Abundance of Siegel balls in a family of Hénon maps

Shigehiro Ushiki
Graduate School of Human and Environmental Studies
Kyoto University

January 9, 2016

Abstract

There exists a real two-parameter family of complex Hénon maps such that the number of coexisting cycles of Siegel balls is unbounded in the subfamily of any open set of parameters.

Complex Hénon map

parameters $(\alpha, \beta) \in \mathbb{C}^2$

$$H_{\alpha,\beta} : \mathbb{C}^2 \to \mathbb{C}^2$$

$$H_{\alpha,\beta}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ \beta(y^2 + \alpha) - \beta^2 x \end{pmatrix}$$

Complex Hénon map

parameters $(\alpha, \beta) \in \mathbb{C}^2$

$$H_{\alpha,\beta} : \mathbb{C}^2 \to \mathbb{C}^2$$

$$H_{\alpha,\beta} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ \beta(y^2 + \alpha) - \beta^2 x \end{pmatrix}$$

$$h_{b,c} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + c + by \\ x \end{pmatrix}$$

Complex Hénon map

parameters $(\alpha, \beta) \in \mathbb{C}^2$

$$H_{\alpha,\beta} : \mathbb{C}^2 \to \mathbb{C}^2$$

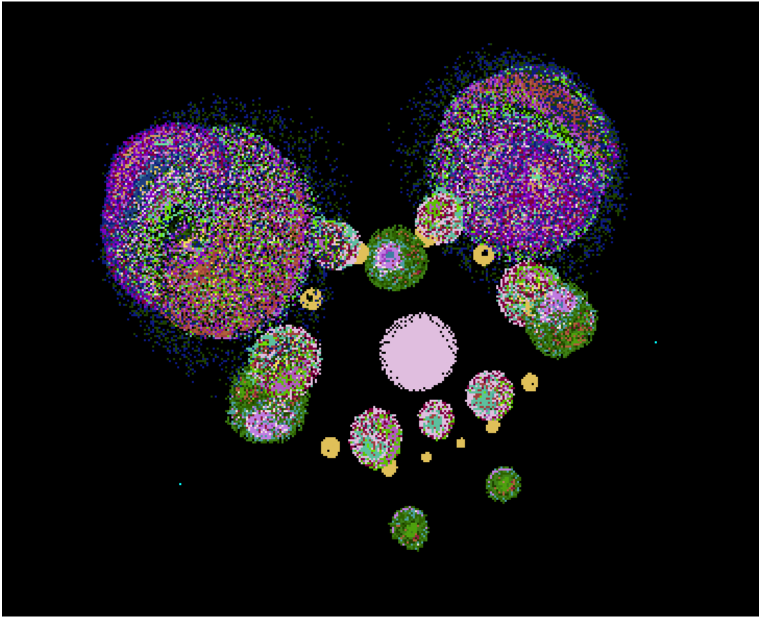
$$H_{\alpha,\beta}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ \beta(y^2+\alpha) - \beta^2 x \end{pmatrix}$$

$$h_{b,c}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2+c+by \\ x \end{pmatrix}$$

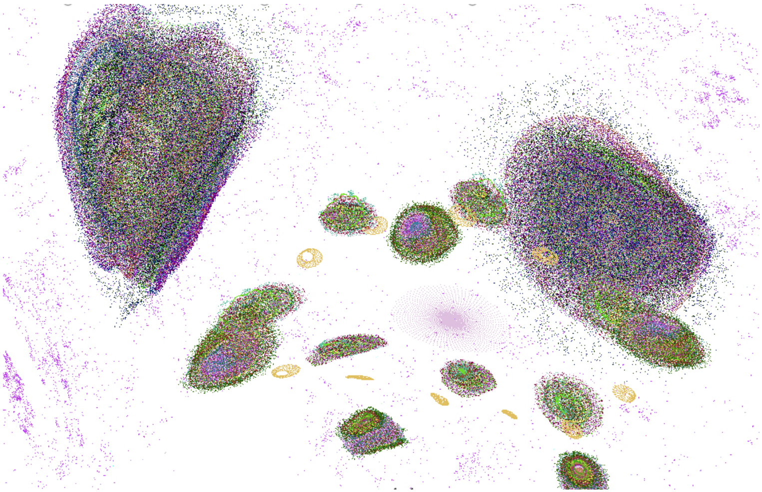
$$\phi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \beta y \\ \beta x \end{pmatrix}, \quad b = -\beta^2, \quad c = -\alpha b$$

$$\phi \circ H_{\alpha,\beta} \circ \phi^{-1} = h_{b,c}$$

Siegel ball



Siegel ball



Fixed point

Fixed point $P_* = \begin{pmatrix} y_* \\ y_* \end{pmatrix}$ is given by

$$y_* = \beta(y_*^2 + \alpha) - \beta^2 y_*,$$

or

$$y_*^2 - (\beta + \beta^{-1})y_* + \alpha = 0.$$

Fixed point

Fixed point $P_* = \begin{pmatrix} y_* \\ y_* \end{pmatrix}$ is given by

$$y_* = \beta(y_*^2 + \alpha) - \beta^2 y_*,$$

or

$$y_*^2 - (\beta + \beta^{-1})y_* + \alpha = 0.$$

The Jacobian matrix at the fixed point is as follows.

$$DH_{\alpha,\beta}|_{P_*} = \begin{pmatrix} 0 & 1 \\ -\beta^2 & 2\beta y_* \end{pmatrix} = \beta \begin{pmatrix} 0 & \beta^{-1} \\ -\beta & 2y_* \end{pmatrix}.$$

$$\text{trace } DH_{\alpha,\beta} = 2\beta y_*, \quad \det DH_{\alpha,\beta} = \beta^2.$$

