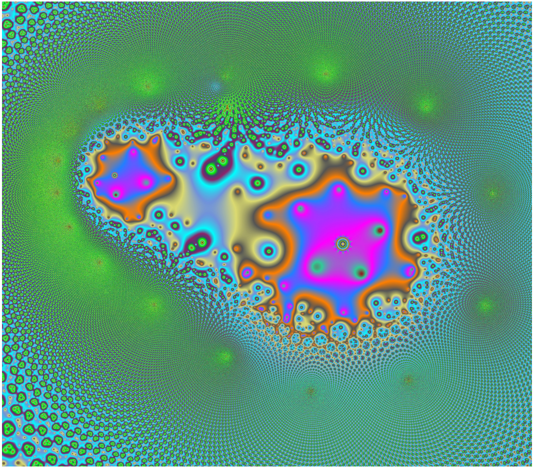
Rational Elliptic Fibration without Section



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Abstract

There exist rational elliptic surfaces which don't admit sections.

In [DM](2022), possible multiple fibers for rational elliptic fibrations are described.

We construct concrete examples of rational elliptic surfaces, whose generic fibers are elliptic curves representing cohomology class $-mK$, $m > 1$.

Contents

0. Introduction

Elliptic surface

Let S be a complex manifold of complex dimension 2. Suppose there is an elliptic fibration onto $\mathbb{P}^1$:

$$\psi : S \to \mathbb{P}^1.$$

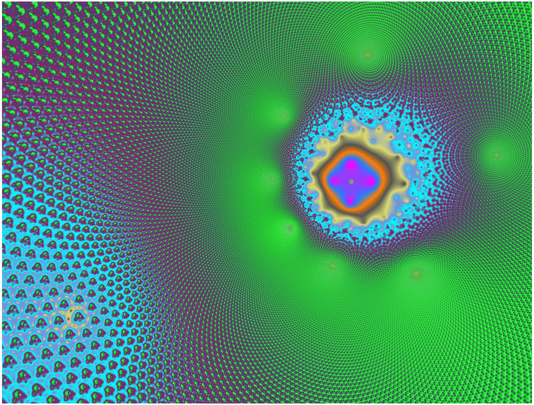
If there is a cross section

$$\sigma : \mathbb{P}^1 \to S, \qquad \psi \circ \sigma = id,$$

we can define Mordell-Weil group $MW(S)$ as the set of all sections.

However, there are elliptic surfaces which don't admit sections.

Picture of a section (QLc135t2BB)



Theorem of Gizatullin

Let $F : S \to S$ be an automorphism of rational surface S.

The **dynamical degree** λ_1 of F is defined as

$$\lambda_1 = \lim_{n\to\infty} ||(F^n)^*||^{1/n}.$$

THEOREM(Gizatullin [1980], Cantat [1999])
Assume $F \in \mathrm{Aut}(S)$, $\lambda_1 = 1$, and $\{||(F^n)^*||\}_{n\in\mathbb{N}}$ is unbounded. Then F preserves an elliptic fibration.

Elliptic fibration

Proposition(Gizatullin[Gi],1980). Let S be a minimal rational elliptic surface. Then for m large enough, we have $\dim|-mK_S| \geq 1$. For m minimal with this property, $|-mK_S|$ is a pencil without base point whose generic fiber is a smooth and reduced elliptic curve.

Remark(Grivaux[Gr], 2019). The elliptic fibration $S \to |-mK_S|^*$ doesn't have a rational section if $m \geq 2$. Indeed, the existence of multiple fibers ($m\mathcal{D}$) is an obstruction for the existence of a section.

An elliptic surface

We consider a surface automorphism with invariant elliptic curve of modulus i for orbit data $(1, 4, 4)$, *cyclic*, choosing multiplier i.

The configuration of the singular fibers is $\mathrm{IV}\ \mathrm{I}_1^8$.

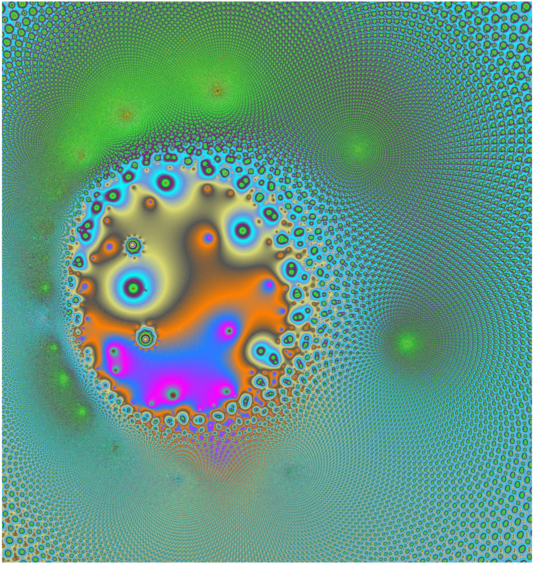
By choosing extra parameters, we find surface automorphisms with

$$\dim|-K| = 0, \qquad \dim|-2K| = 1.$$

REM. This seems to be the case (b) of theorem 3.3 in [DM] with

$$m = 2, n = 4, p = 0.$$

A (double) section ? (EIc144a10b12B)



Another elliptic surface

Consider a surface automorphism with invariant elliptic curve of modulus $\epsilon = \exp(\frac{\pi i}{3})$ for orbit data $(3, 3, 3)$, *cyclic*, choosing multiplier $\omega = \exp(\frac{2\pi i}{3})$.

The configuration of the singular fibers is $\mathrm{III}\ \mathrm{I}_1^9$.

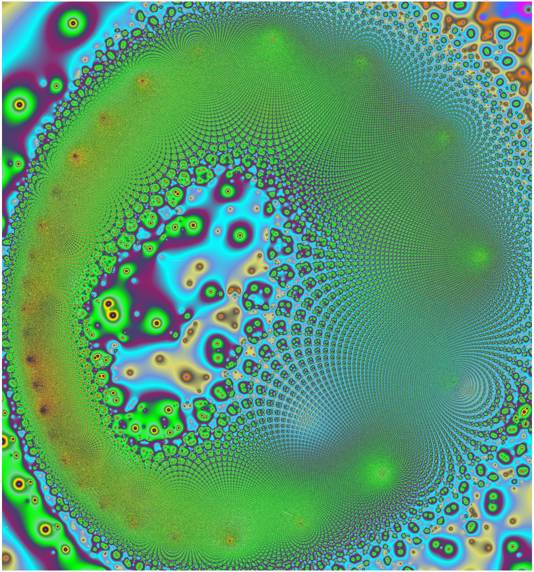
By choosing extra parameters, we find surface automorphisms with

$$\dim|-K| = 0, \quad \dim|-2K| = 0, \quad \dim|-3K| = 1.$$

REM. This seems to be the case (a) of theorem3.3 in [DM] with

$$m = n = 3, p = 0.$$

A (triple) section ? (EWc333b20BB)



There exist ...

