Parabolic bifurcation of area-preserving Hénon maps

Periods 3 and 4

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Abstract

In the area preserving real Hénon maps, cycles bifurcate from a parabolic fixed point whose eigenvalues are prime roots of unity. Cases of 3-cycles and 4-cycles are studied.

1. Area-preserving complex Hénon map

$$H_\alpha : \mathbb{C}^2 \to \mathbb{C}^2, \qquad \alpha \in \mathbb{C},$$

$$H_\alpha(x, y) \;=\; (y, y^2 + \alpha - x).$$

$$\det DH_\alpha = 1$$

If $\alpha \in \mathbb{R}$, then H_α is a diffeomorphism of $\mathbb{R}^2$.

Fixed point

Fixed point $\quad P_* = (y_*, y_*)$ given by

$$y_*^2 - 2y_* + \alpha \;=\; 0.$$

$$DH_\alpha\big|_{P_*} \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 2y_* \end{pmatrix},$$

$$\text{trace } DH_\alpha\big|_{P_*} \;=\; 2y_*, \quad \det DH_\alpha \;=\; 1.$$

Parabolic bifurcation of order 3

Let us consider the case where $\omega = \frac{-1+\sqrt{3}i}{2}$ and $\bar{\omega} = \frac{-1-\sqrt{3}i}{2}$ are the eigenvalues of DH_α at the fixed point P_*. Then,

$$y_* \;=\; \frac{1}{2}(\omega+\bar{\omega}) = -\frac{1}{2} \quad \text{and} \quad \alpha_* \;=\; 2y_* - y_*^2 = -\frac{5}{4}.$$

Let $y_n = u_0 + \omega^n u_1 + \bar{\omega}^n u_2$, and suppose $y_{n+1} = y_n^2 + \alpha - y_{n-1}$ holds.

$$(F) \qquad \begin{cases} 2u_0 &= u_0^2 + 2u_1u_2 + \alpha \\ (\omega+\bar{\omega})u_1 &= 2u_0u_1 + u_2^2 \\ (\bar{\omega}+\omega)u_2 &= 2u_0u_2 + u_1^2 \end{cases}$$

When $\alpha = \alpha_*$, then we have a solution $u_0 = y_*, u_1 = u_2 = 0$. We fix constants $\alpha_* = -\frac{4}{5}$, $y_* = -\frac{1}{2}$ and set $u_0 = u_0(\varepsilon) = y_* - \frac{\varepsilon}{2}$. The second and third equations of (F) are rewritten as follows.

$$\begin{cases} \varepsilon u_1 & = & u_2^2 \\ \varepsilon u_2 & = & u_1^2 \end{cases}$$

We obtain $u_1 = \varepsilon\omega^k, u_2 = \varepsilon\bar{\omega}^k, \quad (k = 0, 1, 2)$. The choice of k corresponds to the choice of initial point in the periodic orbit. We choose $k = 0$ and obtain the solution

$$u_0 = y_* - \frac{\varepsilon}{2}, \qquad u_1 = \varepsilon, \qquad u_2 = \varepsilon.$$

It follows that

$$\alpha = \alpha_* - \frac{3}{2}\varepsilon - \frac{9}{4}\varepsilon^2 \;=\; -\frac{9}{4}(\varepsilon + \frac{1}{3})^2 - 1,$$

$$y_0 = -\frac{1}{2} + \frac{3}{2}\varepsilon, \quad y_1 = -\frac{1}{2} - \frac{3}{2}\varepsilon, \quad y_2 = -\frac{1}{2} - \frac{3}{2}\varepsilon.$$

The trace of the Jacobian matrix of the 3-cycle is given by:

$$\tau(\varepsilon) \;=\; 8y_2y_1y_0 - 2(y_2 + y_1 + y_0) \;=\; 2 + 9\varepsilon^2 + 27\varepsilon^3.$$

And

$$\tau(0) \;=\; 2, \quad \frac{d\tau}{d\varepsilon} \;=\; 9\varepsilon(9\varepsilon+2), \quad \tau(-\frac{1}{3}) = 2, \quad \tau(\frac{2}{3}) = -2.$$

