Visual dynamics in $\mathbb{C}^2$

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Formula for periodic points of period upto five are obtained.

Introduction

Hénon map $H_{b,c} : (x, y) \mapsto (X, Y)$:

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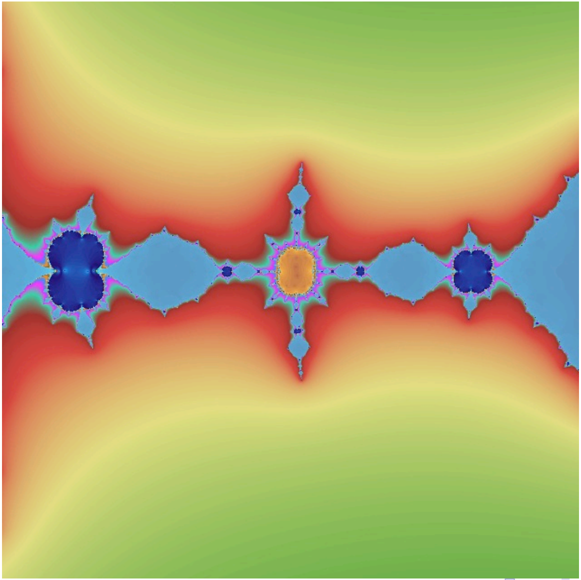
A variable will be called a **cycle variable**, if a cycle can be computed from the variable, and the value can be computed from a cycle.

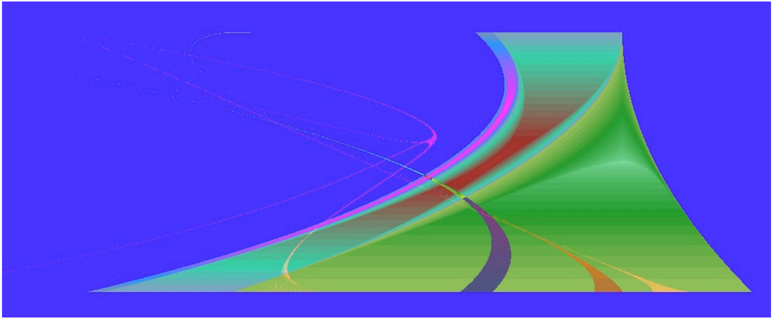
Equation of cycle variable and parameters will be called a **cycle equation**.

Three basins

A portion of unstable manifold of a saddle fixed point.



Real slice of the parameter space



Fixed points

Fixed point is of the form (x_0, x_0) with x_0 given as a root of **cycle equation** :

$$(\text{CE1}) \qquad x_0^2 - (1-b)x_0 + c = 0.$$

x_0 is a **cycle variable**.

Cycles of period 2

Let $\{(x_n, x_{n-1})\}$ be the periodic orbit of period 2, and set

$$x_n = \alpha_0 + (-1)^n \alpha_1.$$

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$$x_n = \alpha_0 + (-1)^n \alpha_1.$$

α_0 can be regarded as a **cycle variable** of period 2, and the **cycle equation** is

$$\text{(CE2)} \qquad 2\alpha_0 + 1 - b = 0.$$

α_1 is obtained from $\alpha_1^2 = -\frac{3}{4}(1-b)^2 - c$.

Cycles of period 3

For 3-cycle $\{(x_n, x_{n-1})\}$, let

$$x_n = u_0 + \omega^n u_1 + \omega^{2n} u_2.$$

Here, ω denotes a cubic root of unity. Assume $(u_1, u_2) \neq (0, 0)$.

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It is an orbit of $H_{b,c}$ if

$$x_{n+1} - b x_{n-1} = (x_n)^2 + c.$$

Cycles of period 3

We obtain :

$$\left\{\begin{array}{ccc} (1-b)u_0 & = & u_0^2 + 2u_1u_2 + c, \\ (\omega - b\omega^2)u_1 & = & u_2^2 + 2u_0u_1, \\ (\omega^2 - \omega b)u_2 & = & u_1^2 + 2u_0u_2. \end{array}\right.$$

These yield

$$u_1u_2 = 4u_0^2 + 2(1-b)u_0 + b^2 + b + 1,$$

$$\frac{u_1^3 + u_2^3}{u_1u_2} = -(1-b) - 4u_0.$$

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The **cycle equation** of period 3 is :

$$\text{(CE3)} \quad 9u_0^2 + 3(1-b)u_0 + 2(b^2+b+1) + c = 0.$$

u_0 is the **cycle variable** of 3-cycle.