Our family

We specify the eigenvalues at the fixed point.
For eigenvalues $\beta\mu, \beta\mu^{-1}$, we have

$$y_* = \frac{\mu + \mu^{-1}}{2}$$

and

$$\alpha = (\beta + \beta^{-1})y_* - y_*^2.$$

Our family is given by

Our family

We specify the eigenvalues at the fixed point.
For eigenvalues $\beta\mu, \beta\mu^{-1}$, we have

$$y_* = \frac{\mu + \mu^{-1}}{2}$$

and

$$\alpha = (\beta + \beta^{-1})y_* - y_*^2.$$

Our family is given by

$$\beta = \cos\theta + i\sin\theta,$$

$$\mu = \cos\varphi + i\sin\varphi.$$

Our family

We specify the eigenvalues at the fixed point.
For eigenvalues $\beta\mu, \beta\mu^{-1}$, we have

$$y_* = \frac{\mu + \mu^{-1}}{2}$$

and

$$\alpha = (\beta + \beta^{-1})y_* - y_*^2.$$

Our family is given by

$$\beta = \cos\theta + i\sin\theta,$$

$$\mu = \cos\varphi + i\sin\varphi.$$

with

$$y_* = \cos\varphi, \quad \alpha = 2\cos\theta\cos\varphi - \cos^2\varphi.$$

Abundance of Siegel balls

THEOREM

For any open set $U \subset (\mathbb{R}/2\pi\mathbb{Z})^2$ and any integer $N > 1$, there exists a point $(\theta_N, \varphi_N) \in U$ such that the Hénon map $H_{\alpha,\beta}$ for this parameter has more than N cycles of Siegel balls.

Elliptic-Parabolic fixed point

Let $p > 1$ be an integer, and let ν be a prime p-th root of unity.

$$\nu^k \neq 1 \quad (k = 1, \cdots, p-1),$$

$$\nu^p = 1.$$

Elliptic-Parabolic fixed point

Let $p > 1$ be an integer, and let ν be a prime p-th root of unity.

$$\nu^k \neq 1 \quad (k = 1, \cdots, p-1),$$

$$\nu^p \ = \ 1.$$

If ν is an eigenvalue of $DH_{\alpha,\beta}|_{P_*}$, then

$$\text{trace } DH_{\alpha,\beta}|_{P_*} \ = \ \nu + \beta^2 \bar{\nu}, \qquad y_* \ = \ \frac{\beta \bar{\nu} + \beta^{-1} \nu}{2},$$

$$\alpha_0 = (\beta + \beta^{-1}) y_* - y_*^2.$$

Periodic orbit

Suppose $\{P_n\} = \left\{ \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right\}$ is p-periodic. From

$$x_{n+1} = y_n, \qquad y_{n+1} = \beta(y_n^2 + \alpha) - \beta^2 x_n,$$

we have

$$\beta^{-1} y_{n+1} = y_n^2 + \alpha - \beta y_{n-1}.$$

Elliptic-parabolic bifurcation

Recall

$$y_* \; = \; \frac{\beta\bar{\nu} + \beta^{-1}\nu}{2}, \qquad \alpha_0 = (\beta + \beta^{-1})y_* - y_*^2.$$

THEOREM

For prime p-th root ν of 1, and for all $\beta \in \mathbb{C}$, except for a finite number of values, there exists a constant $\alpha_1 \neq 0$ and a family $y_n(\varepsilon)$ of p-cycles for parameter $\alpha = \alpha(\varepsilon^p)$, such that

$$\alpha(\varepsilon^p) = \alpha_0 + \alpha_1\varepsilon^p + O(\varepsilon^{2p}), \qquad y_n(\varepsilon) = y_* + \nu^n\varepsilon + O(\varepsilon^2),$$

holds for $\varepsilon \in \mathbb{C}$ near $\varepsilon = 0$.

PROPOSITION

The constant α_1 in the previous theorem depends upon β and is a non trivial rational function of β.

Self-anti-conjugate cycles

Recall

$$\alpha(\varepsilon^p) = \alpha_0 + \alpha_1\varepsilon^p + O(\varepsilon^{2p}), \qquad y_n(\varepsilon) = y_* + \nu^n\varepsilon + O(\varepsilon^2).$$

Theorem

In the previous theorem, if $|\beta| = 1$ then $\alpha_0 \in \mathbb{R}$ and $\alpha_1 \in \mathbb{R}$. Moreover, if $\alpha \in \mathbb{R}$, near α_0, then

$$y_n = \overline{y_{-n}} \qquad \text{or} \qquad y_n = \overline{y_{1-n}}$$

for some ε.

Self-anti-conjugate jacobian matrix

THEOREM

If p-periodic orbit is self-anti-conjugate, *i.e.*

$$y_n = \overline{y_{-n}} \qquad \text{or} \qquad y_n = \overline{y_{1-n}},$$

then the jacobian matrix along the orbit is of the form

$$D(H_{\alpha,\beta}^{\circ p})\big|_{P_0} \;=\; \beta^p A,$$

with $\det A = 1$ and trace $A \in \mathbb{R}$.

Trace function

Let

$$\tau(\beta, \alpha) \;=\; \beta^{-p} \text{ trace } D(H_{\alpha,\beta}^{\circ p})\big|_{P_0}.$$

Proposition

$\tau(\beta, \alpha)$ is holomorphic near (β_0, α_0), and non-constant with respect to α.

Proposition

If $|\beta| = 1$ and $\alpha \in \mathbb{R}$, then $\tau(\beta, \alpha) \in \mathbb{R}$ near $(\beta, \alpha_0(\beta))$, and $-2 < \tau(\beta, \alpha_0(\beta)) < 2$. (Except for finitely many values of β.)

