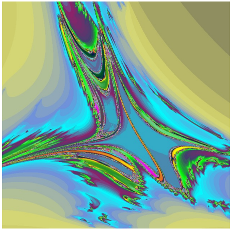
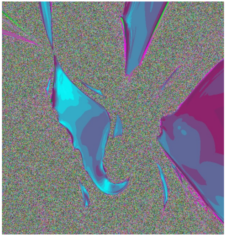
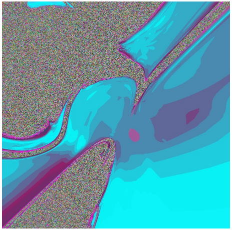
Anti-conjugacy of complex dynamical systems and slices of rotation domains

Shigehiro Ushiki, Kyoto

August 24, 2017

Abstract

Rotation domains under Hénon maps and automorphisms of complex surfaces are visualized.

Slices of Siegel balls and rotation domains are observed in the axis of involution.

Multiply-connected rotation domains of rank 2,

and rotation domain of rank 1 with non-trivial homology are observed numerically.

Swap-conjugacy and anti-conjugacy

Let $T(x,y) = (\bar{y}, \bar{x})$ be an involution.
Let us call this map the **swap-conjugacy** map.

Swap-conjugacy and anti-conjugacy

Let $T(x, y) = (\bar{y}, \bar{x})$ be an involution.
Let us call this map the **swap-conjugacy** map.

Let

$$x = p + iq, \quad y = p - iq.$$

Swap conjugacy T corresponds to the complex conjugacy

$$S(p, q) = (\bar{p}, \bar{q}).$$

Self-anti-conjugacy

DEFINITION Rational automorphism $F : \mathbb{C}^2 \to \mathbb{C}^2$ is said to be **self-anti-conjugate**, if $T \circ F \circ T = F^{-1}$.

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{F} & \mathbb{C}^2 \\ \updownarrow T & & \updownarrow T \\ \mathbb{C}^2 & \xleftarrow{F} & \mathbb{C}^2 \end{array}$$

Self-anti-conjugacy

DEFINITION Rational automorphism
$F : \mathbb{C}^2 \to \mathbb{C}^2$ is said to be **self-anti-conjugate**, if $T \circ F \circ T = F^{-1}$.

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{F} & \mathbb{C}^2 \\ \updownarrow T & & \updownarrow T \\ \mathbb{C}^2 & \xleftarrow{F} & \mathbb{C}^2 \end{array}$$

Note that $T \circ F$, $F \circ T$, $T \circ F \circ F$, $F \circ F \circ T$, *etc.* are involutions, too.

Self-anti-conjugate maps

Volume-preserving Hénon map of the form

$$h(x, y) = (y, \beta P(y) - \beta^2 x),$$

is self-anti-conjugate if $|\beta| = 1$ and $\overline{P(\bar{y})} = P(y)$.

Self-anti-conjugate maps

Volume-preserving Hénon map of the form

$$h(x, y) = (y, \beta P(y) - \beta^2 x),$$

is self-anti-conjugate if $|\beta| = 1$ and $\overline{P(\bar{y})} = P(y)$.

Rational automorphism

$$f(x, y) = (y, \ \frac{P(y)}{x + i\beta} + i\beta)$$

is self-anti-conjugate if $\beta \in \mathbb{R}$ and $\overline{P(\bar{y})} = P(y)$.

Axis of involution

The set of fixed points of involution T ,

$$\Sigma_T = \{(x, y) | T(x, y) = (x, y)\}$$

will be called the **axis of involution** T.

Axis of involution

The set of fixed points of involution T ,

$$\Sigma_T = \{(x, y) | T(x, y) = (x, y)\}$$

will be called the **axis of involution** T.

Also, second axis of involution

$$\Sigma_{T \circ F} = \{(x, y) | T \circ F(x, y) = (x, y)\}$$

will be used.

Self-anti-conjugate orbit

Let $z_n = F^n(z_0)$.

If $z_0 \in \Sigma_T$, then

$$z_n = T(z_{-n}), \quad \text{for} \quad n \in \mathbb{Z}.$$

Self-anti-conjugate orbit

Let $z_n = F^n(z_0)$.

If $z_0 \in \Sigma_T$, then

$$z_n = T(z_{-n}), \quad \text{for} \quad n \in \mathbb{Z}.$$

If $z_0 \in \Sigma_{T \circ F}$, then

$$z_n = T(z_{1-n}), \quad \text{for} \quad n \in \mathbb{Z}.$$

Self-anti-conjugate cycles

Suppose $z_0 \in \Sigma_T$ is a periodic point of period p.

If p is even, then $z_{p/2} \in \Sigma_T$.

If p is odd, then $z_{(p-1)/2} \in \Sigma_{T \circ F}$.