Fig.1

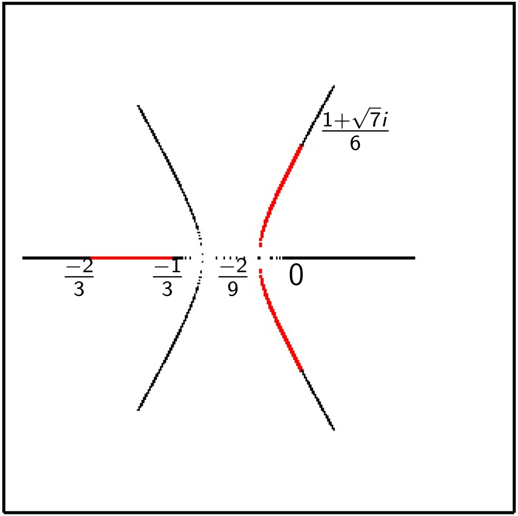

Fig.1 $\{\varepsilon \mid \tau(\varepsilon) \in [-2, 2]\}$ is drawn in red.

Fig.2

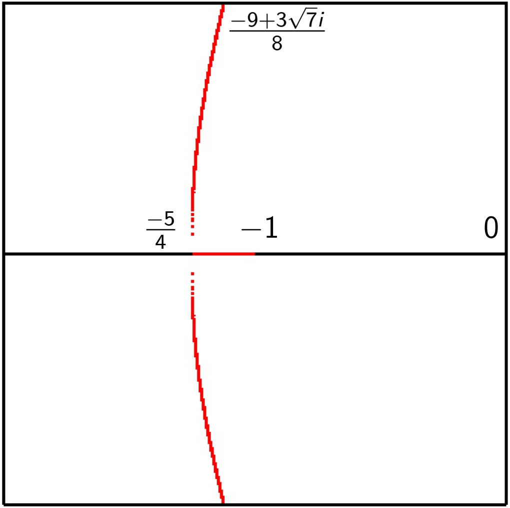

Fig.2 $\{\alpha(\varepsilon) \mid \tau(\varepsilon) \in [-2, 2]\}$ is drawn in red.

Fig.3

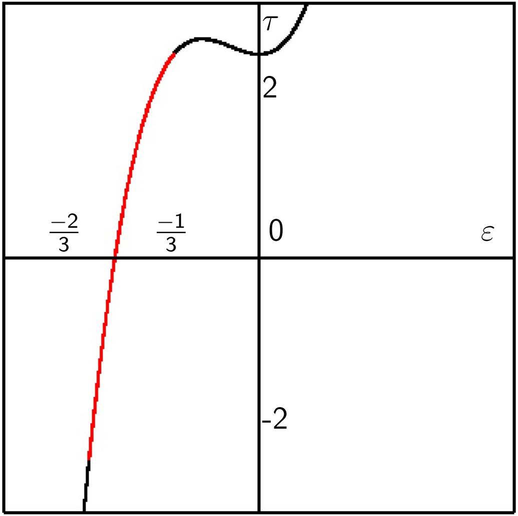

Fig.3. graph of $\tau(\varepsilon)$ for $-1 \leq \varepsilon \leq 1$.

Fig.4

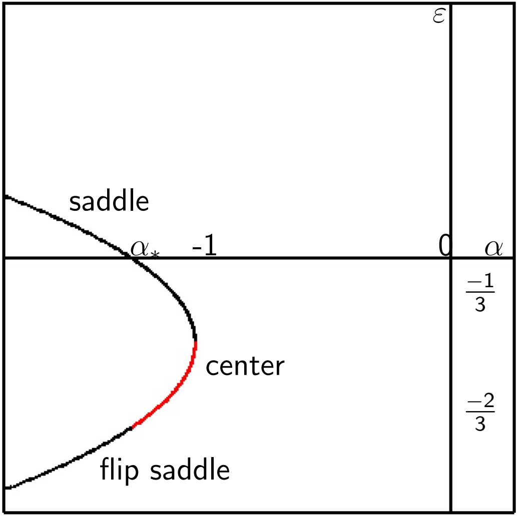

Fig.4. Bifurcation diagram of cycles of period 3.