Discrete Fourier expansion

Recall the equation of p-periodic point. $(y_{n+p} = y_n)$

$$\beta^{-1} y_{n+1} \; = \; y_n^2 + \alpha - \beta y_{n-1}, \qquad n = 0, \cdots, p-1.$$

Discrete Fourier expansion

$$\begin{aligned} y_n \; &= \; u_0 + \nu^n u_1 + \nu^{2n} u_2 + \cdots + \nu^{kn} u_k + \cdots + \nu^{(p-1)n} u_{p-1} \\ &= \sum_{k=0}^{p-1} \nu^{kn} u_k \end{aligned}$$

gives rise to the following equation.

Equation (F)

$$
\begin{aligned}
(F_0) \quad (\beta + \beta^{-1})u_0 &= u_0^2 + \sum_{\ell=1}^{p-1} u_\ell u_{p-\ell} + \alpha \\
(F_1) \quad (\beta\bar{\nu} + \beta^{-1}\nu)u_1 &= 2u_0 u_1 + \sum_{\ell=2}^{p-1} u_\ell u_{p+1-\ell} \\
(F_k) \quad (\beta\bar{\nu}^k + \beta^{-1}\nu^k)u_k &= 2u_0 u_k + \sum_{\ell=1}^{k-1} u_\ell u_{k-\ell} + \sum_{\ell=k+1}^{p-1} u_\ell u_{p+k-\ell}.
\end{aligned}
$$

$$k = 2, \cdots, p-1.$$

Equation (F)

$$(F_0) \quad (\beta + \beta^{-1})u_0 \;=\; u_0^2 + \sum_{\ell=1}^{p-1} u_\ell u_{p-\ell} + \alpha$$

$$(F_1) \quad (\beta\bar{\nu} + \beta^{-1}\nu)u_1 \;=\; 2u_0 u_1 + \sum_{\ell=2}^{p-1} u_\ell u_{p+1-\ell}$$

$$(F_k) \quad (\beta\bar{\nu}^k + \beta^{-1}\nu^k)u_k \;=\; 2u_0 u_k + \sum_{\ell=1}^{k-1} u_\ell u_{k-\ell} + \sum_{\ell=k+1}^{p-1} u_\ell u_{p+k-\ell}.$$

$$k = 2, \cdots, p-1.$$

This equation has a solution, corresponding to the elliptic-parabolic fixed point P_*.

$$u_0 = y_*, \quad u_1 = u_2 = \cdots = u_{p-1} = 0, \quad \alpha = \alpha_0.$$

Equation (F')

Rewite (F) as follows.

$$(F_0) \quad (\beta + \beta^{-1})u_0 \;=\; u_0^2 + \sum_{\ell=1}^{p-1} u_\ell u_{p-\ell} + \alpha$$

$$(F_1') \quad (\beta\bar{\nu} + \beta^{-1}\nu - 2u_0)u_1 \;=\; \sum_{\ell=2}^{p-1} u_\ell u_{p+1-\ell}$$

$$(F_k') \quad (\beta\bar{\nu}^k + \beta^{-1}\nu^k - 2u_0)u_k \;=\; \sum_{\ell=1}^{k-1} u_\ell u_{k-\ell} + \sum_{\ell=k+1}^{p-1} u_\ell u_{p+k-\ell}.$$

$$k = 2, \cdots, p-1.$$

Emanating branch of periodic points

Parameter α appears only in (F_0).

For each β, let $\varepsilon \in \mathbb{C}$ be a small parameter and let $\delta \in \mathbb{C}$ be a constant to be determined. Dependence upon β will be considered later.

Suppose $u_0 \;=\; y_* - \dfrac{\delta}{2}\varepsilon^p$, and $u_1 \;=\; \varepsilon v_1$.
From equation (F'), we may suppose, inductively, $u_k \;=\; \varepsilon^k v_k$, $(k = 2, \cdots, p-1)$.
Here, $v_1, v_2, \cdots, v_{p-1}$ are functions of ε. (Later, we see they are functions of ε^p.)

Emanating branch of periodic points

Parameter α appears only in (F_0).

For each β, let $\varepsilon \in \mathbb{C}$ be a small parameter and let $\delta \in \mathbb{C}$ be a constant to be determined. Dependence upon β will be considered later.

Suppose $u_0 = y_* - \dfrac{\delta}{2}\varepsilon^p$, and $u_1 = \varepsilon v_1$.
From equation (F'), we may suppose, inductively, $u_k = \varepsilon^k v_k$, $(k = 2, \cdots, p-1)$.
Here, $v_1, v_2, \cdots, v_{p-1}$ are functions of ε. (Later, we see they are functions of ε^p.)

$$(E_0) \qquad u_0 = y_* - \frac{\delta}{2}\varepsilon^p,$$

$$(E_k) \qquad u_k = \varepsilon^k v_k, \qquad k = 1, \cdots, p-1.$$

Equation (G)

Rewrite equation (F) using (E), with $y_* = \dfrac{\beta\bar{\nu} + \beta^{-1}\nu}{2}$, to get the following equation.

$$(G_1) \qquad \delta v_1 = \sum_{\ell=2}^{p-1} v_\ell v_{p+1-\ell},$$

$$(G_k) \qquad (\beta(\bar{\nu}^k - \bar{\nu}) + \beta^{-1}(\nu^k - \nu) + \delta\varepsilon^p)v_k$$

$$= \sum_{\ell=1}^{k-1} v_\ell v_{k-\ell} + \varepsilon^p \sum_{\ell=k+1}^{p-1} v_\ell v_{p+k-\ell},$$

$$k = 2, \cdots, p-1.$$

Note that α will be computed by (F_0), afterwards.

