Parabolic bifurcation of area-preserving Hénon maps

(Period 5)

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Abstract

In the area preserving real Hénon maps, pair of a cycle of saddle type and a cycle of center type appears from a parabolic fixed point whose eigenvalues are prime fifth roots of unity.

1. Area-preserving complex Hénon map

$$H_\alpha : \mathbb{C}^2 \to \mathbb{C}^2, \qquad \alpha \in \mathbb{C},$$

$$H_\alpha(x, y) \;=\; (y, y^2 + \alpha - x).$$

$$\det DH_\alpha = 1$$

If $\alpha \in \mathbb{R}$, then H_α is a diffeomorphism of $\mathbb{R}^2$.

Fixed point

Fixed point $\quad P_* = (y_*, y_*)$ given by

$$y_*^2 - 2y_* + \alpha \;=\; 0.$$

$$DH_\alpha\big|_{P_*} \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 2y_* \end{pmatrix},$$

$$\text{trace } DH_\alpha\big|_{P_*} \;=\; 2y_*, \quad \det DH_\alpha \;=\; 1.$$

Parabolic fixed point of order 5

$$\Omega = e^{\frac{2\pi i}{5}} \quad \text{or} \quad \Omega = e^{\frac{4\pi i}{5}}$$

Suppose eigenvalues at the fixed point P_* are

$$\Omega, \quad \bar{\Omega}.$$

$$\omega_1 = \Omega + \bar{\Omega} = \frac{\pm\sqrt{5} - 1}{2},$$

$$\omega_2 = \Omega^2 + \bar{\Omega}^2 = \frac{\mp\sqrt{5} - 1}{2},$$

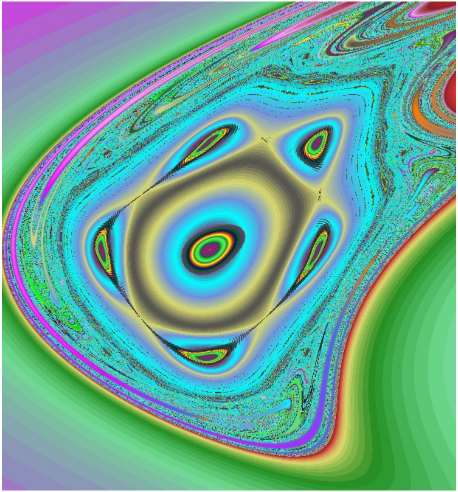
$$\omega_1 \omega_2 = \omega_1 + \omega_2 = -1.$$

$$y_* = \frac{\Omega + \bar{\Omega}}{2}, \qquad \alpha_0 = 2y_* - y_*^2 = \frac{-7 \pm \sqrt{5}}{8}.$$

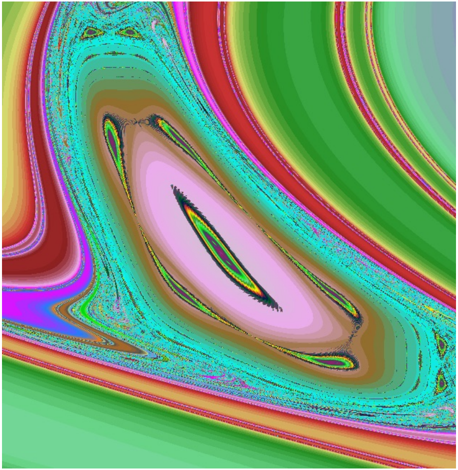
Main Theorem

THEOREM A A pair of cycles of period 5 bifurcates from the fixed point P_* for $\alpha < \alpha_0$ near α_0. One of the cycles is saddle type and the other is center type.