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Each solution u_0 corresponds to a 3-cycle.
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The **saddle-node locus equation**:

$$(\mathrm{SN3}_L) \qquad c = -\frac{1}{4}(7b^2 + 10b + 7).$$

And the **saddle-node cycle equation**:

$$(\mathrm{SN3}_C) \qquad u_0 = -\frac{1}{6}(1 - b).$$

Trace function

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The trace $\tau_3 : \tilde{\Gamma}_3 \to \mathbb{C}$, is called the **trace function**.

We use (b, u_0) as a coordinate of $\tilde{\Gamma}_3$. $c = c(b, u_0)$.

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We use (b, u_0) as a coordinate of $\tilde{\Gamma}_3$. $c = c(b, u_0)$.

$$\tau_3 = -8\left(27u_0^3 + 18(1-b)u_0^2 + 9(b^2 + \frac{b}{4} + 1)u_0 + 1 - b^3\right).$$

Regurarity of trace function

Trace function $\tau_3 : \tilde{\Gamma}_3 \to \mathbb{C}$ is regular near the saddle-node locus if $|b| < 1$.

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PROOF. Partial derivative (in $\tilde{\Gamma}_3$) at saddle-node cycle is given by

$$\left.\frac{\partial \tau_3}{\partial u_0}\right|_{u_0 = -\frac{(1-b)}{6}} = -6(7b^2 + 13b + 7).$$

Note that $|b| = 1$ if $7b^2 + 13b + 7 = 0$.

Cusps at saddle-node locus

For each parameter b with $|b| < 1$, locus of attracting 3-cycles has a cusp point at saddle-node locus $c = -\frac{1}{4}(7b^2 + 10b + 7)$.

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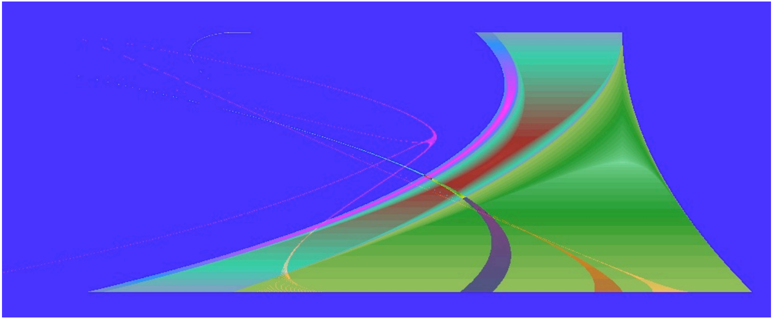
Saddle-node locus of 3-cycle and period-doubling locus of fixed point intersect at two points $(b, c) = (-2 \pm \sqrt{3}, 9(1 \mp \frac{1}{2}\sqrt{3}))$.

Real slice of the parameter space



Cycles of period 4

Let

$$x_n = v_0 + i^n v_1 + (-1)^n v_2 + (-i)^n v_3.$$

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In terms of Fourier coefficients:

$$\text{(EF4)} \quad \left\{ \begin{array}{ccc} (1-b)v_0 & = & v_0^2 + v_2^2 + 2v_1 v_3 + c \\ i(1+b)v_1 & = & 2(v_0 v_1 + v_2 v_3) \\ -(1-b)v_2 & = & v_1^2 + v_3^2 + 2v_0 v_2 \\ -i(1+b)v_3 & = & 2(v_1 v_2 + v_0 v_3) \end{array} \right.$$

Cycle equation of 4-cycles

The **cycle equation** of period four:

$$\text{(CE4)}\ v_0^3 + \frac{1}{4}(c + \frac{3}{4}(b+1)^2)v_0 - \frac{1}{16}(b^2-1)(b+1) = 0.$$

v_0 is a **cycle variable** of 4-cycles.