Principal part equation (L)

Let $v_k \;=\; a_k + O(\varepsilon), \quad k = 1, \cdots, p-1.$ And let

$$\gamma_k \;=\; \beta(\bar{\nu}^k - \bar{\nu}) + \beta^{-1}(\nu^k - \nu),$$

for $k = 2, \cdots, p-1$.
Equation (G), as $\varepsilon \to 0$, yields the following equation.

Principal part equation (L)

Let $v_k \; = \; a_k + O(\varepsilon), \quad k = 1, \cdots, p-1.$ And let

$$\gamma_k \; = \; \beta(\bar{\nu}^k - \bar{\nu}) + \beta^{-1}(\nu^k - \nu),$$

for $k = 2, \cdots, p-1$.
Equation (G), as $\varepsilon \to 0$, yields the following equation.

$$(L_1) \qquad \delta a_1 \; = \; \sum_{\ell=2}^{p-1} a_\ell a_{p+1-\ell},$$

$$(L_k) \qquad \gamma_k a_k \; = \; \sum_{\ell=1}^{k-1} a_\ell a_{k-\ell}, \qquad k = 2, \cdots, p-1.$$

Constant δ

Inductively from $(L_2), \cdots, (L_{p-1})$, we have

$$a_k \;=\; \frac{1}{\gamma_k}\sum_{\ell=1}^{k-1} a_\ell a_{k-\ell} \;=\; \eta_k a_1^k,$$

with $\eta_k = \eta_k(\beta)$ rational function of β, for $k = 2, \cdots, p-1$.
Or

$$\eta_1 = 1, \quad \eta_k = \frac{1}{\gamma_k}\sum_{\ell=1}^{k-1} \eta_\ell \eta_{k-\ell}.$$

Constant δ

Inductively from $(L_2), \cdots, (L_{p-1})$, we have

$$a_k \;=\; \frac{1}{\gamma_k}\sum_{\ell=1}^{k-1} a_\ell a_{k-\ell} \;=\; \eta_k a_1^k,$$

with $\eta_k = \eta_k(\beta)$ rational function of β, for $k = 2, \cdots, p-1$.
Or

$$\eta_1 = 1, \quad \eta_k = \frac{1}{\gamma_k}\sum_{\ell=1}^{k-1} \eta_\ell \eta_{k-\ell}.$$

From (L_1),

$$\delta a_1 \;=\; \Phi(\beta) a_1^{p+1}$$

with a rational function $\Phi(\beta)$.

Non-triviality

Proposition.
$\Phi(\beta)$ is a non-trivial rational function of β.

Non-triviality

PROPOSITION.
$\Phi(\beta)$ is a non-trivial rational function of β.

PROOF.
Obviously, $\Phi(\beta)$ is a rational function of β. We show that $\Phi(-\nu) > 0$.
If $\beta = -\nu$, then

$$\gamma_k \ = \ 2 - (\bar{\nu}^{k-1} + \nu^{k-1}), \quad k = 2, \cdots, p-1.$$

Therefore, $0 < \gamma_k(-\nu) \leq 4, \quad k = 2, \cdots, p-1.$
This imply $\eta_k(-\nu) > 0, \quad k = 2, \cdots, p-1.$
And $\Phi(-\nu) > 0$, since

$$\Phi(\beta) \ = \ \sum_{\ell=2}^{p-1} \eta_\ell(\beta)\eta_{p+1-\ell}(\beta).$$

Reality of solutions

If $|\beta| = 1$, then

$$\gamma_k \in \mathbb{R}, \qquad (k = 2, \cdots, p-1),$$

$$\eta_k \in \mathbb{R}, \qquad (k = 1, \cdots, p-1),$$

and $$\Phi(\beta) \in \mathbb{R}.$$
These are non-zero except for a finite number of values of β.

Now, we determine the constant δ by

$$\delta \;=\; \Phi(\beta).$$

Equation (L) has a solution

$$a_1 = 1, \quad a_k = \eta_k(\beta), \quad k = 2, \cdots, p-1.$$

If $|\beta| = 1$, then $a_1, \cdots, a_{p-1}$ are all real.
Other solutions give the same periodic orbit.

Equation (G)

Now, we go back to equation (G).

$$(G_1) \qquad \delta v_1 \;=\; \sum_{\ell=2}^{p-1} v_\ell v_{p+1-\ell},$$

$$(G_k) \qquad (\gamma_k + \delta\varepsilon^p)v_k \;=\; \sum_{\ell=1}^{k-1} v_\ell v_{k-\ell} \;+\; \varepsilon^p \sum_{\ell=k+1}^{p-1} v_\ell v_{p+k-\ell},$$
$$k = 2, \cdots, p-1.$$

Observe that ε appears only as ε^p in this equation.
Let $\kappa = \varepsilon^p$ and rewrite the equation as follows.

Equation (Γ)

$$(\Gamma_1) \qquad w_1 \;=\; \sum_{\ell=2}^{p-1} v_\ell v_{p+1-\ell} - \delta v_1,$$

$$(\Gamma_k) \qquad w_k \;=\; \sum_{\ell=1}^{k-1} v_\ell v_{k-\ell} - \gamma_k v_k \;+\; \kappa \left(\sum_{\ell=k+1}^{p-1} v_\ell v_{p+k-\ell} - \delta v_k \right),$$

$$k = 2, \cdots, p-1.$$

These define a map

Equation (Γ)

$$(\Gamma_1) \qquad w_1 \;=\; \sum_{\ell=2}^{p-1} v_\ell v_{p+1-\ell} - \delta v_1,$$

$$(\Gamma_k) \qquad w_k \;=\; \sum_{\ell=1}^{k-1} v_\ell v_{k-\ell} - \gamma_k v_k \;+\; \kappa \left(\sum_{\ell=k+1}^{p-1} v_\ell v_{p+k-\ell} - \delta v_k \right),$$

$$k = 2, \cdots, p-1.$$

These define a map

$$\Gamma : \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{p-1} \to \mathbb{C}^{p-1}$$

by

$$\Gamma(\beta, \kappa, v_1, \cdots, v_{p-1}) \;=\; (w_1, \cdots, w_{p-1}).$$

Γ is a quadratic polynomial in $v_1, \cdots, v_{p-1}$, with coefficients rational in β, and affine in κ.