Self-anti-conjugate cycles

Suppose $z_0 \in \Sigma_T$ is a periodic point of period p.

If p is even, then $z_{p/2} \in \Sigma_T$.

If p is odd, then $z_{(p-1)/2} \in \Sigma_{T \circ F}$.

Suppose $z_0 \in \Sigma_{T \circ F}$ is a periodic point of period p.

If p is even, then $z_{p/2} \in \Sigma_{T \circ F}$.

If p is odd, then $z_{(p+1)/2} \in \Sigma_T$.

Quasi-unitary matrix and self-anti-conjugate matrix

Let U be a 2×2-matrix.

DEFINITION U is **quasi-unitary** if U can be written as

$$U = \lambda A,$$

with $|\lambda| = 1$, $\det(A) = 1$ and $\mathrm{trace}(A) \in \mathbb{R}$.

Quasi-unitary matrix and self-anti-conjugate matrix

Let U be a 2×2-matrix.

DEFINITION U is **quasi-unitary** if U can be written as

$$U = \lambda A,$$

with $|\lambda| = 1$, $\det(A) = 1$ and $\mathrm{trace}(A) \in \mathbb{R}$.

DEFINITION U is **self-anti-conjugate** if

$$T \circ U \circ T = U^{-1}.$$

Quasi-unitary matrix

PROPOSITION Self-anti-conjugate matrix is quasi-unitary.

Quasi-unitary matrix

PROPOSITION Self-anti-conjugate matrix is quasi-unitary.

PROOF Let $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $T \circ U \circ T = U^{-1}$ inplies

$$ad - bc = a/\bar{a} = d/\bar{d} = -b/\bar{c} = -c/\bar{b}, \quad |ad - bc| = 1.$$

Take a $\lambda \in \mathbb{C}$, $|\lambda| = 1$, with $\lambda^2 = \det(U)$. Then $U = \lambda \begin{pmatrix} r & w \\ -\bar{w} & s \end{pmatrix}$ for some $r, s \in \mathbb{R}$ and $w \in \mathbb{C}$, with $w\bar{w} = 1 - rs$.

Product of self-anti-conjugate matrices

PROPOSITION Product of self-anti-conjugate matrices is quasi-unitary.

Product of self-anti-conjugate matrices

Proposition Product of self-anti-conjugate matrices is quasi-unitary.

Proof Let $U_1 = \lambda_1 \begin{pmatrix} r_1 & w_1 \\ -\bar{w}_1 & s_1 \end{pmatrix}$, $U_2 = \lambda_2 \begin{pmatrix} r_2 & w_2 \\ -\bar{w}_2 & s_2 \end{pmatrix}$ be self-anti-conjugate matrices. Let $A = \bar{\lambda}_1 \bar{\lambda}_2 U_1 U_2$. Then,

$$U_1 U_2 = \lambda_1 \lambda_2 A$$

with $\det(A) = 1$ and $\mathrm{trace}(A) = r_1 r_2 + s_1 s_2 - 2\mathrm{Re}(w_1 \bar{w}_2) \in \mathbb{R}$.

Jacobian matrix of self-anti-conjugate cycle

Suppose $F : \mathbb{C}^2 \to \mathbb{C}^2$ is self-anti-conjugate.

Theorem

If $z_0 \in \Sigma_T \cup \Sigma_{T \circ F}$ is a periodic point of period p, then the Jacobian matrix of the cycle, $(DF^p)_{z_0}$ is quasi-unitary.

self-anti-conjugate cycle

PROOF (CASE I) If $z_0 \in \Sigma_T$ is a periodic point of period p, then $T(z_0) = z_0$ and $F^p(z_0) = z_0 = F^{-p}(z_0)$. As F is self-anti-conjugate, $T \circ F^p \circ T = F^{-p}$ hols. Hence, we have

$$T \circ (DF^p)_{T(z_0)} \circ T = (DF^{-p})_{z_0}.$$

As $T(z_0) = z_0$ and $(DF^{-p})_{z_0} = ((DF^p)_{F^{-p}(z_0)})^{-1} = ((DF^p)_{z_0})^{-1}$, we have

$$T \circ (DF^p)_{z_0} \circ T = ((DF^p)_{z_0})^{-1}.$$

Hence, $(DF^p)_{z_0}$ is self-anti-conjugate.

self-anti-conjugate cycle

PROOF (CASE II) If $z_0 \in \Sigma_{T\circ F}$ is a periodic point of period p, then $z_1 = F(z_0) = T(z_0)$ and $F^{p-1}(z_1) = z_0$. As F is self-anti-conjugate, $T \circ F \circ T = F^{-1}$ hols. Hence, we have

$$T \circ (DF)_{T(z_1)} \circ T = (DF^{-1})_{z_1}.$$

As $T(z_1) = z_0$ and $(DF^{-1})_{z_1} = ((DF)_{F^{-1}(z_1)})^{-1} = ((DF)_{z_0})^{-1}$, $(DF)_{z_0}$ is self-anti-conjugate.