THEOREM. There exist automorphisms of elliptic surface, induced by quadratic Cremona transformations, such that the elliptic fibration don't admit sections.

1. Elliptic surface

Birational map

Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map. Under certain conditions, birational map induces a holomorphic automorphism $F : S \to S$ of rational surface S, which is obtained by successive blowing ups of $\mathbb{P}^2$, with projection $\pi : S \to \mathbb{P}^2$.

$$\begin{array}{ccc} S & \xrightarrow{F} & S \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{P}^2 & \overset{f}{\dashrightarrow} & \mathbb{P}^2. \end{array}$$

Elliptic fibration

A surjective holomorphic map $\psi : S \to \mathbb{P}^1$ is an **elliptic fibration** if almost all fibers, $\psi^{-1}(\xi)$, are smooth curves of genus 1, and no fiber contains an exceptional (-1)-curve.

An **elliptic surface** S over $\mathbb{P}^1$ is a smooth projective surface with an elliptic fibration over $\mathbb{P}^1$.

For fixed S, fibration $\psi : S \to \mathbb{P}^1$ is unique (up to Möbius transformation).

Kodaira names

Singular fibers are classified by Kodaira. (smooth fiber is indicated by I_0)

$$\mathrm{I}_n,\ n \geq 1, \quad \mathrm{II}, \quad \mathrm{III}, \quad \mathrm{IV}, \quad \mathrm{I}_n^*,\ n \geq 0, \quad \mathrm{IV}^*, \quad \mathrm{III}^*, \quad \mathrm{II}^*.$$

Euler number:

$$e(\mathrm{I}_n) = n, \quad e(\mathrm{II}) = 2, \quad e(\mathrm{III}) = 3, \quad e(\mathrm{IV}) = 4,$$

$$e(\mathrm{I}_n^*) = n + 6, \quad e(\mathrm{IV}^*) = 8, \quad e(\mathrm{III}^*) = 9, \quad e(\mathrm{II}^*) = 10.$$

$$\sum_{F_v:\ \text{singular fiber}} e(F_v) = 12.$$

Preservation of elliptic fibration

We say that automorphism $F : S \to S$ **preserves elliptic fibration** $\psi : S \to \mathbb{P}^1$, if commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{F} & S \\ \downarrow \psi & & \downarrow \psi \\ \mathbb{P}^1 & \xrightarrow{\Omega} & \mathbb{P}^1. \end{array}$$

holds for some Möbius transformation $\Omega : \mathbb{P}^1 \to \mathbb{P}^1$.

S can have other automorphisms. Every automorphism of S preserves the fibration.

2. Elliptic curve

Weierstraß $\wp$-function

We use Weierstraß $\wp$-function as parametrization of invariant smooth cubic curve.

Let $\tau \in \mathbb{C} \setminus \mathbb{R}$ and $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ be a lattice.

Weierstraß $\wp$-function $\wp : \mathbb{C}/\Lambda_\tau \to \mathbb{P}$ is defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda'_\tau} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right),$$

where $\Lambda'_\tau = \Lambda_\tau \setminus \{0\}$.

THEOREM The Weierstraß $\wp$-function satisfies a Weierstraß equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3,$$

$$\text{with} \quad g_2 = 60 \sum_{\omega \in \Lambda'_\tau} \omega^{-4}, \quad \text{and} \quad g_3 = 140 \sum_{\omega \in \Lambda'_\tau} \omega^{-6}.$$

Parametrization

The parametrization of elliptic curve $\{y^2 = 4x^3 - g_2 x - g_3\}$ is given by

$$p(t) = (\wp(t), \wp'(t)), \qquad t \in \mathbb{C}/\Lambda_\tau.$$

THEOREM(Diller, 2011) Let $X \subset \mathbb{P}^2$ be an irreducible cubic curve. Suppose we are given points $p(p_1^+), p(p_2^+), p(p_3^+) \in X_{reg}$, a multiplier $a \in \mathbb{C}^\times$, and a translation $b \in \mathbb{C}/\Lambda$. Then there exists at most one quadratic transformation f properly fixing X with $I(f) = \{p(p_1^+), p(p_2^+), p(p_3^+)\}$ and $f(p(t)) = p(at + b)$. This f exists if and only if the following hold.

$p_1^+ + p_2^+ + p_3^+ \not\equiv 0$;
a is a multiplier for X_{reg};
$a(p_1^+ + p_2^+ + p_3^+) \equiv 3b$.

Finally, the points of indeterminacy for f^{-1} are given by $p_j^- = ap_j^+ - 2b$, $j = 1, 2, 3$.

Elliptic curve

Diller [D] stated the existence of surface automorphisms with positive entropy preserving a smooth cubic curve.

PROPOSITION(Diller, 2011). Suppose that f is a quadratic transformation properly fixing a smooth cubic curve X. If f has positive entropy and lifts to an automorphism of some modification $S \to \mathbb{P}^2$, then either

$X \cong \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ and the multiplier for $f|_X$ is $\pm i$; or

$X \cong \mathbb{C}/(\mathbb{Z} + e^{\pi i/3}\mathbb{Z})$ and the multiplier for $f|_X$ is a prime cube root of -1.

Rem. In the case of zero entropy, similar construction is possible. In the latter case, also with a prime cube root of 1 as multiplier. There are surface automorphisms with 1 as multiplier.

Cremona transformation

The most basic non-linear birational transformation $J : \mathbb{P}^2 \to \mathbb{P}^2$ (Cremona involution) can be expressed as

$$[x : y : z] \mapsto [yz : zx : xy].$$

J acts by blowing up points $e_1 = [1 : 0 : 0]$, $e_2 = [0 : 1 : 0]$,$e_3 = [0 : 0 : 1]$ and then collasping the lines $\{x = 0\}$,$\{y = 0\}$,$\{z = 0\}$ to e_1, e_2, e_3 respectively.