Island of period 5



Island of period 5



Notations

Let $\gamma = \omega_2 - \omega_1 = \mp\sqrt{5}$ and

$$\rho_1(n) = \Omega^n + \bar{\Omega}^n = \begin{cases} 2 & (n \equiv 0, \mod 5) \\ \omega_1 & (n \equiv 1 \text{ or } 4) \\ \omega_2 & (n \equiv 2 \text{ or } 3) \end{cases},$$

$$\rho_2(n) = \Omega^{2n} + \bar{\Omega}^{2n} = \begin{cases} 2 & (n \equiv 0, \mod 5) \\ \omega_2 & (n \equiv 1 \text{ or } 4) \\ \omega_1 & (n \equiv 2 \text{ or } 3) \end{cases}.$$

Bifurcation Theorem

THEOREM B There exists a function

$$\alpha(\varepsilon) = \alpha_0 + \gamma\omega_1\varepsilon^2 + \gamma\varepsilon^3 + \cdots$$

and a family of periodic sequences

$$y_n(\varepsilon) = y_* - \frac{\gamma}{5}\varepsilon^2 + \rho_1(n)(\varepsilon - \frac{\gamma}{4}\varepsilon^2 + \cdots)$$

$$+ \rho_2(n)(\frac{\gamma}{5}\varepsilon^2 - \frac{1}{10}\varepsilon^3 + \cdots)$$

holomorphic in ε near $0 \in \mathbb{C}$, such that for each ε, $H_{\alpha(\varepsilon)}$ has a cycle $\{P_n(\varepsilon) = (y_n(\varepsilon), y_{n+1}(\varepsilon))\}$ of period 5.

Trace Theorem

THEOREM C The trace function of the cycle

$$\tau(\varepsilon) = \text{ trace } DH^{\circ 5}_{\alpha(\varepsilon)}\Big|_{(y_0, y_1)}$$

is holomorphic in ε and not constant near $\varepsilon = 0$, with $\tau(0) = 2$. Moreover, if $\varepsilon \in \mathbb{R}$, then $\alpha(\varepsilon) \in \mathbb{R}$ and $\tau(\varepsilon) \in \mathbb{R}$.

Discrete Fourier expansion

$$y_n = u_0 + \Omega^n u_1 + \Omega^{2n} u_2 + \bar{\Omega}^{2n} u_3 + \bar{\Omega}^n u_4,$$
$$y_{n+1} + y_{n-1} = y_n^2 + \alpha.$$

We get a system of equations:

$$\begin{aligned}
(F_0) \quad 2u_0 &= u_0^2 + 2u_1u_4 + 2u_2u_3 + \alpha, \\
(F_1) \quad \omega_1 u_1 &= 2u_0u_1 + u_3^2 + 2u_2u_4, \\
(F_2) \quad \omega_2 u_2 &= 2u_0u_2 + u_1^2 + 2u_3u_4, \\
(F_3) \quad \omega_2 u_3 &= 2u_0u_3 + u_4^2 + 2u_1u_2, \\
(F_4) \quad \omega_1 u_4 &= 2u_0u_4 + u_2^2 + 2u_1u_3.
\end{aligned}$$

Introduction of ε

α appears only in (F_0).
Constant : δ (to be fixed as $\delta = \frac{2}{\gamma}$ later).

$$u_0 = y_* - \frac{\delta}{2}\varepsilon^2 = \frac{\omega_1}{2} - \frac{\delta}{2}\varepsilon^2.$$

System of algebraic equations parametrized by ε:

$$\begin{array}{llcl}
(F_{\varepsilon,1}) & \delta\varepsilon^2 u_1 & = & u_3^2 + 2u_2u_4, \\
(F_{\varepsilon,2}) & (\gamma + \delta\varepsilon^2)u_2 & = & u_1^2 + 2u_3u_4, \\
(F_{\varepsilon,3}) & (\gamma + \delta\varepsilon^2)u_3 & = & u_4^2 + 2u_1u_2, \\
(F_{\varepsilon,4}) & \delta\varepsilon^2 u_4 & = & u_2^2 + 2u_1u_3.
\end{array}$$

Algebraic variety

Family of polynomial mappings parametrized by ε :

$$F_\varepsilon : \mathbb{C}^4 \to \mathbb{C}^4.$$

Family of algebraic varieties parametrized by ε :

$$F_\varepsilon(\mathbf{u}) = \mathbf{0}, \qquad \mathbf{u} = (u_1, u_2, u_3, u_4).$$

$\mathbf{u} = \mathbf{0}$ is a solution for all ε near 0.