Matrix Λ_β

Except for a finite number of values of β,

$$\Gamma(\beta, 0, a_1, \cdots, a_{p-1}) = (0, \cdots, 0)$$

holds.The jacobian matrix $\Lambda_\beta = \left(\dfrac{\partial w_i}{\partial v_j}\right)$ at $(\beta, 0, a_1, \cdots, a_{p-1})$ is as follows.

$$\Lambda_\beta = \begin{pmatrix} -\delta & 2a_{p-1} & 2a_{p-2} & \cdots & \cdots & 2a_2 \\ 2a_1 & -\gamma_2 & 0 & \cdots & \cdots & 0 \\ 2a_2 & 2a_1 & -\gamma_3 & 0 & \cdots & 0 \\ 2a_3 & 2a_2 & 2a_1 & -\gamma_4 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 2a_{p-2} & 2a_{p-3} & \cdots & 2a_2 & 2a_1 & -\gamma_{p-1} \end{pmatrix}.$$

All components of Λ_β are rational functions of β.

Regularity of Λ_β

PROPOSITION
If $p \geq 3$, the matrix Λ_β is regular except for a finite number of values of β.

Regularity of Λ_β

PROPOSITION

If $p \geq 3$, the matrix Λ_β is regular except for a finite number of values of β.

PROOF

Obviously, the determinant of Λ_β is a rational function of β. We show that it is non-trivial. Let

$$M_\beta = \begin{pmatrix} -\delta & a_{p-1} & a_{p-2} & \cdots & \cdots & a_2 \\ a_1 & -\gamma_2 & 0 & \cdots & \cdots & 0 \\ a_2 & a_1 & -\gamma_3 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & -\gamma_4 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ a_{p-2} & a_{p-3} & \cdots & a_2 & a_1 & -\gamma_{p-1} \end{pmatrix}.$$

Equation (L) is equivalent to

$$M_\beta \begin{pmatrix} a_1 \\ \vdots \\ a_{p-1} \end{pmatrix} = \mathbf{0}.$$

As we have non-trivial solutions for all values of β, except for a finite number of values,

$$\det M_\beta = 0$$

holds as a rational function of β.

Observe the sweeping-out process of M_β. To sweep out the off-diagonal components of the first line, other lines of M_β are used with diagonal components $\gamma_2, \cdots, \gamma_{p-1}$ as pivots. These pivots are non-trivial rational functions of β.

To suppress a term in $(1, k)$-component of M_β, say t_k, we add the k-th line multiplied by t_k/γ_k to the first line. Then the $(1, j)$-component, say c_j, becomes

$$c_j + \frac{a_{k-j}}{\gamma_k} t_k, \qquad j = 1, \cdots, k-1.$$

In this process, all the components of the first line, except for $-\delta$, are sums of terms of the form

$$\frac{1}{\gamma_{k_1}^{m_1} \cdots \gamma_{k_\ell}^{m_\ell}}.$$

When all the off-diagonal components of the first line are swept out, the first line vanishes.

Next, let us compute $\det \Lambda_\beta$ in a similar way. To compare the sweeping-out process, let $b_k = a_k$ and rewrite the off-diagonal components fof Λ_β as

$$2a_k \;=\; a_k + b_k, \qquad k = 1, \cdots, p-1.$$

Sweep-out the off-diagonal components of the first line of Λ_β to get a lower triangle matrix. The terms without b_k's are exactly same as in the sweep-out procedure of M_β. The terms without a_k's contribute exactly same. There are other terms consisting of a_k's and b_k's. Anyway, all the terms are always sums of terms of the form

$$\frac{1}{\gamma_{k_1}^{m_1} \cdots \gamma_{k_\ell}^{m_\ell}}.$$

After the sweeping-out, in the $(1,1)$-component of the triangle matrix, $-\delta$ cancells the a_k-only terms. And b_k-only terms gives δ. The remaining terms are a sum of terms of the form

$$\frac{1}{\gamma_{k_1}^{m_1}\cdots\gamma_{k_\ell}^{m_\ell}},$$

with positive coefficients.

The $(1,1)$-component of the triangle matrix is obviously a rational function of β. To prove the non-triviality of the rational function, we show that it does not vanish for $\beta=-\nu$. As

$$\begin{aligned}\gamma_k(\beta) &= \beta(\bar{\nu}^k-\bar{\nu})+\beta^{-1}(\nu^k-\nu),\\ \gamma_k(-\nu) &= 2-(\nu^{k-1}+\bar{\nu}^{k-1})>0, \qquad k=2,\cdots,p-1.\end{aligned}$$

These imply the positivity of $a_1, \cdots, a_{p-1}$ and δ. Furthermore, terms of the form

$$\frac{1}{\gamma_{k_1}^{m_1} \cdots \gamma_{k_\ell}^{m_\ell}}$$

are all positive. Hence the $(1,1)$-cpmponent of the triangle matrix is strictly positive and greater than δ.