Trace function of 4-cycle

Let $\tau_4 = \operatorname{tr} DH^4_{b,c}$ denote the trace of the derivative along the cycle.
We have

$$
\text{(TF4)} \quad \tau_4 = 16(-16 b_1 v_0^3 + 8 c v_0^2 + 2 b_1 (6 b_2^2 - 2c) v_0 + 4 b_2^4 + b_1^2 b_2^2 + c^2 - 4 b_2^2 c) + 16 b b_2^2 + 2b^2.
$$

Trace function is a function on the surface defined by (CE4).

Saddle-node locus of 4-cycle

The discriminant of the cycle equation gives the **saddle-node-locus equation**:

$$(\mathrm{SN4}_L) \qquad (c - 3b_2^2)^3 = -27 b_1^2 b_2^4.$$

The cycle variable of the saddle-node v_0 is the double root of the cycle equation.

Saddle-node-cycle equation is

$$(\mathrm{SN4}_C) \qquad v_0^3 = -\frac{1}{8}\, b_1 b_2^2.$$

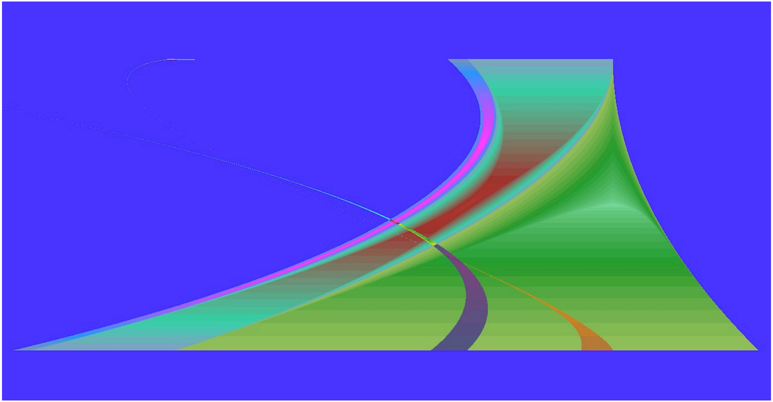
Regularity of trace

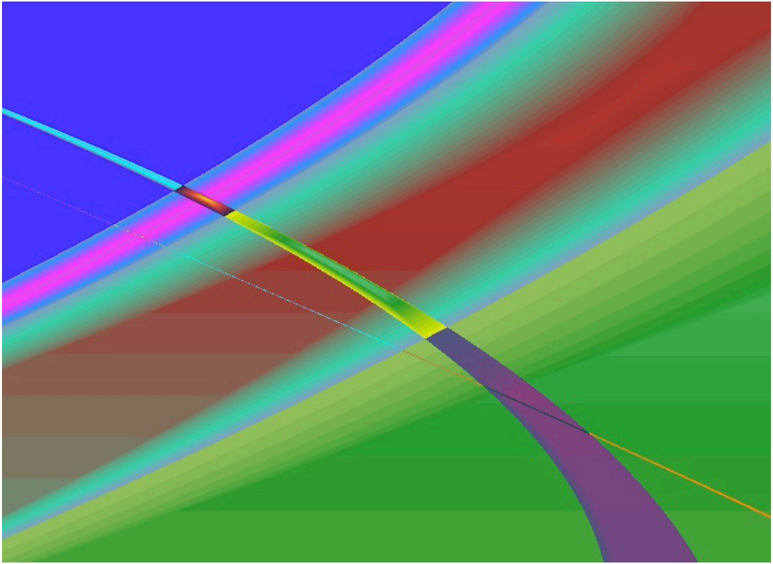
Trace function $\tau_4 : \tilde{\Gamma}_4 \to \mathbb{C}$ is regular on saddle-node locus $(\mathrm{SN4}_L)$ for $0 < |b| < 1$.