Solving process is essentially a resolution of singularities.

Weighted scaling

$$u_1 = \varepsilon v_1, \quad u_2 = \varepsilon^2 v_2, \quad u_3 = \varepsilon^2 v_3, \quad u_4 = \varepsilon v_4,$$

and assuming $\varepsilon \neq 0$:

$$\begin{array}{llcl}
(G_{\varepsilon,1}) & \delta v_1 & = & 2 v_2 v_4 + \varepsilon v_3^2, \\
(G_{\varepsilon,2}) & (\gamma + \delta \varepsilon^2) v_2 & = & v_1^2 + 2\varepsilon v_3 v_4, \\
(G_{\varepsilon,3}) & (\gamma + \delta \varepsilon^2) v_3 & = & v_4^2 + 2\varepsilon v_1 v_2, \\
(G_{\varepsilon,4}) & \delta v_4 & = & 2 v_1 v_3 + \varepsilon v_2^2.
\end{array}$$

Equation $G_\varepsilon(\mathbf{v}) = \mathbf{0}$, with $\mathbf{v}(\varepsilon) = (v_1, v_2, v_3, v_4)$.

Principal part

Let $\mathbf{a} = \mathbf{v}(0)$.

$$
\begin{aligned}
(G_{0,1}) \qquad & \delta a_1 = 2a_2a_4, \\
(G_{0,2}) \qquad & \gamma a_2 = a_1^2, \\
(G_{0,3}) \qquad & \gamma a_3 = a_4^2, \\
(G_{0,1}) \qquad & \delta a_4 = 2a_1a_3.
\end{aligned}
$$

constant δ

Now, determine the constant $\delta = \frac{2}{\gamma}$, as noticed above, to obtain non-trivial solutions in a simple form. Suppose $a_1 \neq 0$. then we have

$$a_2 = \frac{a_1^2}{\gamma}, \quad a_3 = \frac{a_1^{-2}}{\gamma}, \quad a_4 = a_1^{-1}.$$

Here, **a** is not uniquely determined.

Second jet

$\mathbf{a}$ was not uniquely determined. Let $a_1 = \sigma$, and

$$\mathbf{a} = \mathbf{a}(\sigma) = (\sigma, \frac{\sigma^2}{\gamma}, \frac{\sigma^{-2}}{\gamma}, \sigma^{-1}).$$

$G_0(\mathbf{a}(\sigma)) = 0$ holds for $\sigma \in \mathbb{C} \setminus \{0\}$.
Let

$$\mathbf{v}(\varepsilon) = \mathbf{a} + \varepsilon \mathbf{w}(\varepsilon), \quad v_i(\varepsilon) = a_i + \varepsilon w_i(\varepsilon),$$

and rewrite the equation (G_ε).

Equation (M)