A generic quadratic Cremona transformation can be obtained from J by pre- and post- composing with linear transformations $f = L_1 \circ J \circ L_2^{-1}$.

Conditions

We see that

$$I(f) = \{L_2(e_1), L_2(e_2), L_2(e_3)\}, \quad I(f^{-1}) = \{L_1(e_1), L_1(e_2), L_1(e_3)\}.$$

The choice of L_1 and L_2 is not unique, since specification of three points does not determine a linear transformation uniquely. We need a supplementary condition to determine the transformation with uniqueness.

A unique biquadratic transformation $f = L_1 \circ K \circ J \circ L_2^{-1}$ is obtained by specifying a linear transformation $K : \mathbb{P}^2 \to \mathbb{P}^2$, which fixes e_1,e_2, e_3, and setting

$$\tilde{K} = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix},$$

$$\tilde{L}_1 = \begin{pmatrix} \wp(p_1^-) & \wp(p_2^-) & \wp(p_3^-) \\ \wp'(p_1^-) & \wp'(p_2^-) & \wp'(p_3^-) \\ 1 & 1 & 1 \end{pmatrix},$$

$$\tilde{L}_2 = \begin{pmatrix} \wp(p_1^+) & \wp(p_2^+) & \wp(p_3^+) \\ \wp'(p_1^+) & \wp'(p_2^+) & \wp'(p_3^+) \\ 1 & 1 & 1 \end{pmatrix}.$$

Take a fixed point t_0 of the inner dynamics $t \mapsto at + b$. Then point $p(t_0)$ must be a fixed point of f. We can choose $\tilde{K}$ by

$$\tilde{L}_1^{-1}(p(t_0)) = \tilde{K} \circ \tilde{J} \circ \tilde{L}_2^{-1}(p(t_0)).$$

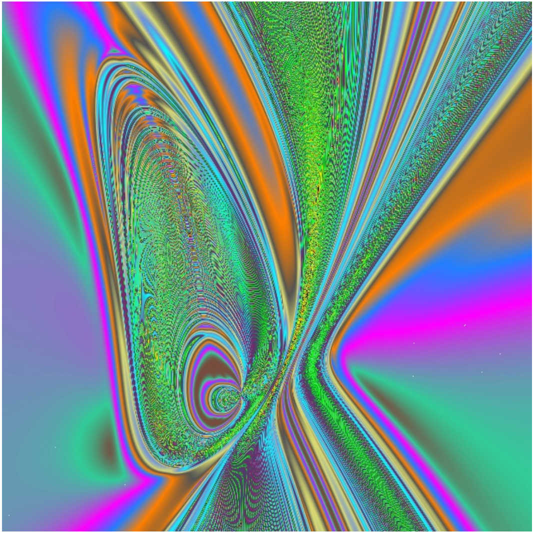
Obtained biquadratic transformation $f = L_1 \circ K \circ J \circ L_2^{-1}$ is the unique one satisfying

$$I(f) = \{p(p_1^+), p(p_2^+), p(p_3^+)\}, \quad I(f^{-1}) = \{p(p_1^-), p(p_2^-), p(p_3^-)\},$$

$$\text{and} \quad f(p(t_0)) = p(t_0).$$

As at most one quadratic transformation properly fixing X, this f is the quadratic transformation described in the above theorem.

Surface with elliptic curve (Elc144a10b10RR)



3. Orbit data $(1, 4, 4)$, *cyclic*

From orbit data to Cremona transformation

Let $\Lambda_i = \mathbb{Z} + i\mathbb{Z}$.
Let us construct a surface automorphism with

$$\text{orbit data} : (1, 4, 4), \textit{cyclic},$$

$$X \cong \mathbb{C}/\Lambda_i,$$

and the multiplier for $f|_X$ is i.

Suppose the translation of the inner dynamics is $b \in \mathbb{C}/\Lambda_i$. And the inner dynamics $t \mapsto it + b$.

Conditions for orbit data $(n_1, n_2, n_3), \sigma$ are as follows (mod Λ_i).

$$p_1^+ + p_2^+ + p_3^+ \equiv -3ib \not\equiv 0,$$

$$p_j^- \equiv ip_j^+ - 2b, \quad j = 1, 2, 3.$$

$$p_{\sigma(j)}^+ \equiv i^{n_j - 1}(p_j^- - \frac{1+i}{2}b) + \frac{1+i}{2}b, \quad j = 1, 2, 3.$$

$X \cong \mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$

Conditions for orbit data $(1,4,4), \sigma=(1,2,3)$ are as follows $\pmod{\Lambda_i}$.

$$p_1^+ + p_2^+ + p_3^+ \equiv -3ib \not\equiv 0,$$

$$p_1^- \equiv ip_1^+ - 2b, \quad p_2^- \equiv ip_2^+ - 2b, \quad p_3^- \equiv ip_3^+ - 2b,$$

$$p_2^+ \equiv ip_1^+ - 2b, \quad p_3^+ \equiv p_2^+ + 3ib, \quad p_1^+ \equiv p_3^+ + 3ib.$$

From the last three equations, we get

$$(1-i)p_1^+ \equiv (-2+6i)b.$$

We put

$$(1-i)p_1^+ = (-2+6i)b + \alpha, \qquad \alpha \in \Lambda_i.$$

And we get

$$p_1^+ + p_2^+ + p_3^+ = (-12-3i)b + \frac{-1+3i}{2}\alpha \equiv -3ib.$$

We put

$$-12b + \frac{-1+3i}{2}\alpha = \beta, \qquad \beta \in \Lambda_i,$$

to obtain

$$b = -\frac{1}{12}\beta + \frac{-1+3i}{24}\alpha.$$

If $-3ib \not\equiv 0$, then we get solutions :

$$p_1^+ \equiv \frac{4-2i}{12}\beta + \frac{-14-2i}{24}\alpha, \quad p_1^- \equiv \frac{4+4i}{12}\beta + \frac{4+4i}{24}\alpha,$$

$$p_2^+ \equiv \frac{4+4i}{12}\beta + \frac{4+4i}{24}\alpha, \quad p_2^- \equiv \frac{-2+4i}{12}\beta + \frac{-2-2i}{24}\alpha,$$

$$p_3^+ \equiv \frac{4+i}{12}\beta + \frac{-5+i}{24}\alpha, \quad p_3^- \equiv \frac{1+4i}{12}\beta + \frac{1-11i}{24}\alpha.$$

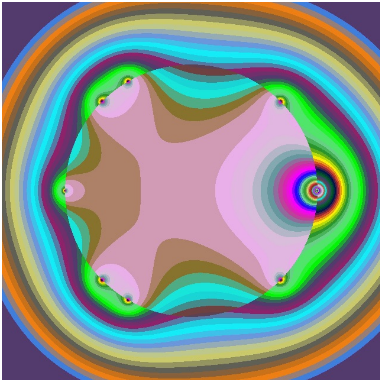
Let $F_{\alpha,\beta} : S_{\alpha,\beta} \to S_{\alpha,\beta}$ denote our surface automorphism.

