1)^k \tau_k(S_{\mathbb{C}})$, and
$\mathbf{1}(k) - \mathbf{2}(k) = -(-1)^k \mathbf{1}(k)$.

Compare

$$\Lambda(F_{\mathbb{C}}^k) = 4 \cdot \mathbf{1}(k) + \mathbf{2}(k) + 2 \cdot \mathbf{4}(k) + (\tau_k(S_{\mathbb{C}}) - \mathbf{1}(k) - \mathbf{2}(k)).$$

$$\Lambda(F_{\mathbb{R}}^k) = \mathbf{1}(k) - 3 \cdot \mathbf{1}(k) - 2 \cdot \mathbf{4}(k) - (-1)^k(\tau_k(S_{\mathbb{C}}) - \mathbf{1}(k) - \mathbf{2}(k)).$$

These show that the four fixed points in $X_{\mathbb{C}}$ became one source(or sink) and three non-flip saddles,

Cycle of period two is in $X_{\mathbb{C}} \setminus X_{\mathbb{R}}$.

All other periodic points are in $X_{\mathbb{R}}$, and cycles of odd period are bi-flip saddles and cycles of even period are non-flip saddles.

5. Julia set

The measure of maximal entropy

THEOREM(Bedford-Lyubich-Smilie 1993, Cantat 2003). Let f be a loxodromic automorphism of a complex projective surface X. Let $\mathrm{Per}(f,k)$ be the set of isolated periodic points of f with period at most k. Then

$$\frac{1}{\lambda_f^k} \sum_{p\in\mathrm{Per}(f,k)} \delta_p$$

converges toward μ_f as k goes to ∞. The same result holds if $\mathrm{Per}(f,k)$ is replaced by the set $\mathrm{Per}_{sad}(f,k)$ of saddle periodic points of period at most k.

If p is a saddle point, either p is contained in the support of μ_f, or p is contained in a cycle of periodic rational curves.

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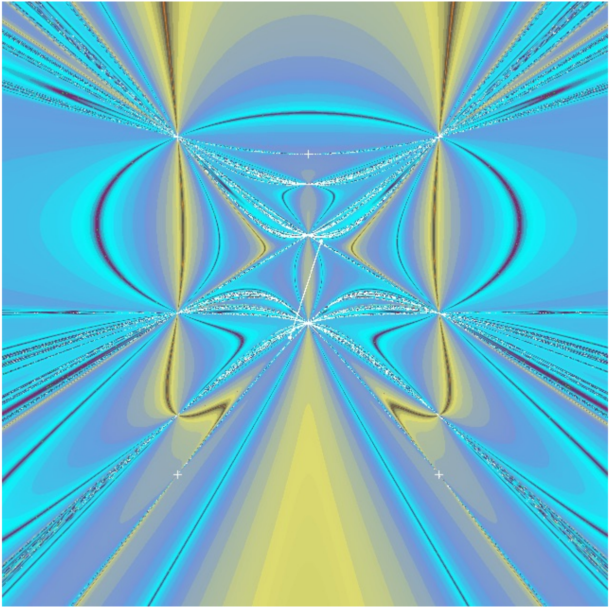
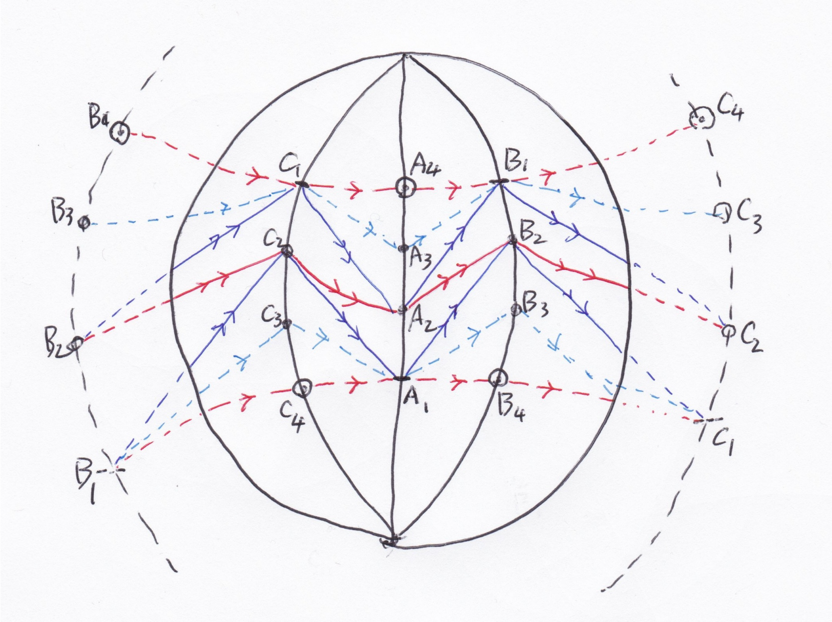


$F_{\mathbb{R}} : X_{\mathbb{R}} \to X_{\mathbb{R}}$



Homology classes

PROPOSITION. There are homology classes in $H_1(X_{\mathbb{R}}, \mathbb{Z})$, such that the subspace spanned by the classes is invariant under $F_{\mathbb{R}*}$, and the characteristic polynomial of the action of $F_{\mathbb{R}*}$ in the subspace is the Salem factor $S_{\mathbb{R}}$.