For each parameter b with $|b| < 1$,the locus of attracting cycles $\Omega_4 \cap (\{b\} \times \mathbb{C})$ has cusp points at saddle-node loci.

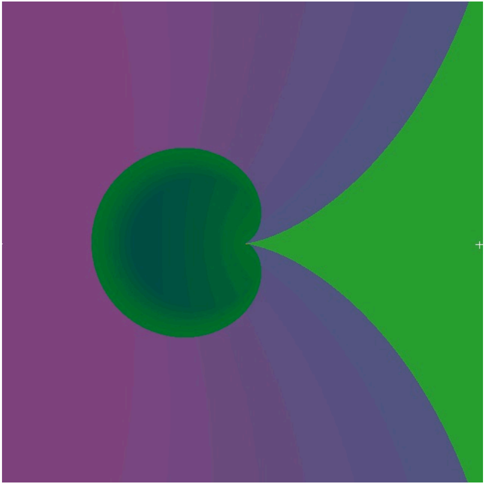
Coexistence of attractive cycles

The intersection $\Omega_1 \cap \Omega_3 \cap \Omega_4$ is not empty.

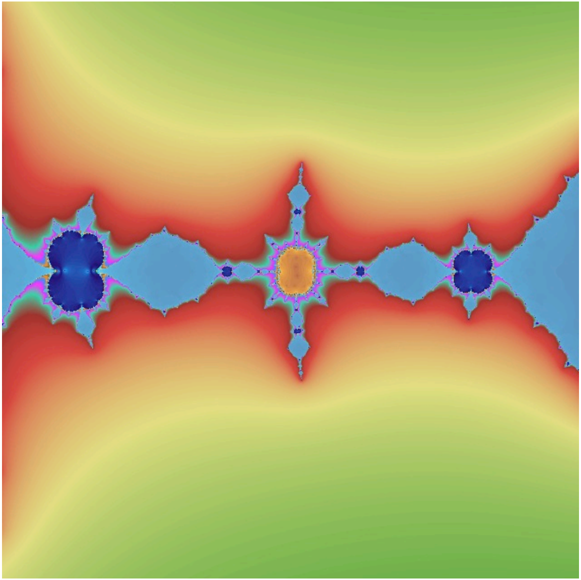




Complex slice



$b = -0.3946$, $c = -1.0362$.



Cycles of period 5

Let κ denote a quintic root of unity and let

$$x_n = \sum_{k=0}^{4} a_k \kappa^{nk}.$$

From our difference scheme

$$x_{n+1} - b x_{n-1} = x_n^2 + c,$$

$$(\text{EF5}) \quad \left\{ \begin{array}{ccc} (1-b)a_0 & = & a_0^2 + 2a_1a_4 + 2a_2a_3 + c \\ (\kappa - b\kappa^4)a_1 & = & 2a_0a_1 + 2a_2a_4 + a_3^2 \\ (\kappa^2 - b\kappa^3)a_2 & = & 2a_0a_2 + 2a_3a_4 + a_1^2 \\ (\kappa^3 - b\kappa^2)a_3 & = & 2a_0a_3 + 2a_1a_2 + a_4^2 \\ (\kappa^4 - b\kappa)a_4 & = & 2a_0a_4 + 2a_1a_3 + a_2^2 \end{array} \right.$$

Variable ρ

Now, we introduce a new variable ρ by

$$\rho = \frac{a_1 a_4}{a_2 a_3}.$$

Then, we get a cubic equation of ρ, with coefficients being polynomial functions of b and a_0.

$$W(b, a_0, \rho) = 0.$$

This equation is of degree 2 in b, of degree 3 in ρ and of degree 2 in a_0.