Similarly, from $T \circ F^{p-1} \circ T = F^{1-p}$,

$$T \circ (DF^{p-1})_{T(z_0)} \circ T = (DF^{1-p})_{z_0}.$$

As $(DF^{1-p})_{z_0} = ((DF^{p-1})_{F^{1-p}(z_0)})^{-1} = ((DF^{p-1})_{z_1})^{-1}$, $(DF^{p-1})_{z_1}$ is self-anti-conjugate. Therefore, product of self-anti-conjugate matrices, $(DF^p)_{z_0}$, is a quasi-unitary matrix.

Dilatation

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2-matrix.
Define the **dilatation** of matrix A by

$$\delta(A) = \frac{\max_{|z|=1} |Az|}{\min_{|z|=1} |Az|}.$$

Dilatation

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2-matrix.
Define the **dilatation** of matrix A by

$$\delta(A) = \frac{\max_{|z|=1} |Az|}{\min_{|z|=1} |Az|}.$$

Dilatation can be computed explicitly. Let
$e = |a|^2 + |b|^2 + |c|^2 + |d|^2$, $f = |a|^2 - |b|^2 + |c|^2 - |d|^2$ and
$g = |a\bar{b} + c\bar{d}|$. Then

$$\delta(A) = \sqrt{\frac{\frac{1}{2}e + \sqrt{(\frac{1}{2}f)^2 + g^2}}{\frac{1}{2}e - \sqrt{(\frac{1}{2}f)^2 + g^2}}}.$$

Dilatation growth along orbit

For initial point z_0, define the dilatation along the orbit by

$$\delta_n = \delta((DF^n)_{z_0}).$$

If the orbit has a positive Lyapounov exponent, then δ_n grows exponentially.

If the orbit behaves in a parabolic manner (for example orbit in a KAM circle), the sequence of dilatations diverges to the infinity by a linear growth.

If the orbit is in a rotation domain, then the sequence δ_n remains bounded.

Conjugacy of involutions

Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ and $S : \mathbb{C}^2 \to \mathbb{C}^2$ be involutions, and $H : \mathbb{C}^2 \to \mathbb{C}^2$ be a holomorphic isomorphism satisfying

$$T \circ H = H \circ S.$$

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{H} & \mathbb{C}^2 \\ \updownarrow S & & \updownarrow T \\ \mathbb{C}^2 & \xrightarrow{H} & \mathbb{C}^2 \end{array}$$

Conjugacy of self-anti-conjugate maps

PROPOSITION If $F : \mathbb{C}^2 \to \mathbb{C}^2$ is T-self-anti-conjugate, then $G = H^{-1} \circ F \circ H$ is S-self-anti-conjugate.

$$
\begin{array}{cccc}
 & \mathbb{C}^2 & \xrightarrow{F} & \mathbb{C}^2 \\
H\nearrow & \updownarrow T & & \nearrow H \\
\mathbb{C}^2 & \xrightarrow{G} & \mathbb{C}^2 & \updownarrow T \\
\updownarrow S & \mathbb{C}^2 & \xleftarrow{F} & \mathbb{C}^2 \\
 & \nearrow H & \updownarrow S & \nearrow H \\
\mathbb{C}^2 & \xleftarrow{G} & \mathbb{C}^2 &
\end{array}
$$

PROOF $S \circ G \circ S = S \circ H^{-1} \circ F \circ H \circ S = H^{-1} \circ T \circ F \circ T \circ H = H^{-1} \circ F^{-1} \circ H = G^{-1}$.

Self-conjugate diffeomorphism

Let $S : \mathbb{C}^2 \to \mathbb{C}^2$ be the usual complex conjugation $S(x, y) = (\bar{x}, \bar{y})$.

THEOREM If $G : \mathbb{C}^2 \to \mathbb{C}^2$ is S-self-anti-conjugate and G has a linearizable fixed point in $\Sigma_S (= \mathbb{R}^2)$, then there is an S-self-conjugate linearizing map $\Phi : U \to \mathbb{C}^2$ defined in a neighborhood U of the fixed point , *i.e.*,

$$\Phi \circ G \circ \Phi^{-1} = DG_{\textit{fixed point}}, \quad S \circ \Phi \circ S = \Phi.$$