PROOF. Let $\alpha, \beta, \gamma, \delta \in H_1(X_{\mathbb{R}}, \mathbb{Z})$ be as follows.

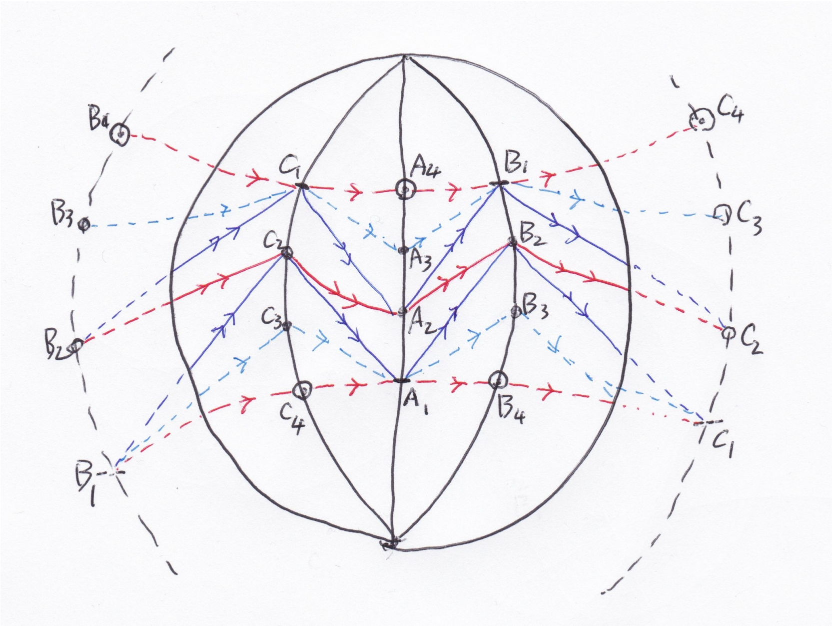
$$\alpha = Q + A_1 - A_3 - A_4 - B_1 + B_3 + B_4 - C_1 + C_3 + C_4,$$

$$\beta = A_1 - A_2 - 2A_3 - 2A_4 - B_1 + B_2 + 2B_3 + 2B_4 - C_1 + C_2 + 2C_3 + 2C_4,$$

$$\gamma = A_1 - A_3 - 2A_4 - B_1 + B_3 + 2B_4 - C_1 + C_3 + 2C_4,$$

$$\delta = A_1 - A_4 - B_1 + B_4 - C_1 + C_4.$$

$F_{\mathbb{R}} : X_{\mathbb{R}} \to X_{\mathbb{R}}$



These classes are represented by lifting smooth loops passing through base points in $\mathbb{RP}^2$, respectively,

$$\alpha: \quad \mathbf{a}_2 \to \mathbf{b}_2 \to \mathbf{c}_2 \to \mathbf{a}_2,$$

$$\beta: \quad \mathbf{a}_1 \to \mathbf{b}_2 \to \mathbf{c}_1 \to \mathbf{a}_2 \to \mathbf{b}_1 \to \mathbf{c}_2 \to \mathbf{a}_1,$$

$$\gamma: \quad \mathbf{a}_1 \to \mathbf{b}_3 \to \mathbf{c}_1 \to \mathbf{a}_3 \to \mathbf{b}_1 \to \mathbf{c}_3 \to \mathbf{a}_1,$$

$$\delta: \quad \mathbf{a}_1 \to \mathbf{b}_4 \to \mathbf{c}_1 \to \mathbf{a}_4 \to \mathbf{b}_1 \to \mathbf{c}_4 \to \mathbf{a}_1.$$

They are mapped by $F_{\mathbb{R}*}$ as

$$\alpha \mapsto -\gamma, \quad \beta \mapsto -\beta - \gamma, \quad \gamma \mapsto -\beta - \delta, \quad \delta \mapsto -\alpha.$$

In matrix form:

$$F_{\mathbb{R}*}\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1\\ 0 & -1 & -1 & 0\\ -1 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{pmatrix}.$$