$$
\begin{aligned}
(M_{\varepsilon,1})\quad \frac{2}{\gamma} w_1 &= \frac{\sigma^{-4}}{\gamma^2} + \frac{2\sigma^2}{\gamma} w_4 + 2\sigma^{-1} w_2 + \varepsilon\Big(\frac{2\sigma^{-2}}{\gamma} w_3 + 2 w_2 w_4\Big) + \varepsilon^2 w_3^2, \\
(M_{\varepsilon,2})\quad \gamma w_2 + \varepsilon\Big(\frac{2\sigma^2}{\gamma^2} + \frac{2\varepsilon}{\gamma} w_2\Big) &= 2\sigma w_1 + \frac{2\sigma^{-3}}{\gamma} \\
&\quad + \varepsilon\Big(w_1^2 + \frac{2\sigma^{-2}}{\gamma} w_4 + 2\sigma^{-1} w_3\Big) + 2\varepsilon^2 w_3 w_4, \\
(M_{\varepsilon,3})\quad \gamma w_3 + \varepsilon\Big(\frac{2\sigma^{-2}}{\gamma^2} + \frac{2\varepsilon}{\gamma} w_3\Big) &= 2\sigma^{-1} w_4 + \frac{2\sigma^3}{\gamma} \\
&\quad + \varepsilon\Big(w_4^2 + \frac{2\sigma^2}{\gamma} w_1 + 2\sigma w_2\Big) + 2\varepsilon^2 w_1 w_2, \\
(M_{\varepsilon,4})\quad \frac{2}{\gamma} w_4 &= \frac{\sigma^4}{\gamma^2} + \frac{2\sigma^{-2}}{\gamma} w_1 + 2\sigma w_3 + \varepsilon\Big(\frac{2\sigma^2}{\gamma} w_2 + 2 w_1 w_3\Big) + \varepsilon^2 w_2^2.
\end{aligned}
$$

Principal part

Here, $\mathbf{w}$ is supposed to be an analytic function of ε, and let $\mathbf{w} = \mathbf{b} + O(\varepsilon)$, $\mathbf{b} = (b_1, b_2, b_3, b_4)$, with $w_i = b_i + O(\varepsilon)$. The principal part of (M_ε) is obtained by letting $\varepsilon \to 0$.

$$
\begin{aligned}
(M_{0,1}) \quad \frac{2}{\gamma} b_1 &= \frac{\sigma^{-4}}{\gamma^2} + \frac{2\sigma^2}{\gamma} b_4 + 2\sigma^{-1} b_2, \\
(M_{0,2}) \quad \gamma b_2 &= 2\sigma b_1 + \frac{2\sigma^{-3}}{\gamma}, \\
(M_{0,3}) \quad \gamma b_3 &= 2\sigma^{-1} b_4 + \frac{2\sigma^3}{\gamma}, \\
(M_{0,4}) \quad \frac{2}{\gamma} b_4 &= \frac{\sigma^4}{\gamma^2} + \frac{2\sigma^{-2}}{\gamma} b_1 + 2\sigma b_3.
\end{aligned}
$$

Equation for **b**

Rewrite this system of equations as follows.

$$\begin{pmatrix} -\frac{2}{\gamma} & 2\sigma^{-1} & 0 & \frac{2\sigma^2}{\gamma} \\ 2\sigma & -\gamma & 0 & 0 \\ 0 & 0 & -\gamma & 2\sigma^{-1} \\ \frac{2\sigma^{-2}}{\gamma} & 0 & 2\sigma & -\frac{2}{\gamma} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} -\frac{\sigma^{-4}}{\gamma^2} \\ -\frac{2\sigma^{-3}}{\gamma} \\ -\frac{2\sigma^3}{\gamma} \\ -\frac{\sigma^4}{\gamma^2} \end{pmatrix}.$$

$$\text{rank } A = 3,$$

$$\text{rank } A = \text{rank } (A\ \mathbf{b}) \quad \Rightarrow \sigma^5 - \sigma^{-5} = 0.$$

b is not uniquely determined, here.

$\sigma = 1$

PROPOSITION. Without loss of generalities, we can choose $\sigma = 1$. Other choices of σ give the same family of cycles.

PROOF. $a_1 \to \Omega^k a_1$: choice of initial point of the periodic orbit.

Let $\sigma' = -\sigma$, $\varepsilon' = -\varepsilon$, and

$$w_1'(\varepsilon') = w_1(-\varepsilon'), \; w_2'(\varepsilon') = -w_2(-\varepsilon'),$$

$$w_3'(\varepsilon') = -w_3(-\varepsilon'), \; w_4'(\varepsilon') = w_4(-\varepsilon').$$

Equations (M_ε^σ) and $(M_{\varepsilon'}^{\sigma'})$ give same solutions for $\mathbf{u}$.

In the following, we treat only the case of $\sigma = 1$.