We conclude that

$$\det \Lambda_{-\nu} \neq 0.$$

Equation (Γ)

Now, we go back to equation (Γ).

$$(\Gamma_1) \qquad w_1 = \sum_{\ell=2}^{p-1} v_\ell v_{p+1-\ell} - \delta v_1,$$

$$(\Gamma_k) \qquad w_k = \sum_{\ell=1}^{k-1} v_\ell v_{k-\ell} - \gamma_k v_k + \kappa \left(\sum_{\ell=k+1}^{p-1} v_\ell v_{p+k-\ell} - \delta v_k \right),$$

$$k = 2, \cdots, p-1.$$

$$\Gamma(\beta, \kappa, v_1, \cdots, v_{p-1}) = (w_1, \cdots, w_{p-1}).$$

PROPOSITION

For any $\beta_0 \in \mathbb{C}$, except for a finite number of values, there exists a neighborhood, U, of $(\beta_0, 0) \in \mathbb{C}^2$, such that implicit functions

$$v_1(\beta,\kappa), v_2(\beta,\kappa), \cdots, v_{p-1}(\beta,\kappa)$$

defined by

$$\Gamma(\beta,\kappa,v_1,\cdots,v_{p-1}) \;=\; (0,\cdots,0)$$

with

$$v_k(\beta_0,0) \;=\; a_k(\beta_0), \qquad k=1,\cdots,p-1,$$

exist and holomorphic in U.

Proof

Except for a finite number of values of β,

$$\Gamma(\beta, 0, a_1, \cdots, a_{p-1}) \;=\; (0, \cdots, 0)$$

holds.

The jacobian matrix $\Lambda_\beta \;=\; \left(\dfrac{\partial w_i}{\partial v_j}\right)$ at $(\beta, 0, a_1, \cdots, a_{p-1})$ is regular. Apply the implicit function theorem.

Parameter α

Functions $v_1, \cdots, v_{p-1}$, with $\kappa = \varepsilon^p$ give solutions of equation (G).

Next, let us go back to equation (F).
We introduced redundant parameters ε and δ. Parameter δ is determined as a rational function of β. The redundant parameter $\kappa = \varepsilon^p$ is related to the remaining parameter α by equation (F_0). From equation (F_0), we have

$$(K) \qquad \alpha = (\beta + \beta^{-1})(y_* - \frac{\delta}{2}\kappa) - (y_* - \frac{\delta}{2}\kappa)^2 - \kappa \sum_{\ell=1}^{p-1} v_\ell v_{p-\ell},$$

which is a function of β and κ.

As

$$y_* = \frac{\beta\bar{\nu} + \beta^{-1}\nu}{2}, \qquad \alpha_0 = (\beta + \beta^{-1})y_* - y_*^2,$$

$$\alpha - \alpha_0 = -\kappa\left(\sum_{\ell=1}^{p-1} v_\ell v_{p-\ell} - y_*\delta + \frac{\delta}{2}(\beta + \beta^{-1})\right) - \frac{\delta^2}{4}\kappa^2$$

$$= -\kappa\left(\sum_{\ell=1}^{p-1} a_\ell a_{p-\ell} + \frac{\delta}{2}(\beta(1-\bar{\nu}) + \beta^{-1}(1-\nu))\right) + o(\kappa).$$

By setting

$$\alpha_1 = -\left(\sum_{\ell=1}^{p-1} a_\ell a_{p-\ell} + \frac{\delta}{2}(\beta(1-\bar{\nu}) + \beta^{-1}(1-\nu))\right),$$

we have

$$\alpha = \alpha_0 + \alpha_1\kappa + o(\kappa).$$

Recall

$$\alpha_1 \;=\; -\left(\sum_{\ell=1}^{p-1} a_\ell a_{p-\ell} + \frac{\delta}{2}(\beta(1-\bar{\nu}) + \beta^{-1}(1-\nu))\right).$$

PROPOSITION

α_1 is a non-trivial rational function of β, and takes real value if $|\beta| = 1$.

Recall

$$\alpha_1 \;=\; -\left(\sum_{\ell=1}^{p-1} a_\ell a_{p-\ell} + \frac{\delta}{2}(\beta(1-\bar{\nu}) + \beta^{-1}(1-\nu))\right).$$

PROPOSITION

α_1 is a non-trivial rational function of β, and takes real value if $|\beta| = 1$.

PROOF

If $\beta = -\nu$, then, as in the preceeding proposition, $a_1, \cdots, a_{p-1}$ and δ are real and positive.

Moreover, $\beta(1-\bar{\nu}) + \beta^{-1}(1-\nu) \;=\; 2 - (\nu + \bar{\nu}) > 0$.

Hence, $\alpha_1(-\nu) < 0$, which shows α_1 is non-trivial.

If $|\beta| = 1$, then $\beta^{-1} = \bar{\beta}$. The reality of $\gamma_1, \cdots, \gamma_{p-1}$, $a_1, \cdots, a_{p-1}$, and δ is obvious.

Parameters α and κ

Recall

$$y_*(\beta) \;=\; \frac{\beta\bar{\nu} + \beta^{-1}\nu}{2}, \quad \alpha_0(\beta) \;=\; ((\beta + \beta^{-1}) - y_*(\beta))y_*(\beta).$$

$$(K) \qquad \alpha \;=\; (\beta + \beta^{-1})(y_* - \frac{\delta}{2}\kappa) - (y_* - \frac{\delta}{2}\kappa)^2 - \kappa \sum_{\ell=1}^{p-1} v_\ell v_{p-\ell}.$$

$$\alpha \;=\; \alpha_0 + \alpha_1\kappa + o(\kappa).$$

Proposition

For all $\beta_0 \in \mathbb{C}$, except for a finite number of values, there exists a neighborhood, U, of $(\beta_0, \alpha_0(\beta_0)) \in \mathbb{C}^2$, such that the implicit function $\kappa = \kappa(\beta, \alpha)$ satisfying $\kappa(\beta_0, \alpha_0(\beta_0)) = 0$ defined by equation (K) exists and holomorphic in U.