Eigenvalues for orbit data $(1,4,4)$*cyclic*

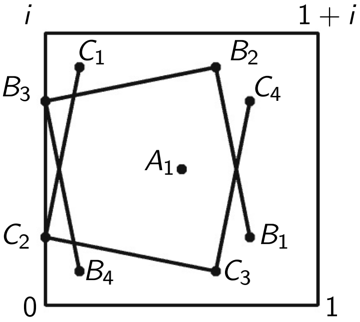
The characteristic polynomial for orbit data $(1,4,4)$, *cyclic* is

$$P(\lambda) = (\lambda - 1)(\lambda^2 - 1)(\lambda^3 - 1)(\lambda^4 + 1),$$

and i is not an eigenvalue.



Base points for $\alpha = \beta = 1$



4. Configuration

Singular fibers

Numerical observations suggest the existence of invariant curves other than the invariant elliptic curve $X \cong \mathbb{C}/\Lambda$.

And the surface $S_{\alpha,\beta}$ seems to have an elliptic fibration, invariant under $F_{\alpha,\beta} : S_{\alpha,\beta} \to S_{\alpha,\beta}$.

By numerical observations, we guess the configuration of this fibration is

$$\mathrm{IV}\ \mathrm{I}_1^8.$$

In the following, we verify it by finding the effective divisors.

Nodal root

For Rational surface, following commutative diagram holds.

$$\begin{array}{ccccccc} & 0 \longrightarrow \mathrm{Pic}(S) & \xrightarrow{c_1} & H^2(S,\mathbb{Z}) \longrightarrow 0, \\ & \downarrow r & & \downarrow \iota^* \\ 0 \to \mathrm{Pic}_0(X) \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{\mathrm{deg}} & H^2(X,\mathbb{Z}) \longrightarrow 0. \end{array}$$

X : cuspidal cubic, three lines through a point, quadric with a tangent line

$$\mathrm{Pic}_0(X) \simeq \mathbb{C},$$

X : nodal cubic (one, two, or three nodes)

$$\mathrm{Pic}_0(X) \simeq \mathbb{C}/\mathbb{Z},$$

X : elliptic cubic

$$\mathrm{Pic}_0(X) \simeq \mathbb{C}/\Lambda.$$

Nodality

If $\mathcal{P} \in H^2(S, Z)$ is a cohomology class of a (strict transform of) curve $C \subset \mathbb{P}^2$, then

$$\iota^*(\mathcal{P}) = 0 \quad \text{and} \quad r \circ c_1^{-1}(\mathcal{P}) = 0.$$

With our choice of Picard coordinates, we have the following fact.

THEOREM. $3d$ (not necessarily distinct) points $p_1, \cdots, p_{3d} \in X_{reg}$ comprise the intersection of X with a curve of degree d if and only if

each irreducible $V \subset X$ contains $d \cdot \deg V$ of the points; and $\sum p_j \sim 0$.

Genus formula

If $\mathcal{R} \in H^2(S, \mathbb{Z})$ is a cohomology class of an irreducible component of a reducible singular fiber of the fibration, then

$$\mathcal{R}^2 = -2, \quad \text{and} \quad r \circ c_1^{-1}(\mathcal{R}) = 0.$$

The condition $r \circ c_1^{-1}(\mathcal{R}) = 0$ implies $\mathcal{R}$ is nodal, *i.e.* it represents the class of a curve.

And $\mathcal{R}^2 = -2$ implies the curve is isomorphic to a Riemann sphere.

The **arithmetic genus** of a curve C representing class $\mathcal{R}$ is

$$g(C) = \frac{1}{2}\mathcal{R} \cdot (\mathcal{R} + K_S) + 1.$$

Our map case with orbit data $(1, 4, 4)$, *cyclic*

Now, let $A_1 \in H^2(S, \mathbb{Z})$ denote the cohomology class of the exceptional fiber $[\pi^{-1}(p(p_1^-))]$. Let $B_i = [\pi^{-1}(f^{i-1}(p(p_2^-)))]$, $i = 1, 2, 3, 4$, and $C_i = [\pi^{-1}(f^{i-1}(p(p_3^-)))]$, $i = 1, 2, 3, 4$.

Let $H \in H^2(S, \mathbb{Z})$ denote the class of a generic line $[\pi^{-1}(L)]$.

A basis of $H^2(S, \mathbb{Z})$ is given by classes

$$H, A_1, B_1, B_2, B_3, B_4, C_1, C_2, C_3, C_4.$$

Automorphism $F^* : H^2(S, \mathbb{Z}) \to H^2(S, \mathbb{Z})$ acts as follows.

$$H \mapsto 2H - A_1 - B_4 - C_4,$$

$$A_1 \mapsto H - A_1 - B_4,$$

$$B_4 \mapsto B_3 \mapsto B_2 \mapsto B_1 \mapsto H - B_4 - C_4,$$

$$C_4 \mapsto C_3 \mapsto C_2 \mapsto C_1 \mapsto H - A_1 - C_4.$$

Periodic roots

Let

$$\mathcal{X} = 3H - A_1 - B_1 - B_2 - B_3 - B_4 - C_1 - C_2 - C_3 - C_4$$

denote the class of anticanonical curve, represented by our invariant elliptic curve $X \cong \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$.