Parabolic bifurcation of order 4

Let us consider the case where $\pm i$ are the eigenvalues of DH_α at the fixed point P_*. Then $y_* = 0$ and $\alpha_* = 0$.

Recall the equation of 4-periodic point. $(y_{n+4} = y_n)$

$$y_{n+1} \;=\; y_n^2 + \alpha - y_{n-1}, \qquad n = 0, \cdots, 3.$$

Discrete Fourier expansion

$$y_n \;=\; u_0 + i^n u_1 + (-1)^n u_2 + (-i)^n u_3$$

gives rise to the following system of equations.

$$\begin{aligned}
(F_0) \quad 2u_0 &= u_0^2 + u_2^2 + 2u_1u_3 + \alpha, \\
(F_1) \quad 0 &= 2u_0u_1 + 2u_2u_3, \\
(F_2) \quad -2u_2 &= 2u_0u_2 + u_1^2 + u_3^2, \\
(F_3) \quad 0 &= 2u_0u_3 + 2u_1u_2.
\end{aligned}$$

Necessary condition

From (F_1) and (F_3), we have

$$\begin{pmatrix} u_0 & u_2 \\ u_2 & u_0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_3 \end{pmatrix} = \mathbf{0}.$$

To have a non-trivial 4-cycle, it is necessary to have $u_0^2 = u_2^2$.

Case I

Case I $u_0 = u_2 = 0.$
In this case, from (F_2) and (F_0), we have two sub-cases

$$u_3 = iu_1, \quad \alpha = 2iu_1^2,$$

and

$$u_3 = -iu_1, \quad \alpha = -2iu_1^2,$$

They give the same 4-cycle. And the trace of the 4-cycle is given by

$$\tau = 2 - 64u_1^4.$$

In this case, real cycles for real α are all saddles.

Fig. 5

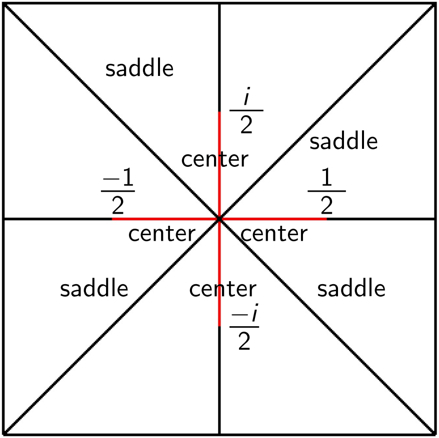

Fig.5. $\{u_1 \mid \tau(u_1) \in [-2, 2]\}$ is drawn in red in u_1 space

Fig.6

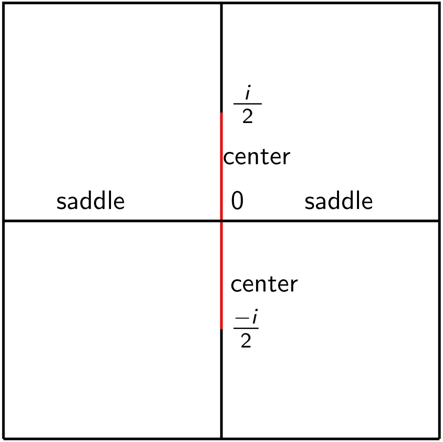

Fig.6. α space for CASE I.

Case II,III

Case II $u_2 = -u_0$ and $u_3 = u_1$.
From (F_2), $u_1 = \pm\sqrt{u_0^2 + u_0}$.

$$y_0 = 2u_1, \quad y_1 = 2u_0, \quad y_2 = -2u_1, \quad y_3 = 2u_0.$$

Case III $u_0 = u_2$ and $u_3 = -u_1$.
From (F_2), $u_1 = \pm\sqrt{-u_0^2 - u_0}$.

$$y_0 = 2u_0, \quad y_1 = 2iu_1, \quad y_2 = 2u_0, \quad y_3 = -2iu_1.$$

This gives the same orbit as in Case II.

Trace function

In these cases the trace of the 4-cycle is given by $\tau = 2 - 256u_0^3(1+u_0)$. And $\alpha = -4u_0^2$. For real u_0, the trace τ is real and plotted in Fig.7.
Location of the u_0 values with $\tau(u_0) \in [-2, 2]$ is plotted in Fig.8.
And the corresponding values of α are plotted in Fig.9 and Fig.10.
In Figs 9 and 10, segment $[\frac{-i}{2}, \frac{i}{2}]$ of $\textsc{Case}$ I is plotted,too.