The characteristic polynomial of this linear map in the sub-lattice spanned by these is same as the Salem factor

$$S_{\mathbb{R}}(t) = t^4 + t^3 - t^2 + t + 1.$$

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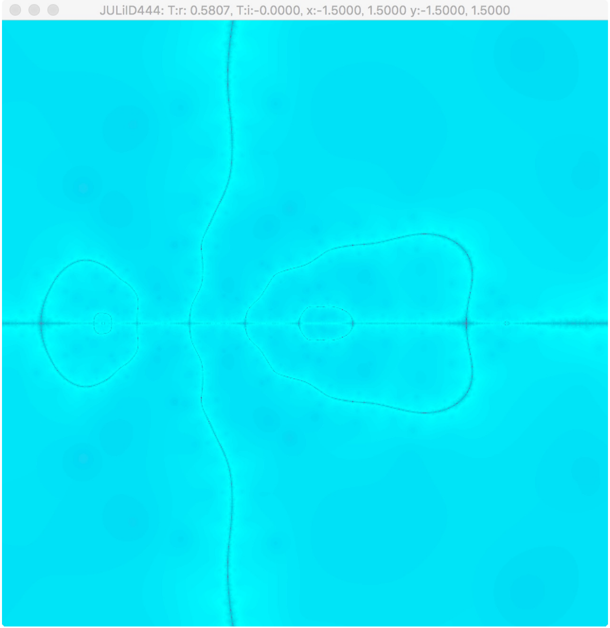

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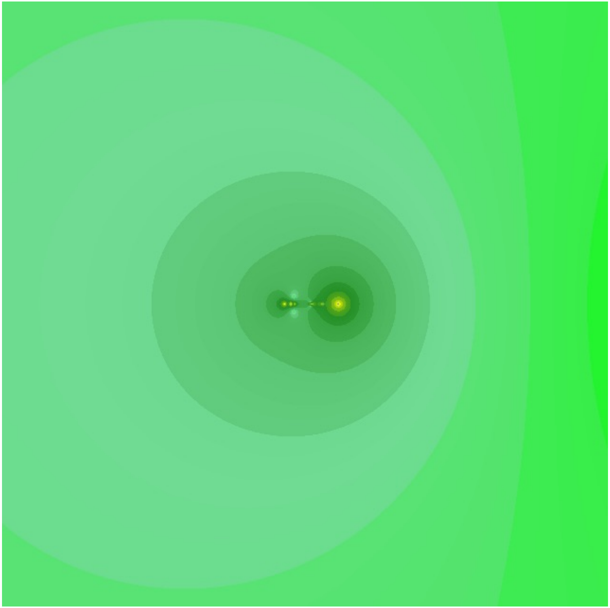

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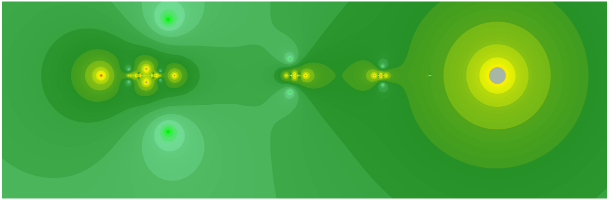

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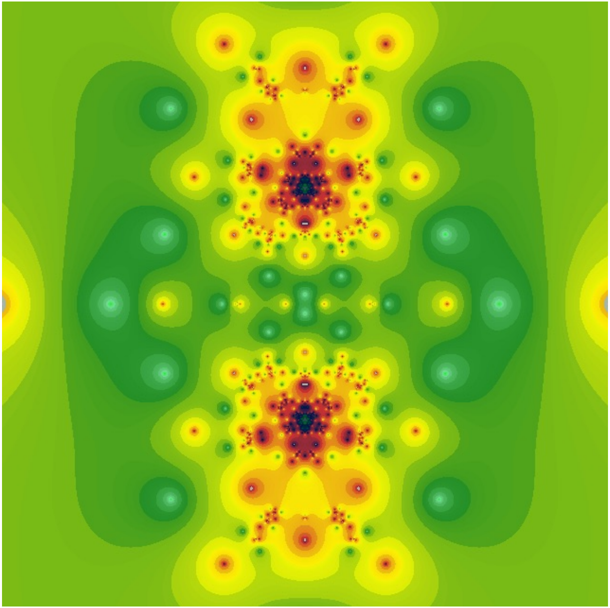

References

[BBD] D. Barrett, E. Bedford and J. Dadok, $\mathbb{T}^n$-actions on holomorphically separable complex manifolds. Math. Z. 202(1989), no. 1, 65-82.