Self-conjugate linearizer

PROOF As G is self-anti-conjugate, $S \circ G \circ S = G^{-1}$ holds. Hence, its derivative $L = DF_{\mathit{fixed\ point}}$ is also self-anti-conjugate, so, $S \circ L \circ S = L^{-1}$ hols. Therefore, by setting $\Psi = S \circ \Phi \circ S$, we see that

$$\Phi \circ G \circ \Phi^{-1} = L$$

induces

$$\Psi \circ G \circ \Psi^{-1} = L.$$

As the linearizing map is determined uniquely by prescribing the linear part, we can choose the linearizing maps satisfying

$$\Psi = \Phi.$$

Self-anti-conjugate Hénon map

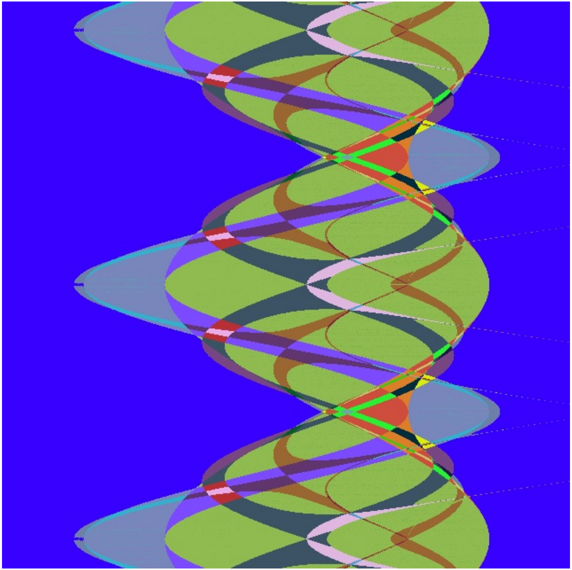
Volume-preserving Hénon map of the form

$$h(x, y) = (y, \beta(y^2 + \alpha) - \beta^2 x),$$

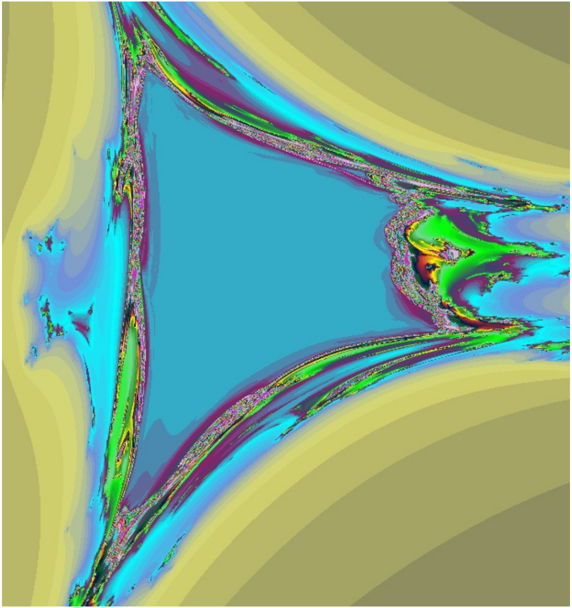
is self-anti-conjugate if $\alpha \in \mathbb{R}$ and $|\beta| = 1$.

Parameter space of $(\alpha, \arg(\beta))$ is drawn in the next slide.

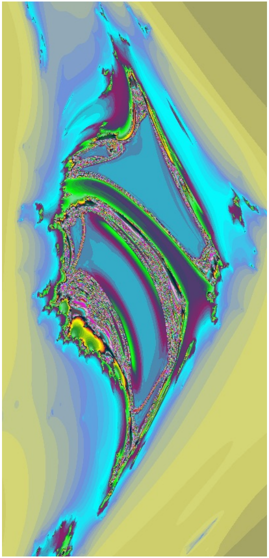
Parameter space of self-anti-conjugate Hénon map



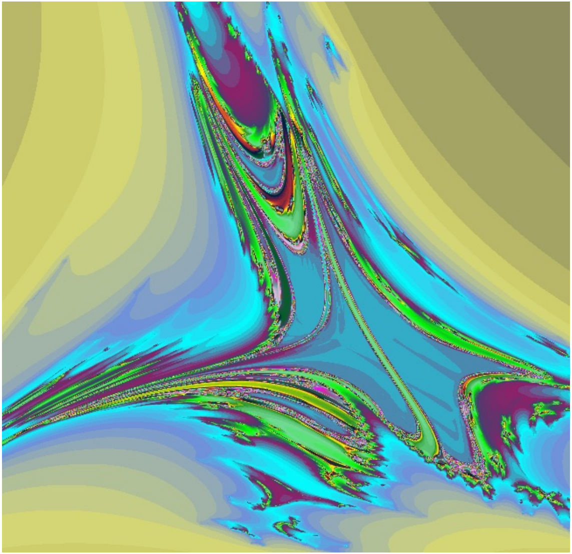
Slice of a Siegel ball around a fixed point