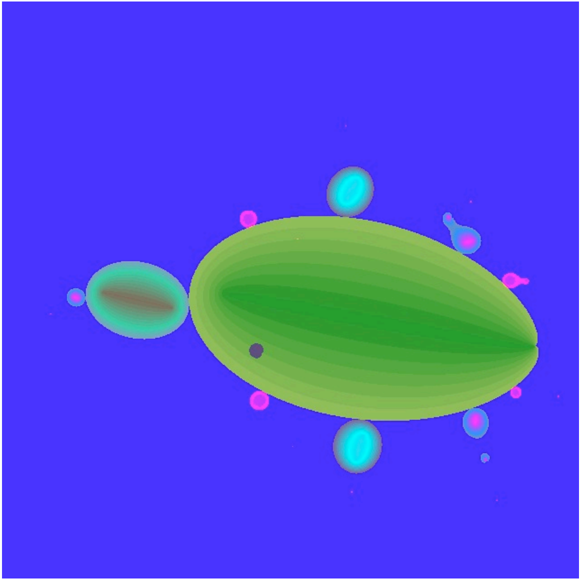
Cycle equation

Our "cycle equation" is given by a system of equations.

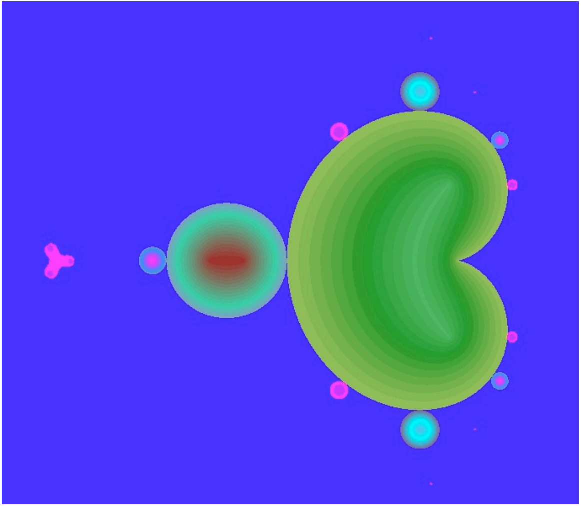
$$\text{(CE5)} \quad \begin{cases} W(b, a_0, \rho) &= 0, \\ c &= b_1 a_0 - a_0^2 - Q(\rho). \end{cases}$$

where, $Q(\rho)$ is a rational function of ρ of degree 2 with coefficients being polynomial functions of b and a_0. One equation to determine the cycle variable, and the other to describe the relation to parameter c.