Fig.7

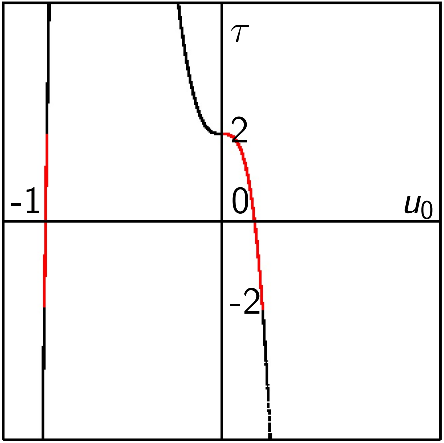

Fig.7. Graph of $\tau(u_0)$ for real u_0.

center locus

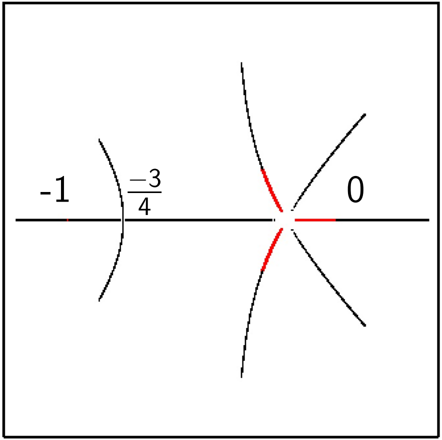

Fig.8. $\{u_0 \mid \tau(u_0) \in [-2, 2]\}$ is drawn in red.
Observe that a short interval near -1 is in red.

α space

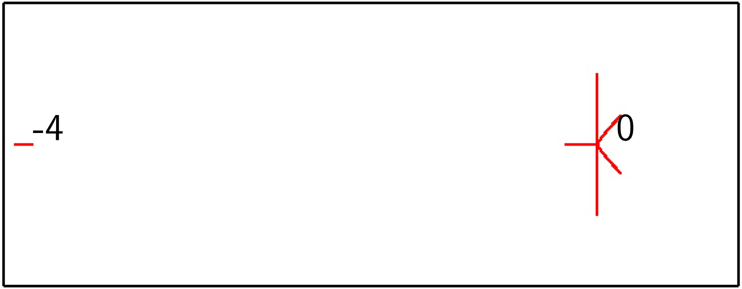

Fig.9. $\{\alpha(\varepsilon) \mid \tau(\varepsilon) \in [-2, 2]\}$ is drawn in red.

Enlarged

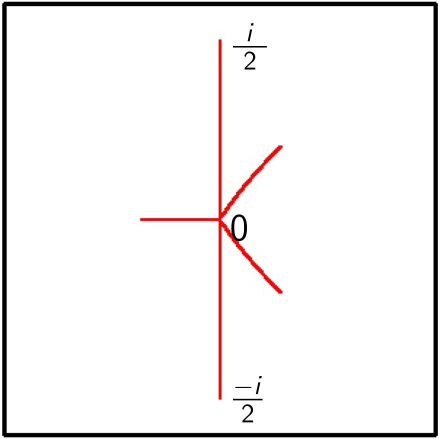

Fig.10. Enlargement of fig.9.

Real Bifurcation diagram

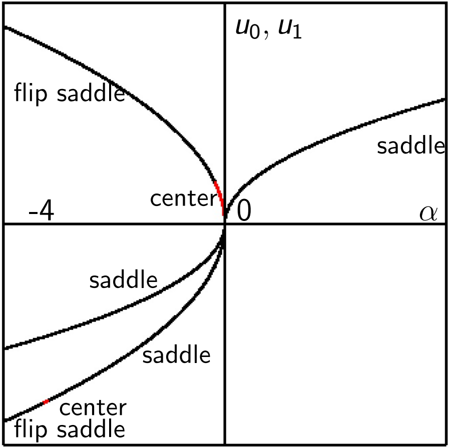

Fig.11. Bifurcation diagram of real 4-cycles for real α.