Choice of initial point

Now, fix β_0 with $|\beta_0| = 1$, and set $\alpha_0 = \alpha_0(\beta_0) \in \mathbb{R}$.
If $\alpha \in \mathbb{R}$, and $|\alpha - \alpha_0|$ is sufficiently small, then $\kappa(\beta_0, \alpha)$ is real, since the preceeding procedure keeps the realities.

We choose a p-th root, ε, of $\kappa(\beta_0, \alpha)$ as wollows.

CASE I If p is odd or $\kappa(\beta_0, \alpha) > 0$, then the equation $\varepsilon^p = \kappa(\beta_0, \alpha)$ has a real root.

CASE II If p is even and $\kappa(\beta_0, \alpha) < 0$, then take a solution of $\varepsilon^p = \kappa(\beta_0, \alpha)$ satisfying $\varepsilon = \bar{\nu}\bar{\varepsilon}$.

Choice of initial point

Now, fix β_0 with $|\beta_0| = 1$, and set $\alpha_0 = \alpha_0(\beta_0) \in \mathbb{R}$.
If $\alpha \in \mathbb{R}$, and $|\alpha - \alpha_0|$ is sufficiently small, then $\kappa(\beta_0, \alpha)$ is real, since the preceeding procedure keeps the realities.

We choose a p-th root, ε, of $\kappa(\beta_0, \alpha)$ as wollows.

CASE I If p is odd or $\kappa(\beta_0, \alpha) > 0$, then the equation $\varepsilon^p = \kappa(\beta_0, \alpha)$ has a real root.

CASE II If p is even and $\kappa(\beta_0, \alpha) < 0$, then take a solution of $\varepsilon^p = \kappa(\beta_0, \alpha)$ satisfying $\varepsilon = \bar{\nu}\bar{\varepsilon}$.

Choice of ε determines the chice of the initial point of the cycle.

Self-anti-conjugate cycle

CASE I

Real ε gives real solutions $v_1, \cdots, v_{p-1}$ of equation (G), and real solutions $u_0, \cdots, u_{p-1}$ of equations (F) and (E).

These give rise to a periodic orbit

$$y_n = u_0 + \nu^n u_1 + \cdots + \nu^{(p-1)n} u_{p-1},$$

with real y_0. We see that $y_{-n} = \bar{y}_n$.

$$\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = H_{\alpha,\beta} \begin{pmatrix} y_{-1} \\ y_0 \end{pmatrix}.$$

$$P_1 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \text{ and } P_0 = \begin{pmatrix} y_{-1} \\ y_0 \end{pmatrix}$$

are swap-conjugate to each other.

The obtained periodic orbit is self-anti-conjugate. The jacobian matrix along the orbit is of the form

$$D(H_{\alpha,\beta}^{\circ p})\big|_{P_0} \;=\; \beta^p A,$$

with $\det A = 1$ and trace $A \in \mathbb{R}$.

Self-anti-conjugate cycle

CASE II
We took ε satisfying $\varepsilon = \bar{\nu}\bar{\varepsilon}$ and $\varepsilon^p = \bar{\varepsilon}^p = \kappa(\beta_0, \alpha) \in \mathbb{R}$.

Solutions $v_1, \cdots, v_{p-1}$ of equation (G) are real, since $\kappa(\beta_0, \alpha) \in \mathbb{R}$.

The solutions of equation (F) are as follows.

$$u_0 \ = \ y_* - \frac{\delta}{2}\varepsilon^p, \qquad u_k \ = \ \varepsilon^k v_k, \quad (k = 1, \cdots, p-1).$$

As u_0 is real, we have

$$
\begin{aligned}
y_n &= \sum_{k=0}^{p-1} \nu^{kn} u_k = u_0 + \sum_{k=1}^{p-1} \nu^{kn} \varepsilon^k v_k = u_0 + \sum_{k=1}^{p-1} \nu^{k(n-1)} \bar{\varepsilon}^k v_k \\
&= u_0 + \overline{\sum_{k=1}^{p-1} \nu^{-k(n-1)} \varepsilon^k v_k} = \overline{\sum_{k=0}^{p-1} \nu^{k(1-n)} u_k} = \overline{y_{1-n}}.
\end{aligned}
$$

The obtained periodic orbit is self-anti-conjugate. The jacobian matrix along the orbit is of the form

$$
D(H_{\alpha,\beta}^{\circ p})\big|_{P_0} = \beta^p A,
$$

with $\det A = 1$ and trace $A \in \mathbb{R}$.

Trace function

Let

$$\tau \;=\; \beta^{-p} \text{ trace } D(H_{\alpha,\beta}^{\circ p})\big|_{P_0}.$$

τ is an algebraic function of (α, β).

Note that τ does not depend on the choice of ε among the p-th root of κ, sice the choice of ε corresponds to the choice of the initial point in the periodic orbit.

τ is locally univalent and continuous near (β_0, α_0). Hence τ is holomorphic in (β, α) near (β_0, α_0).

As we saw, if $|\beta| = 1$ and $\alpha \in \mathbb{R}$, then $\tau(\beta, \alpha) \in \mathbb{R}$ near (β_0, α_0), and $-2 < \tau(\beta_0, \alpha_0) < 2$.

PROPOSITION

$\tau(\beta, \alpha)$ is holomorphic and non-constant near (β_0, α_0).

PROOF

Consider the analytic continuation of τ. As τ is algebraic, continuation along the real axis of α exists by avoiding branch points choosing some branch. If $|\alpha|$ is sufficiently large, then all the periodic points of the Hénon map are hyperbolic, and $|\tau(\beta_0, \alpha)| > 2$ there.