A class $\mathcal{R} \in H^2(S, \mathbb{Z})$ is said to be **a root of positive degree** if

$$\mathcal{R} \cdot \mathcal{X} = 0, \quad \mathcal{R}^2 = -2, \quad \mathcal{R} \cdot H \geq 0.$$

The characteristic polynomial for orbit data $(1, 4, 4)$, *cyclic* is

$$P(\lambda) = (\lambda - 1)(\lambda^2 - 1)(\lambda^3 - 1)(\lambda^4 + 1).$$

If there is a periodic root the period is 1, 2, 3, or 8.

Period 1 and 2

We have

$$\mathrm{Ker}(F^* - id) = \ < \mathcal{X} >,$$

$$\mathrm{Ker}(F^{*2} - id) = \ < \mathcal{E}_1, \ \mathcal{E}_2 > .$$

where

$$\mathcal{E}_1 = 2H - 2A_1 - B_2 - B_4 - C_1 - C_3,$$

$$\mathcal{E}_2 = H + A_1 - B_1 - B_3 - C_2 - C_4.$$

Then

$$F^*\mathcal{E}_1 = \mathcal{E}_2, \quad F^*\mathcal{E}_2 = \mathcal{E}_1, \quad F^*\mathcal{X} = \mathcal{X}, \quad \mathcal{X}^2 = 0,$$

$$\mathcal{E}_1^2 = \mathcal{E}_2^2 = -4, \quad \mathcal{E}_1 \cdot \mathcal{E}_2 = 4, \quad \mathcal{E}_1 \cdot \mathcal{X} = \mathcal{E}_2 \cdot \mathcal{X} = 0.$$

There are no roots in these subspaces.

Period 3

We have

$$\mathrm{Ker}(F^{*3} - id) \;=\; <\mathcal{L}_1,\ \mathcal{L}_2,\ \mathcal{L}_3>,$$

where

$$\mathcal{L}_1 = H - A_1 - B_3 - C_2,$$

$$\mathcal{L}_2 = H - B_1 - B_4 - C_3,$$

$$\mathcal{L}_3 = H - B_2 - C_1 - C_4.$$

And

$$F^*\mathcal{L}_1 = \mathcal{L}_3, \quad F^*\mathcal{L}_2 = \mathcal{L}_1, \quad F^*\mathcal{L}_3 = \mathcal{L}_2,$$

$$\mathcal{L}_1^2 = \mathcal{L}_2^2 = \mathcal{L}_3^2 = -2,$$

$$\mathcal{L}_1 \cdot \mathcal{L}_2 = \mathcal{L}_2 \cdot \mathcal{L}_3 = \mathcal{L}_3 \cdot \mathcal{L}_1 = 1.$$

Moreover,

$$\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 = \mathcal{X}.$$

Another periodic root

There exists another 3-cycle of roots of positive degree.

$$\mathcal{Q}_1 = \mathcal{L}_2 + \mathcal{L}_3,$$

$$\mathcal{Q}_2 = \mathcal{L}_3 + \mathcal{L}_1,$$

$$\mathcal{Q}_3 = \mathcal{L}_1 + \mathcal{L}_2.$$

with

$$F^*\mathcal{Q}_1 = \mathcal{Q}_3, \quad F^*\mathcal{Q}_2 = \mathcal{Q}_1, \quad F^*\mathcal{Q}_3 = \mathcal{Q}_2,$$

$$\mathcal{Q}_1^2 = \mathcal{Q}_2^2 = \mathcal{Q}_3^2 = -2,$$

$$\mathcal{Q}_1 \cdot \mathcal{Q}_2 = \mathcal{Q}_2 \cdot \mathcal{Q}_3 = \mathcal{Q}_3 \cdot \mathcal{Q}_1 = 1.$$

Moreover,

$$\mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3 = 2\mathcal{X}.$$

Singular fiber

If these roots are nodal and there exist three lines (or three quadrics) representing these classes, they form a singular fiber of type I_3 or IV.

To decide the type, recall the Lefschetz formula:

$$\sum_{f(p)=p} \operatorname{sign}(\det(Df_p - I)) = \sum_{i=0}^{\dim M} (-1)^i \operatorname{trace}(f_*|_{H_i(M,\mathbb{R})}).$$

To describe periodic cycles in terms of Lefschetz index, for $m \in \mathbb{N}$ and $k \in \mathbb{Z}$, let

$$\mathbf{m}(k) = \begin{cases} m & k \equiv 0 \pmod{m} \\ 0 & \text{otherwise} \end{cases}.$$

Periodic points

Recall the characteristic polynomial for orbit data $(1, 4, 4)$, *cyclic* :

$$P(\lambda) = (\lambda - 1)(\lambda^2 - 1)(\lambda^3 - 1)(\lambda^4 + 1).$$

The Lefschetz number $\Lambda(F^k)$ is expressed as

$$\Lambda(F^k) = \mathbf{1} + \mathbf{3} + \mathbf{1} + \mathbf{1} + \mathbf{2} - \mathbf{4} + \mathbf{8}.$$

The invariant elliptic curve $X \cong \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$, with inner dynamics $t \mapsto it + b$, has two fixed points and a cycle of period two. The inner dynamics is period four, and the periodic points are not counted in the Lefschetz number if $k \equiv 0 \pmod 4$.

So, the periodic points in X is given by $\mathbf{1} + \mathbf{1} + \mathbf{2} - \mathbf{4}$. The cycle $\mathbf{8}$ of period 8 comes from singular fiber I_1^8, obtained later.

The periodic points in the cycle of period three is described by $\mathbf{1} + \mathbf{3}$, that is, a singular fiber of type IV.

Picard projection

For Rational surface, following commutative diagram holds.

$$
\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & \mathrm{Pic}(S) & \xrightarrow{c_1} & H^2(S,\mathbb{Z}) & \longrightarrow & 0, \\
 & & & & \downarrow r & & \downarrow \iota^* & & \\
0 & \to & \mathrm{Pic}_0(X) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{\deg} & H^2(X,\mathbb{Z}) & \longrightarrow & 0.
\end{array}
$$

In our case, $X \cong \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ is an elliptic cubic curve,

$$\mathrm{Pic}_0(X) \simeq \mathbb{C}/\Lambda_i.$$

For $\mathcal{P} \in H^2(S,\mathbb{Z})$, with $\iota^*(\mathcal{P}) = 0$, we denote

$$\widetilde{\mathcal{P}} = r \circ c_1^{-1}(\mathcal{P}) \in \mathrm{Pic}_0(X).$$

We say $\widetilde{\mathcal{P}}$ is the **Picard projection** of $\mathcal{P}$.