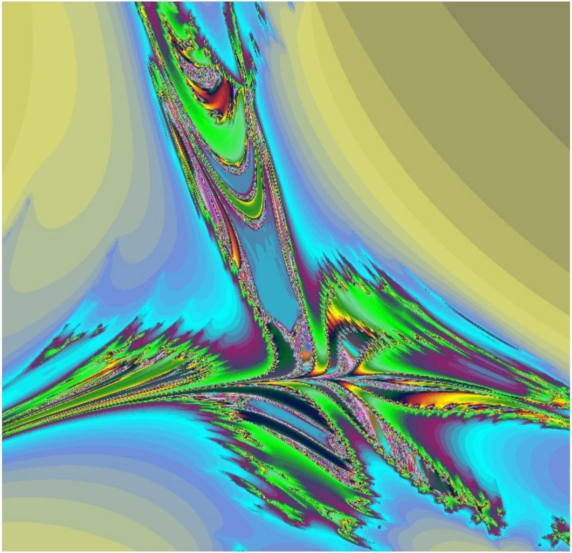
Slice of Siegel balls of periods 1 and 3



Slice of Siegel balls with an exotic rotation domain



Slice of an exotic rotation domain



Self-anti-conjugate rational automorphism

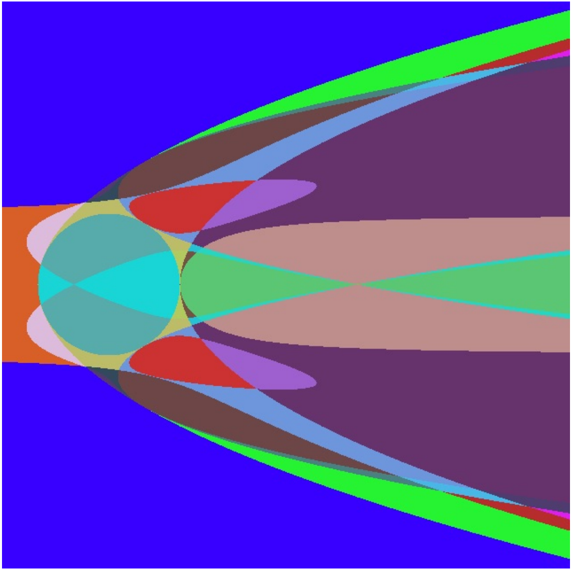
Rational automorphism

$$f(x, y) = (y,\ \frac{y+\alpha}{x+i\beta} + i\beta)$$

is self-anti-conjugate if $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$.

Parameter space of (α, β) is drawn in the next slide.

Parameter space of self-anti-conjugate rational maps



Surface automorphism

$p = (-i\beta, -\alpha)$ is an indeterminate point of f.

$e_1 = [0 : 1 : 0]$ and $e_2 = [0 : 0 : 1]$ are indeterminate points of f.

Line $\{y = -\alpha\}$ is mapped to $q = (-\alpha, i\beta)$.

Bedford and Kim [BK1] showed that f is birationally conjugate to an automorphism of a compact complex surface if and only if $f^n(q) = p$ for some n.

The compact surface is obtained by blowing up the $n + 3$ points $e_1, e_2, f^j(q), 0 \leq j \leq n$.

Parameter correspondence to Bedford-Kim map

Birational map studied in [BK1],[BK2],

$$f_{a,b}(x,y) = (y, \frac{y+a}{x+b})$$

is equivalent to our self-anti-conjugate map

$$f(x,y) = (y, \frac{y+\alpha}{x+i\beta} + i\beta),$$

by parameter correspondence $a = \alpha + i\beta, b = 2i\beta$, and change of coordinates $(x+i\beta, y+i\beta) \leftrightarrow (x,y)$.

Parameter correspondence to McMullen map

Birational map studied in [M],

$$f_{a,b}(x,y) = (a,b) + (y, \frac{y}{x})$$

is equivalent to our self-anti-conjugate map

$$f(x,y) = (y, \frac{y+\alpha}{x+i\beta} + i\beta),$$

by parameter correspondence
$a = -\alpha + i\beta, b = \alpha + i\beta$, and change of coordinates $(x + i\beta, y + \alpha) \leftrightarrow (x, y)$.