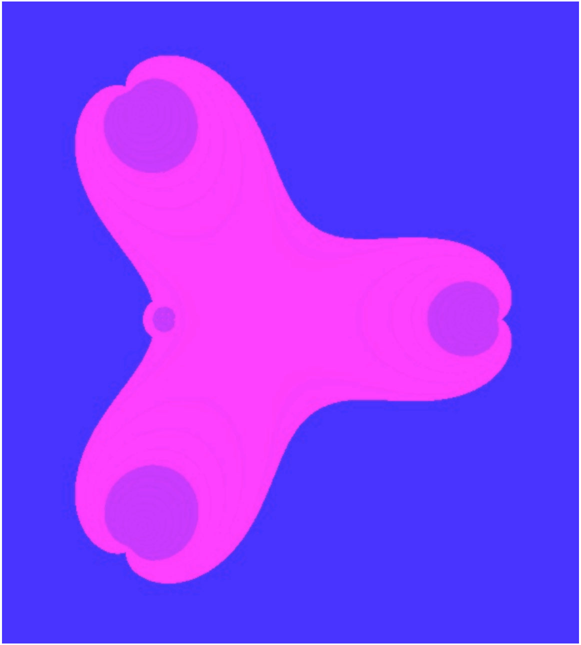
Complex slice of parameter space



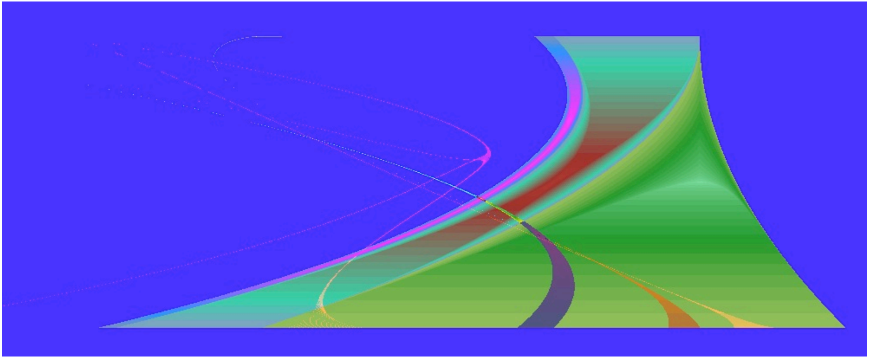
b=0.15



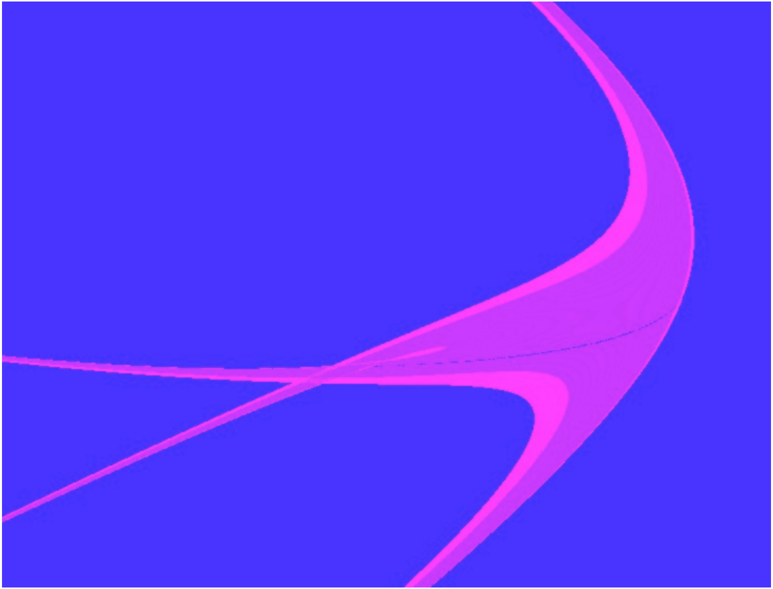
b=0.15



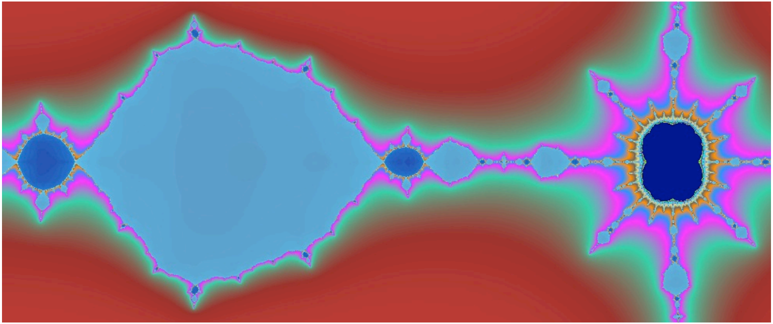
Real slice of the parameter space



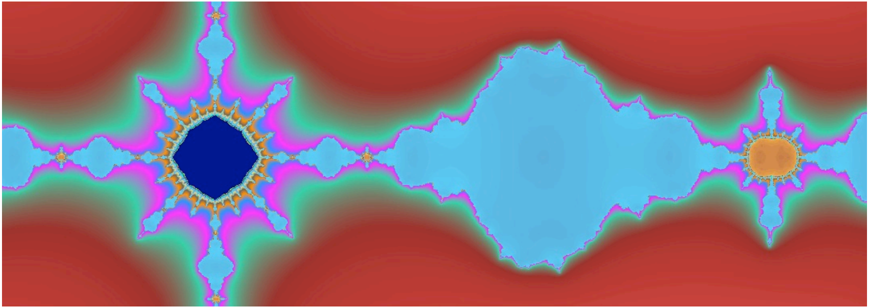
Enlargement of the swallow's tail region



Coexisting attracting cycles of periods 1, 3, and 5.



Attracting cycles of periods 1, 4, and 5.



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