Nodal periodic roots

For our automorphism $F_{\alpha,\beta} : S_{\alpha,\beta} \to S_{\alpha,\beta}$, the Picard projections of periodic roots of positive degree can be computed as follows $\pmod{\Lambda_i}$.

$$b = -\frac{1}{12}\beta + \frac{-1+3i}{24}\alpha, \qquad \alpha, \beta \in \Lambda_i,$$

$$\widetilde{\mathcal{L}_1} \equiv \widetilde{\mathcal{L}_2} \equiv \widetilde{\mathcal{L}_3} \equiv \widetilde{\mathcal{X}} \equiv \frac{1+i}{2}\alpha,$$

$$\widetilde{\mathcal{Q}_1} \equiv \widetilde{\mathcal{Q}_2} \equiv \widetilde{\mathcal{Q}_3} \equiv \widetilde{2\mathcal{X}} \equiv 0.$$

So, we conclude that if $\frac{1+i}{2}\alpha \equiv 0$ then singular fiber of type IV is a cubic curve consisting of three lines passing through a point.

And if $\frac{1+i}{2}\alpha \not\equiv 0$, then singular fiber of type IV comprises three conics passing through a point. In this case $\mathcal{X}$ cannot be the class of generic fibers.

Roots of period 8

$$\mathrm{Ker}(F^{*8} - id) = < \mathcal{U}_1,\ \mathcal{U}_2,\ \mathcal{U}_3,\ \mathcal{U}_4,\ \mathcal{U}_5,\ \mathcal{U}_6,\ \mathcal{U}_7,\ \mathcal{U}_8 >,$$

$$\begin{array}{rcc} \mathcal{U}_1 & = & H - A_1 - B_4 - C_1 \\ \mathcal{U}_2 & = & A_1 - C_2 \\ \mathcal{U}_3 & = & H - A_1 - C_1 - C_3 \\ \mathcal{U}_4 & = & H - B_1 - C_2 - C_4 \\ \mathcal{U}_5 & = & H - A_1 - B_2 - C_3 \\ \mathcal{U}_6 & = & H - B_1 - B_3 - C_4 \\ \mathcal{U}_7 & = & H - A_1 - B_2 - B_4 \\ \mathcal{U}_8 & = & A_1 - B_3 \end{array}.$$

$\mathcal{U}_1, \cdots, \mathcal{U}_8$ are all roots of positive (non-negative) degree, cyclically mapped, and

$$\sum_{k=1}^{8} \mathcal{U}_k = 2\mathcal{X}.$$

Picard projections

The Picard projections of these roots are as follows.

$$\widetilde{\mathcal{U}_k} \equiv \frac{i^k}{4}((1-i)\beta+\alpha), \qquad k = 1, \cdots, 8.$$

So, if $\frac{1}{4}((1-i)\beta+\alpha) \notin \Lambda_i$, then roots $\mathcal{U}_1, \cdots, \mathcal{U}_8$ are not nodal. Other roots in this subspace are not nodal, neither.

On the other hand, Lefschetz formula tells the existence of 8-cycle of (saddle) periodic points.

In the list of possible configurations of singular fibers ([P],[K]), only one configuration is compatible with the above observations :

$$\mathrm{IV}\ \mathrm{I}_1^8.$$

Persson's list of configurations

In the list of configurations of singular fibers given by Persson([P],1990), those containing I_8 or I_1^8 are :

$$\mathrm{IV}\ \mathrm{I}_1^8,\quad \mathrm{II}^2\ \mathrm{I}_8,\quad \mathrm{II}\ \mathrm{I}_8\ \mathrm{I}_1^2,\quad \mathrm{II}\ \mathrm{I}_2\ \mathrm{I}_1^8,$$

$$\mathrm{I}_8\ \mathrm{I}_2\ \mathrm{I}_1^2,\quad \mathrm{I}_8\ \mathrm{I}_1^4,\quad \mathrm{I}_4\ \mathrm{I}_1^8,\quad \mathrm{I}_2^2\ \mathrm{I}_1^8.$$

5. Multiple fibration

Parameters

In this section, we construct a surface automorphism with invariant elliptic fibration of $|-2K|$ type.

Choose $\alpha, \beta \in \Lambda_i$, satisfying

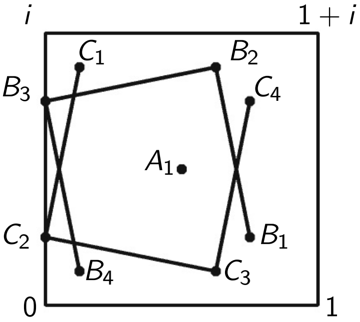
$$\frac{3+i}{8}\alpha + \frac{i}{4}\beta \notin \Lambda_i, \qquad (-3ib \not\equiv 0),$$

$$\frac{1}{2}(1+i)\alpha \notin \Lambda_i, \qquad (per.3,\ \widetilde{\mathcal{L}}_i \not\equiv 0,\ \widetilde{\mathcal{Q}}_i \equiv 0),$$

$$\frac{1}{4}(\alpha + (1-i)\beta) \notin \Lambda_i, \qquad (per.8,\ \widetilde{\mathcal{U}}_i \not\equiv 0).$$

Such α, β exist. For example, $\alpha = 1,\ \beta = 1$.

Base points for $\alpha = \beta = 1$



Surface automorphism

THEOREM(Diller, 2011) Let $X \subset \mathbb{P}^2$ be an irreducible cubic curve. Suppose we are given points $p(p_1^+), p(p_2^+), p(p_3^+) \in X_{reg}$, a multiplier $a \in \mathbb{C}^\times$, and a translation $b \in \mathbb{C}/\Lambda$. Then there exists at most one quadratic transformation f properly fixing X with $I(f) = \{p(p_1^+), p(p_2^+), p(p_3^+)\}$ and $f(p(t)) = p(at + b)$. This f exists if and only if the following hold.

$p_1^+ + p_2^+ + p_3^+ \not\equiv 0$;

a is a multiplier for X_{reg};

$a(p_1^+ + p_2^+ + p_3^+) \equiv 3b$.

Finally, the points of indeterminacy for f^{-1} are given by $p_j^- = ap_j^+ - 2b$, $j = 1, 2, 3$.