[BK1] E. Bedford and K. Kim. Periodicities in Linear Fractional Recurrences: Degree growth of birational surface maps, Mich. Math. J. **54**(2006), 647-670.

[BK2] E. Bedford and K. Kim. Dynamics of Rational Surface Automorphisms: Linear Fractional Recurrences. J. Geomet. Anal. **19**, 553-583(2009).

[BK3] E. Bedford and K. Kim. Dynamics of rational surface automorphisms: rotation domains. Amer. J. Math. 134(2012), no. 2, 379-405.

[BK4] E. Bedford and K. Kim. Continuous families of rational surface automorphisms with positive entropy. Math. Ann. 348(3), 667-688 (2010).

References

[BLS] E. Bedford, M. Lyubich, and J. Smilie. Polynomial diffeomorphisms of $\mathbb{C}^2$. IV: The measure of maximal entropy and laminar currents. Invent. math. 112, 77-125(1993).
[BS1] E. Bedford and J. Smilie. Polynomial diffeomorphisms of $\mathbb{C}^2$: currents, equilibrium measures and hyperbolicity. Invent. Math. 103(1991), no. 1, 69-99.
[BS2] E.Bedford and J. Smilie. Polynomial diffeomorphisms of $\mathbb{C}^2$. Stable manifolds and recurrence. J. Amer. Math. Soc. **4**(1991), no. 4, 657-679.

References

[C1] S. Cantat. Dynamique des automorphisms des surfaces projectives complexes. C.R. Acad. Sci. Paris Sér I Math., 328(10):901-906, 1999.
[C2] S. Cantat. Dynamique des automorphismes des surfaces K3. Acta Math., 187(1):1-57, 2001.
[C3] S. Cantat. Dynamics of automorphisms of compact complex surfaces. "Frontiers in Complex Dynamics – In Celebration of John Milnor's 80th birthday", Eds. A.Bonifant, M. Lyubich, S. Sutherland, Prinston University Press, Princeton and Oxford, pp. 463-509, 2014

References

[D] J. Diller. Cremona transformations, surface automorphisms, and plane cubics. Michigan Math. J. **60**(2011), no. 2, pp409-440, with an appendix by Igor Dolgachev.

[DJS] J. Diller, D. Jackson, A. Sommese. Invariant curves for birational surface maps, Trans. A.M.S., Vol. 359, No. 6, June 2017, pp. 2973-2991.

[M1] C. T. McMullen. Dynamics on K3 surfaces Salem numbers and Siegel disks. J. reine angew. Math. 545(2002),201-233.

[M2] C. T. McMullen. Dynamics on blowups of the projective plane. Publ. Sci. IHES, **105**, 49-89(2007).

[N] M. Nagata. On rational surfaces. II. Mem. Coll. Sci. Univ. Kyoto Ser. A Math., 33:271-293, 1960/1961.

References

[UD1] T. Ueda. Local structure of analytic transformations of two complex variables. I. J. Math. Kyoto Univ., 26(2), 233-261, (1986).

[UD2] T. Ueda. Local structure of analytic transformations of two complex variables. II. J.Math. Kyoto Univ. 31(3), 695-711, (1991).

[UH1] T. Uehara. Rational surface automorphisms preserving cuspidal anticanonical curves. Mathematische Annalen, Band 362, Heft 3-4, 2015.

[UH2] T. Uehara. Rational surface automorphisms with positive entropy. Ann. Inst. Fourier (Grenoble) **66**(2016), 377-432.

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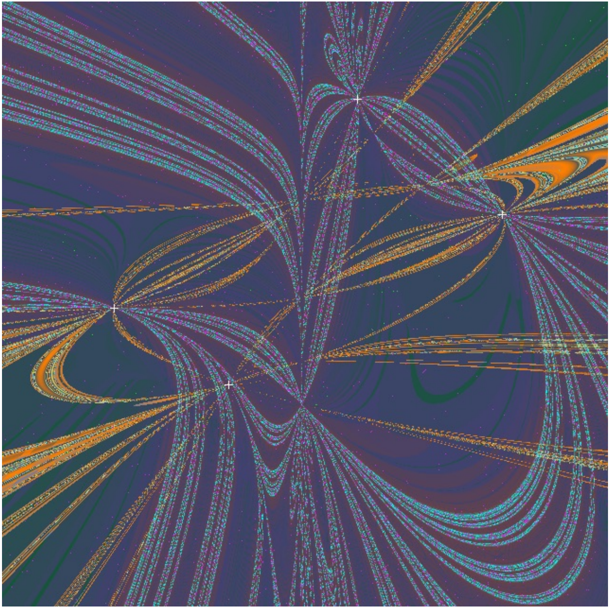

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