Trace function

$$\text{Let} \quad B = \begin{pmatrix} 0 & \beta^{-1} \\ -\beta & 0 \end{pmatrix} \quad \text{and} \quad Y_k = \begin{pmatrix} 0 & 0 \\ 0 & 2y_k \end{pmatrix}.$$

$$\begin{aligned} \tau \; &= \; \beta^{-p} \, \mathrm{trace} \, D(H^{\circ p}_{\alpha,\beta})\big|_{P_0} \\ &= \mathrm{trace} \, \left(\begin{pmatrix} 0 & \beta^{-1} \\ -\beta & 2y_{p-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & \beta^{-1} \\ -\beta & 2y_0 \end{pmatrix} \right) \\ &= \mathrm{trace} \, \left((B + Y_{p-1}) \cdots (B + Y_0) \right). \end{aligned}$$

$$\tau \; = \; \sum_{k=0}^{[\frac{p}{2}]} (-1)^k 2^{p-2k} \left(\sum_{0 \le i_1 < i_2 < \cdots < i_{p-2k} < p} y_{i_1} y_{i_2}, \cdots y_{i_{p-2k}} \right).$$

Here sum in the parenthes is taken over $i_1, \cdots, i_{p-2k}$ with some extra condition.

Coefficients of $\tau(\kappa)$ are rational functions of β.

Family of Hénon maps

For $(\theta, \varphi) \in (\mathbb{R}/2\pi\mathbb{Z})^2$, let

$$\beta \;=\; \cos\theta + i\sin\theta, \qquad \alpha \;=\; 2\cos\theta\cos\varphi - \cos^2\varphi,$$

which defines a family of self-anti-conjugate Hénon maps

$$H_{\alpha,\beta}\begin{pmatrix} x \\ y \end{pmatrix} \;=\; \begin{pmatrix} y \\ \beta(y^2+\alpha) - \beta^2 x \end{pmatrix},$$

with $|\beta| = 1$ and $\alpha \in \mathbb{R}$.

Abundance of Siegel balls

THEOREM

For any open set $U \subset (\mathbb{R}/2\pi\mathbb{Z})^2$ and any integer $N > 1$, there exists a point $(\theta_N, \varphi_N) \in U$ such that the Hénon map $H_{\alpha,\beta}$ for this parameter has more than N cycles of Siegel balls.

PROOF

Recall

$$\beta \;=\; \cos\theta + i\sin\theta, \qquad \alpha \;=\; 2\cos\theta\cos\varphi - \cos^2\varphi.$$

Fixed points of $H_{\alpha.\beta}$ are given by

$$y_* \;=\; \cos\theta \pm (\cos\theta - \cos\varphi).$$

We choose

$$y_* \;=\; \cos\varphi$$

and set

$$\mu \;=\; \cos\varphi + i\sin\varphi.$$

Eigenvalues of jacobian matrix at the fixed point

$$DH_{\alpha,\beta} = \beta \begin{pmatrix} 0 & \beta^{-1} \\ -\beta & 2\cos\varphi \end{pmatrix}$$

are $\beta\mu$ and $\beta\bar{\mu}$.
The arguements of these eigenvalues are

$$\theta + \varphi \qquad \text{and} \qquad \theta - \varphi \qquad (\text{mod } 2\pi).$$

By Siegel's theorem (or Brjuno's theorem), there is a subset $W_1 \subset (\mathbb{R}/2\pi\mathbb{Z})^2$ of full measure, such that the fixed point of the corresponding Hénon map has a Siegel ball.

We set

$$U_1 \;=\; U, \quad p_1 = 1, \quad \text{and} \quad V_1 \;=\; U_1 \cap W_{p_1}.$$

Inductively, we assume U_m is an open subset of U and V_m is a full measure subset of U_m, such that $H_{\alpha,\beta}$ for any $(\theta, \varphi) \in V_m$ has m cycles of Siegel balls of periods $p_1, \cdots, p_m$.

In open set U_m, there is a point $(\theta, \varphi) \in U_m$, such that $\frac{1}{2\pi}(\theta + \varphi) = \frac{q}{p}$ is rational with $p > p_m$ and, p and q are mutally prime.

Then perturb (θ, φ) keeping $\theta + \varphi = \frac{2\pi q}{p}$, so that $\beta = \cos\theta + i \sin\theta$ avoids the values of β forbidden in the preceeding propositions.

There is an open set of parameters containing such a parameter, such that the Hénon map $H_{\alpha,\beta}$ has a neutral cycle of period p, which is self-anti-conjugate with eigenvalues of the form $\beta^p\lambda$ and $\beta^p\bar{\lambda}$, with $-2 < \tau(\beta, \alpha) < 2$.

The trace function $\tau(\beta, \alpha)$ is a non-trivial analytic function with respect to α. Determinant of $H^p_{\alpha,\beta}$ is β^{2p}. Hence, the eigenvalues of the neutral p cycle varies effevtively.

Note that $\tau(\beta, \alpha) \in \mathbb{R}$, if $\alpha \in \mathbb{R}$, and $-2 < \tau(\beta, \alpha_0(\beta)) < 2$.

This implies that there is an open subset $U_{m+1} \subset U_m$ and a full measure set $W_p \subset U_{m+1}$ of parameters, such that the Hénon map has a Siegel ball of period p.

Set $V_{m+1} = V_m \cap W_p \subset U_{m+1}$, and $p_{m+1} = p$. V_{m+1} is a full measure subset of U_{m+1}.

Continue this procedure until $m = N$.

V_N is a set of positive measure. Hence, we can find a parameter $(\theta_N, \varphi_N) \in V_N \subset U$.