Orbit data

As we have constructed, we set

$$a = i \in \mathbb{C}^{\times}, \qquad b = \frac{-1+3i}{24}\alpha - \frac{1}{12}\beta \in \mathbb{C}/\Lambda_i,$$

$$p_1^+ \equiv \frac{4-2i}{12}\beta + \frac{-14-2i}{24}\alpha,$$

$$p_2^+ \equiv \frac{4+4i}{12}\beta + \frac{4+4i}{24}\alpha,$$

$$p_3^+ \equiv \frac{4+i}{12}\beta + \frac{-5+i}{24}\alpha.$$

And apply the theorem above.

The obtained quadratic automorphism satisfies orbit data $(1,4,4)$*cyclic*. And it lifts to an automorphism of a surface.

Multiple fibration

Under the conditions for our α, β, the obtained surface automorphism has a singular fiber of type IV, consisting of three quadrics intersecting at a point.

And periodic singular fiber of type I_1 of period 8, which is a sextic curve with a node, representing class $2\mathcal{X}$.

The configuration of singular fibers is

$$\mathrm{IV}\ \mathrm{I}_1^8.$$

The fibration corresponds to linear system $|-2K|$.

And $|-K|$ is generated by the invariant elliptic curve $X \cong \mathbb{C}/\Lambda_i$ representing $\mathcal{X}$.

The elliptic curve X should support a multiple fiber ${}_2\mathrm{I}_0$.

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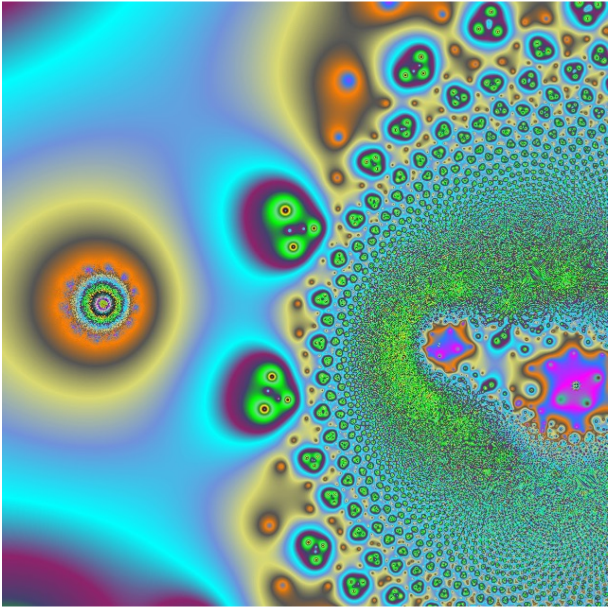

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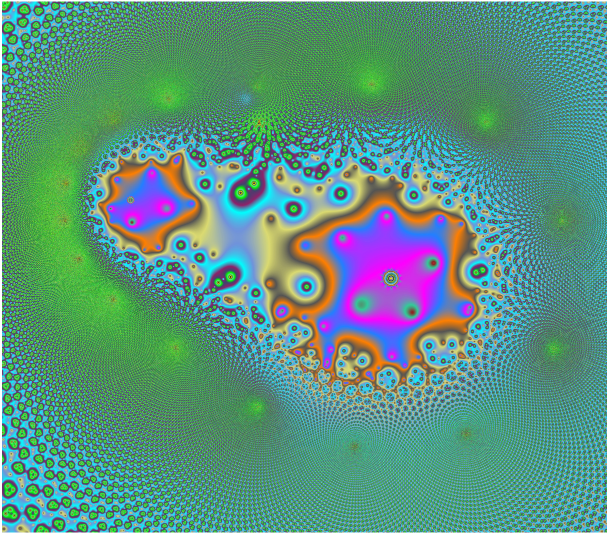

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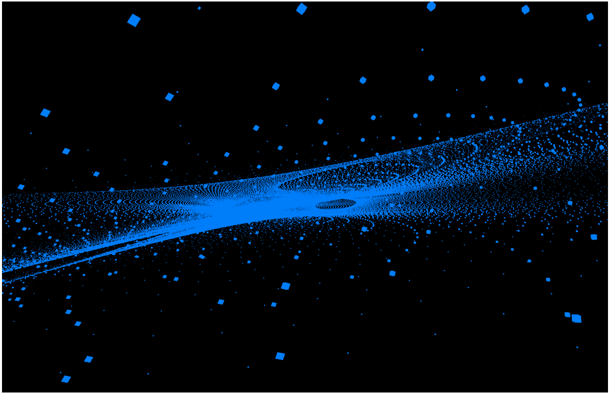

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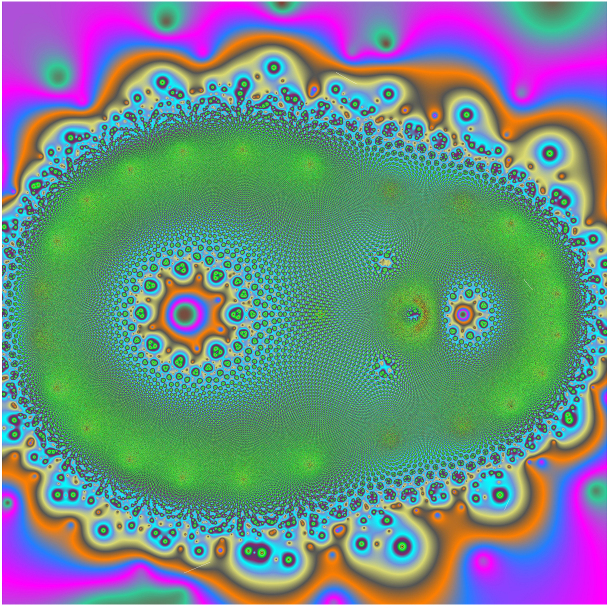