Equations with $\sigma = 1$

$$
\begin{aligned}
(M_1') \quad 0 &= \frac{1}{\gamma^2} - \frac{2}{\gamma} w_1 + 2w_2 + \frac{2}{\gamma} w_4 + \varepsilon\left(\frac{2}{\gamma} w_3 + 2w_2 w_4\right) + \varepsilon^2 w_3^2, \\
(M_2') \quad 0 &= \frac{2}{\gamma} + 2w_1 - \gamma w_2 + \varepsilon\left(-\frac{2}{\gamma^2} + w_1^2 + 2w_3 + \frac{2}{\gamma} w_4\right) \\
&\quad + \varepsilon^2\left(-\frac{2}{\gamma} w_2 + 2w_3 w_4\right), \\
(M_3') \quad 0 &= \frac{2}{\gamma} - \gamma w_3 + 2w_4 + \varepsilon\left(-\frac{2}{\gamma^2} + \frac{2}{\gamma} w_1 + 2w_2 + w_4^2\right) \\
&\quad + \varepsilon^2\left(-\frac{2}{\gamma} w_3 + 2w_1 w_2\right), \\
(M_4') \quad 0 &= \frac{1}{\gamma^2} + \frac{2}{\gamma} w_1 + 2w_3 - \frac{2}{\gamma} w_4 + \varepsilon\left(\frac{2}{\gamma} w_2 + 2w_1 w_3\right) + \varepsilon^2 w_2^2.
\end{aligned}
$$

Change of variables and equations

$$p = w_1 + w_4, \quad q = w_2 + w_3, \quad r = w_2 - w_3, \quad s = w_1 - w_4,$$

$$(P) = (M_1') + (M_4'), \quad (Q) = (M_2') + (M_3'),$$
$$(R) = (M_2') - (M_3'), \quad (S) = (M_1') - (M_4'),$$

$$(P) \quad \frac{2}{\gamma^2} + 2q + O(\varepsilon) = 0,$$

$$(Q) \quad \frac{4}{\gamma} + 2p - \gamma q + O(\varepsilon) = 0,$$

$$(R) \quad 2s - \gamma r + \varepsilon(ps - 2r - \frac{2}{\gamma}s) + O(\varepsilon^2) = 0,$$

$$(S) \quad -\frac{4}{\gamma}s + 2r + \varepsilon(-\frac{2}{\gamma}r + pr - qs) + O(\varepsilon^2) = 0.$$

Principal part

Now, let $\varepsilon \to 0$, to have:

$$\frac{2}{\gamma^2} + 2q_0 = 0, \quad \frac{4}{\gamma} + 2p_0 - \gamma q_0 = 0,$$

$$2s_0 - \gamma r_0 = 0, \quad -\frac{4}{\gamma} s_0 + 2r_0 = 0.$$

We have:

$$q_0 = -\frac{1}{\gamma^2}, \quad p_0 = -\frac{5}{2\gamma}, \quad \text{and} \quad 2s_0 - \gamma r_0 = 0.$$

Here, s_0 and r_0 are not uniquely determined. Remember that $b_1, \cdots, b_4$ were not uniquely determined.

$$p_0 = b_1 + b_4, \quad q_0 = b_2 + b_3, \quad r_0 = b_2 - b_3, \quad s_0 = b_1 - b_4.$$

Cokernel equation

From $(U) = (2(R) + \gamma(S))/\varepsilon$, we have

$$(U) \qquad (\gamma p - 6)r + (2p - \gamma q - \frac{4}{\gamma})s + O(\varepsilon) = 0.$$

We suppose, by letting $\varepsilon \to 0$,

$$(\gamma p_0 - 6)r_0 + (2p_0 - \gamma q_0 - \frac{4}{\gamma})s_0 = 0$$

holds, *i.e.*,

$$-\frac{17}{2}r_0 - \frac{8}{\gamma}s_0 = 0.$$

Together with $2s_0 - \gamma r_0 = 0$, we determine $r_0 = s_0 = 0$.