$\mathcal{V}_n$

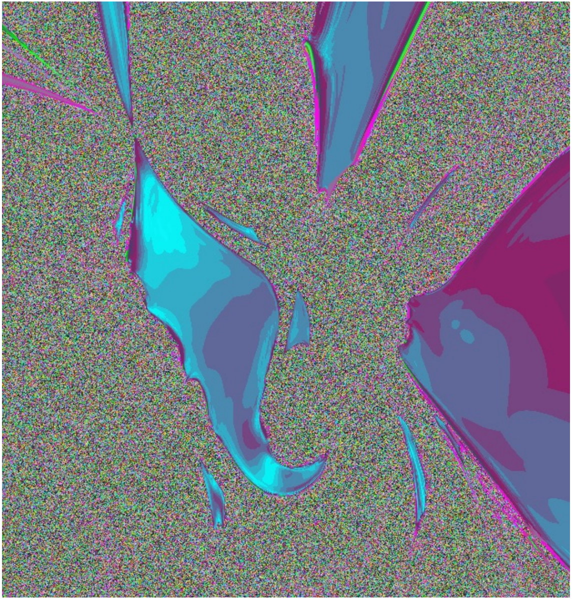
Set of parameters such that the map f induce complex surface automorphisms are specified as follows.

$$\mathcal{V}_n = \{(\alpha, \beta) | p \neq f^j(q), 0 \leq j \leq n{-}1, \mathit{and}\ p = f^n(q)\}$$

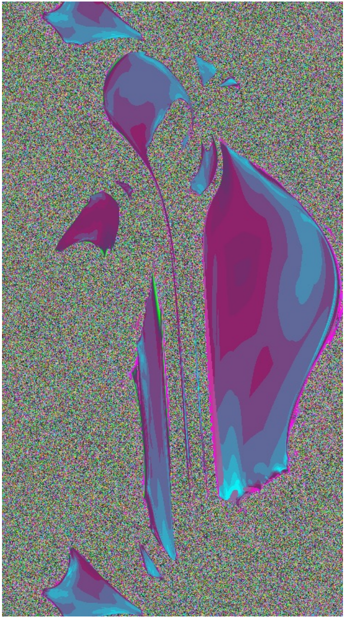
Bedford and Kim found several Siegel balls for a map in $\mathcal{V}_{12}$.
McMullen found Siegel balls for maps in $\mathcal{V}_8$ and in $\mathcal{V}_9$.
McMullen's automorphism for Q_{11} is in $\mathcal{V}_8$, and one for Q_{12} is in $\mathcal{V}_9$.

n(McMullen) $\leftrightarrow$ n(Bedford-Kim) $+ 3$.

Slice of Siegel balls in $\mathcal{V}_{12}$ map



Second Slice of Siegel balls in $\mathcal{V}_{12}$ map



Invariant cubics

The set of parameters whose map has an invariant cubic curve is described in [BK2].

Γ_1: Irreducible cubic with a cusp.

$$(a, b) = \varphi_1(t) = \left(\frac{t - t^3 - t^4}{(1+t)^2}, \frac{1 - t^5}{t^2 + t^3}\right).$$

Γ_2: Line tangent to a quadric.

$$(a, b) = \varphi_2(t) = \left(\frac{t + t^2 + t^3}{(1+t)^2}, \frac{t^3 - 1}{t + t^2}\right).$$

Γ_3: Three lines passing through a point.

$$(a, b) = \varphi_3(t) = (1 + t, t - t^{-1}).$$

Invariant cubics for self-anti-conjugate family

The parameters in our coordinate (α, β) for self-anti-conjugate family of rational maps are as follows.

Γ_1: Irreducible cubic with a cusp.

$$(\alpha, \beta) = \phi_1(t) = \left(\frac{-1 - t + 2t^3 - t^5 - t^6}{2t^2(1+t)^2}, \frac{1 - t^5}{2it^2(1+t)}\right).$$

Γ_2: Line tangent to a quadric.

$$(\alpha, \beta) = \phi_2(t) = \left(\frac{1 + t + 2t^2 + t^3 + t^4}{2t(1+t)^2}, \frac{t^3 - 1}{2it(1+t)}\right).$$

Γ_3: Three lines passing through a point.

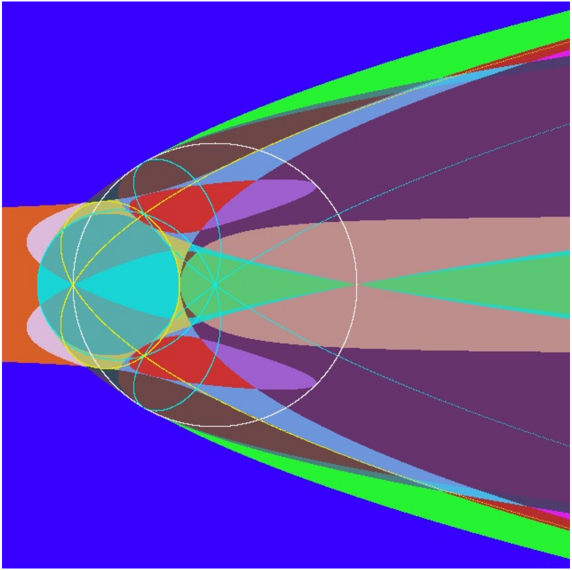
$$(\alpha, \beta) = \phi_3(t) = \left(\frac{t^2 + 2t + 1}{2t}, \frac{t^2 - 1}{2it}\right).$$