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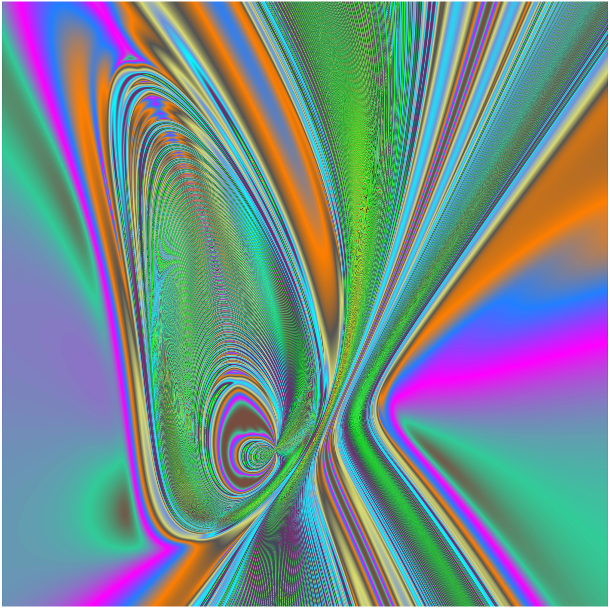

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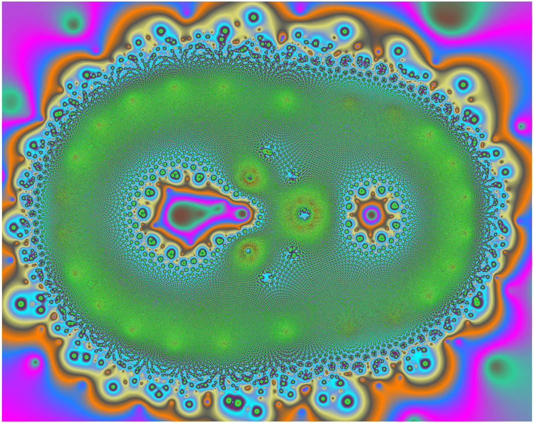

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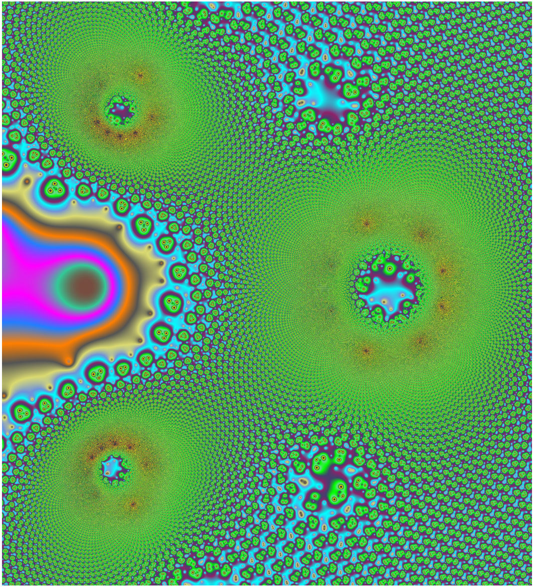

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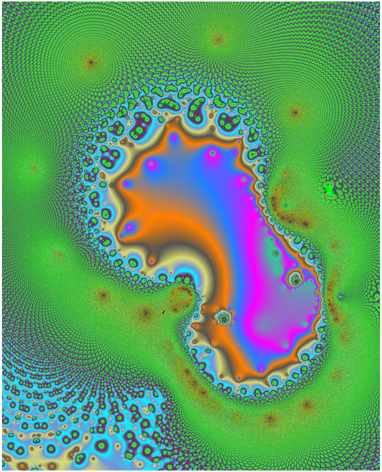

Thank you !

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F_v	J	d	r	e	p
$2\mathrm{I}_0$	1	8	7	0	$\mathbf{1}+\mathbf{1}+\mathbf{2}-\mathbf{4}$
IV	0	8	7	4	$\mathbf{1}+\mathbf{3}$
I_1^8	∞	8×1	0	8×1	$\mathbf{8}$

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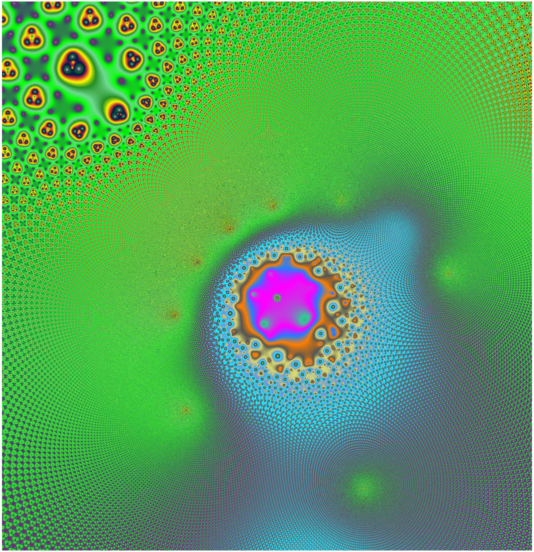

Multiple fiber

THEOREM. (Dolgachev-Martin,[DM]2022) Let $f : X \to B$ be a genus one surface with jacobian $J(f) : J(X) \to B$ and let $\mathrm{Aut}_f(X)$ be the group of automorphisms of X preserving f. Assume that f is cohomologically flat. Then there is a homomorphism $\varphi : \mathrm{Aut}_f(X) \to \mathrm{Aut}_{J(f)}(J(f))$ satisfying the following properties, where $g \in \mathrm{Aut}_f(X)$:

(1) Both g and $\varphi(g)$ induce the same automorphism of B.

(2) $\mathrm{Ker}(\varphi) \cong \mathrm{MW}(J(f))$.

(3) $\varphi(g)$ preserves the zero section of $J(f) : J(X) \to B$.

(4) If g acts trivially on $\mathrm{Num}(X)$, then $\varphi(g)$ acts trivially on $\mathrm{Num}(J(X))$.

(5) Let mF_0 be a fiber of f of multiplicity m and let $(J_0^\sharp)^0$ be the identity component of the smooth part $J_0^\sharp$ of the corresponding fiber J_0 of $J(f)$, then either $\varphi(g)$ acts trivially on $(J_0^\sharp)^0$ or one of the following holds, where $n = \mathrm{ord}(\varphi(g)|_{(J_0^\sharp)^0})$:

- (a) F_0 is smooth, $m = n = 3, p \neq 3$.
- (b) F_0 is smooth, $m = 2, n \in \{2, 4\}, p \neq 2$.
- (c) F_0 is smooth and ordinary, $m = n = p = 2$.
- (d) F_0 is an irreducible nodal curve, $m = n = 2, p \neq 2$.
- (e) F_0 is of type $\tilde{A}_1, m = n = 2, p \neq 2$.

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