Regularization

System of algebraic equations $\{(P),(Q),(R),(U)\}$, in variables (p,q,r,s) and analytically parametrized by ε, has a solution

$$(p_0,q_0,r_0,s_0)=(-\frac{5}{2\gamma},-\frac{1}{\gamma^2},0,0),\quad \text{for} \quad \varepsilon=0.$$

Jacobian at (p_0,q_0,r_0,s_0):

$$\det\begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & -\gamma & 0 & 0 \\ 0 & 0 & -\gamma & 2 \\ 0 & 0 & -\frac{17}{2} & -\frac{8}{\gamma} \end{pmatrix} = -100 \neq 0.$$

Analytic family of solutions

PROPOSITION System of equations $\{(P),(Q),(R),(S)\}$ has a family of solutions $(p(\varepsilon),q(\varepsilon),r(\varepsilon),s(\varepsilon))$, analytic near $\varepsilon=0$, satisfying $p(0)=p_0$, $q(0)=q_0$, and $r(\varepsilon)\equiv 0$, $s(\varepsilon)\equiv 0$.

PROOF System of equations $\{(P),(Q),(R),(S)\}$ is equivalent to the system of equations $\{(P),(Q),(R),(U)\}$, which has the solution. System of equations $\{(P),(Q),(R),(U)\}$ has a family of solutions $(p(\varepsilon),q(\varepsilon),r(\varepsilon),s(\varepsilon))$, analytic near $\varepsilon=0$, satisfying $p(0)=p_0$, $q(0)=q_0$, $r(0)=0$, $s(0)=0$. The terms $O(\varepsilon^2)$ in equations (R) and (S) are computed as follows.

$$-\varepsilon^2(pr+qs+\frac{1}{\gamma}r), \qquad -\varepsilon^2(qr).$$

Hence, (R) and (S) always hold if $r=s=0$.

By assuming $r = s = 0$, we see our system of equations reduces to the following system of equation in p and q only.

$$(P_0) \quad \frac{2}{\gamma^2} + 2q + \varepsilon(\frac{2}{\gamma}q + pq) + \varepsilon^2 \frac{q^2}{2} = 0,$$

$$(Q_0) \quad \frac{4}{\gamma} + 2p - \gamma q + \varepsilon(\frac{1}{2}p^2 + \frac{2}{\gamma}p + 2q - \frac{4}{\gamma^2}) + \varepsilon^2(pq - \frac{1}{\gamma}q) = 0,$$

which has a family of solutions $p(\varepsilon)$ and $q(\varepsilon)$, near $\varepsilon = 0$, satisfying $p(0) = p_0$ and $q(0) = q_0$.

By the uniqueness of the solutions given by the implicit function theorem, these solutions are the same.

Proof of theorem B

As stated in the above, our system of equations has a family of solutions parametrized by ε. Obviously, our solutions give the followings.

$$a_1 = a_4 = 1, \quad a_2 = a_3 = \frac{1}{\gamma},$$

$$b_1 = b_4 = -\frac{5}{4\gamma}, \quad b_2 = b_3 = -\frac{1}{2\gamma^2}.$$

$$u_0 = y_* - \frac{\varepsilon^2}{\gamma},$$

$$u_1 = u_4 = \varepsilon - \frac{5}{4\gamma}\varepsilon^2 + \cdots ,$$

$$u_2 = u_3 = \frac{1}{\gamma}\varepsilon^2 - \frac{1}{2\gamma^2}\varepsilon^3 + \cdots .$$

Hence, we have

$$y_n = y_* - \frac{1}{\gamma}\varepsilon^2 + \rho_1(n)(\varepsilon - \frac{5}{4\gamma}\varepsilon^2 + \cdots) + \rho_2(n)(\frac{1}{\gamma}\varepsilon^2 - \frac{1}{2\gamma^2}\varepsilon^3 + \cdots).$$