Invariant cubics and parameters

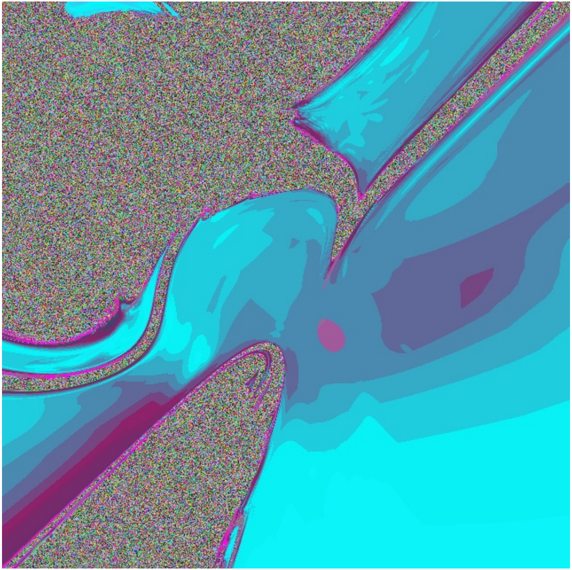
Proposition If $|t| = 1$, then

$$\phi_1(t) \in \mathbb{R}^2, \quad \phi_2(t) \in \mathbb{R}^2, \quad \phi_3(t) \in \mathbb{R}^2.$$

Γ_j set in (α, β)-coordinates



Slice of a rotation domain with two fixed points in $\mathcal{V}_7$ map



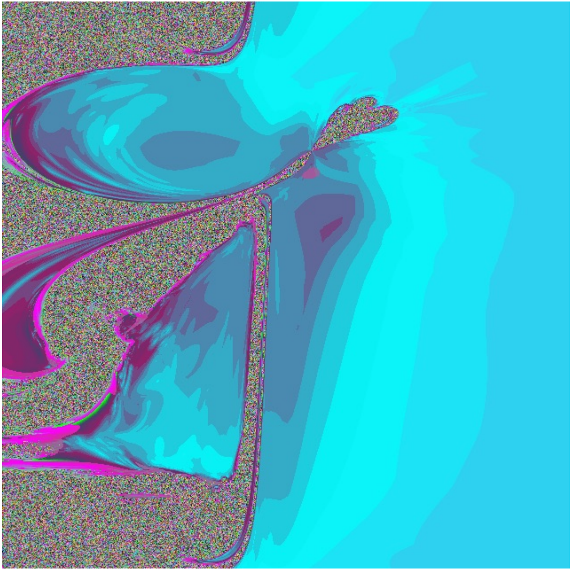
Eigenvalues for $\mathcal{V}_n \cap \Gamma_1$

For parameter $(\alpha, \beta) = \phi_1(t) \in \mathcal{V}_n \cap \Gamma_1$, the eigenvalues at the fixed points are as follows.

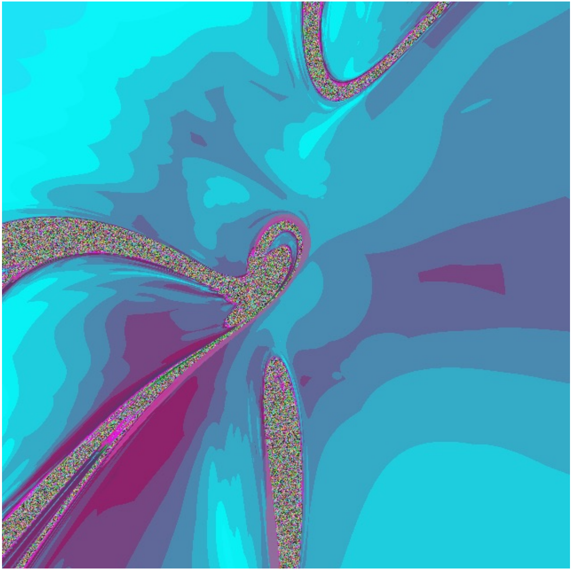
At FP_r, eigenvalues are t^2 and t^3.
At FP_s, eigenvalues are t^{-1} and t^n.

Eigenvalues t^2, t^3, t^{-1} acts in the invariant cubic curve with a cusp at FP_r.