Furthermore, from (F_0),

$$\alpha(\varepsilon) - \alpha_0 = (\frac{\omega_1}{\gamma} - \frac{2}{\gamma} - 2)\varepsilon^2 + \frac{5}{\gamma}\varepsilon^3 + \cdots ,$$

with $\omega_1 - 2 - 2\gamma = \omega_1 - 2 - 2\omega_2 + 2\omega_1 = 5\omega_1$, we get

$$\alpha(\varepsilon) = \alpha_0 + \frac{5}{\gamma}\omega_1\varepsilon^2 + \frac{5}{\gamma}\varepsilon^3 + \cdots .$$

Note that if $\omega_1 = \Omega + \bar{\Omega} > 0$, then
$\gamma = \Omega^2 + \bar{\Omega}^2 - (\Omega + \bar{\Omega}) < 0$. And if $\Omega + \bar{\Omega} < 0$, then $\gamma > 0$.
So, $\frac{5}{\gamma}\omega_1 < 0$. These, with $\gamma^2 = 5$, prove Theorem B.

Proof of Theorem C

The system of equations $\{(P_0), (Q_0)\}$, with conditions $p(0) = p_0$ and $q(0) = q_0$, can be regarded as a real analytic family of systems of real analytic equations. So, for sufficiently small real values of ε, $p(\varepsilon)$ and $q(\varepsilon)$ are real. With real values of **a** and **b**, the corresponding parameter $\alpha(\varepsilon)$ and periodic points are real and real analytic with respect to ε, near $\varepsilon = 0$.

The trace of the Jacobian matrix along the cycle is also real analytic in ε and takes real values. It is also holomorphic in ε, considered as a complex variable, near $\varepsilon = 0$. As $\alpha(0) = \alpha_0$, and the eigenvalues of the fixed point P_* are Ω and $\bar{\Omega}$, we see that $\tau(0) = 2$. On the other hand, the coordinates of the periodic cycle is algebraic with respect to complex parameter α. For sufficiently large value of α, the periodic cycle become hyperbolic, *i.e.*, the absolute value of the analytic continuation of the trace function is larger than 2. Therefore, the trace function is not constant as an algebraic function of α. Hence $\tau(\varepsilon)$ is not constant near $\varepsilon = 0$.

Proof of Theorem A

As is shown in the proof of Theorem B, $\frac{\omega_1}{\gamma} < 0$ holds in both cases of Ω. Parameter α is related to ε by a real analytic function

$$\alpha(\varepsilon) = \alpha_0 + \frac{5}{\gamma}\omega_1\varepsilon^2 + \frac{5}{\gamma}\varepsilon^3 + \cdots .$$

If $\alpha < \alpha_0$ and α is sufficiently near α_0, there exist real values ε_- and ε_+ near $\varepsilon = 0$, such that

$$\alpha = \alpha(\varepsilon_-) = \alpha(\varepsilon_+), \qquad \varepsilon_- < 0 < \varepsilon_+,$$

with

$$\tau(\varepsilon_-) \neq 2, \qquad \tau(\varepsilon_+) \neq 2.$$

If $\alpha > \alpha_0$ and sufficiently near α_0, then $\alpha = \alpha(\varepsilon)$ does not have real solutions near $\varepsilon = 0$.

Index of a fixed point $P \in \mathbb{R}^2$ of mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as follows. Let U denote a small neighborhood of the fixed point. Define a mapping $\varphi : U \setminus \{P\} \to \mathbb{R}^2 \setminus \{O\}$ by $\varphi(X) = f(X) - X$. By an appropriate choice of the neighborhood U, the induced homomorphism, $\varphi_* : \pi_1(U \setminus \{P\}) \to \pi_1(\mathbb{R}^2 \setminus \{O\})$, of the fundamental groups defines an integer. This integer is called the local index of fixed point P.

By Poincaré's index theorem, the sum of the local indices of the fixed points is invariant under continuous perturbations of the mapping f. In the case of area preserving diffeomorphism, the local index of a saddle is -1, and the local index of a center is $+1$. So, the created two cycles cannot be the same type. This proves Theorem A.