Second Slice of the rotation domain in $\mathcal{V}_7$ map



Slice of rotation domain with a 2-cycle in a $\mathcal{V}_8$ map



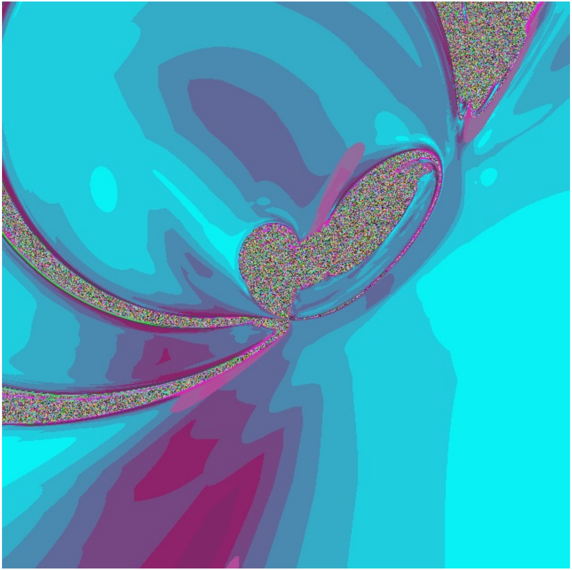
Eigenvalues for $\mathcal{V}_n \cap \Gamma_2$

For parameter $(\alpha, \beta) = \phi_2(t) \in \mathcal{V}_n \cap \Gamma_2$, the eigenvalues are as follows.

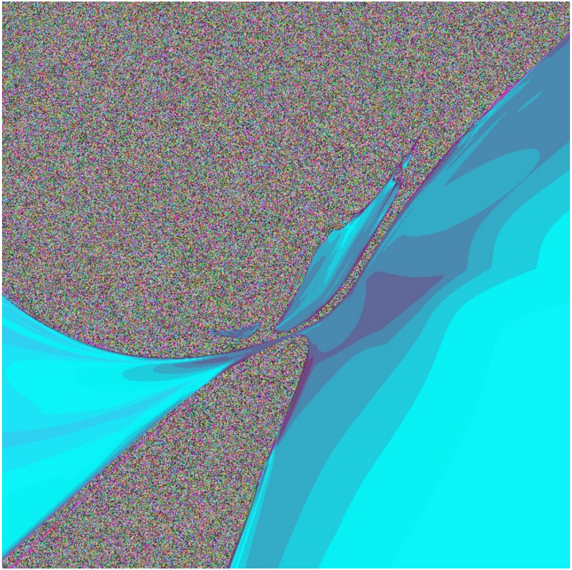
At FP_s, eigenvalues are $-t$ and $-t^2$.
At 2-periodic point, eigenvalues are t^{-2} and t^{n+2}.

Eigenvalues $-t, t^{-2}$ acts in the invariant cubic consisting of a quadric and a line tangent at FP_r.

Slice of rotation domain in another $\mathcal{V}_8$ map



Slice of a rotation domain with 3-cycle in a $\mathcal{V}_9$ map



Eigenvalues for $\mathcal{V}_n \cap \Gamma_3$

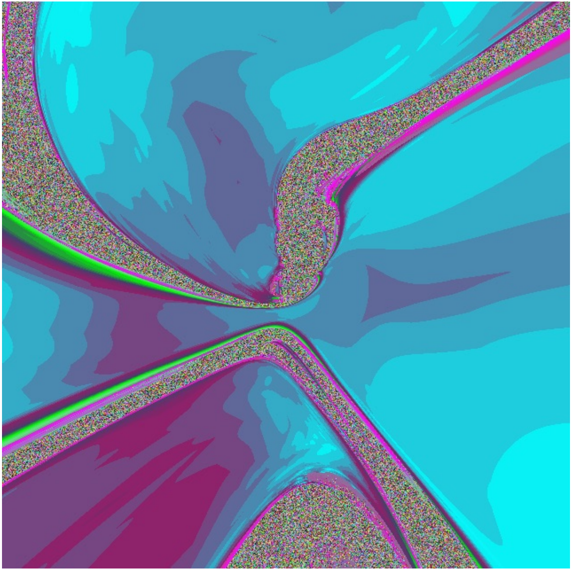
For parameter $(\alpha, \beta) = \phi_3(t) \in \mathcal{V}_n \cap \Gamma_3$, the eigenvalues are as follows.

At FP_s, eigenvalues are ωt and $\omega^2 t$, where ω is a prime cubic root of 1.

At 3-periodic point, eigenvalues are t^{-3} and t^{n+3}.

Eigenvalues t^{-3} acts in the invariant cubic consisting of three lines intersecting at FP_s.

Slice of a rotation domain with 3-cycle in another $\mathcal{V}_9$ map



References

[BK1] E.Bedford and KH. Kim, Periodicities in Linear Fractional Recurrences: Degree growth of birational surface maps, Mich. Math. J. **54**(2006), 647-670.
[BK2] Eric Bedford and Kyounghee Kim, Dynamics of Rational Surface Automorphisms: Linear Fractional Recurrences. J. Geomet. Anal. **19**, 553-583(2009).
[M] Curtis T. McMullen, Dynamics on blowups of the projective plane., Publ. Sci. IHES, **105**, 49-89(2007).