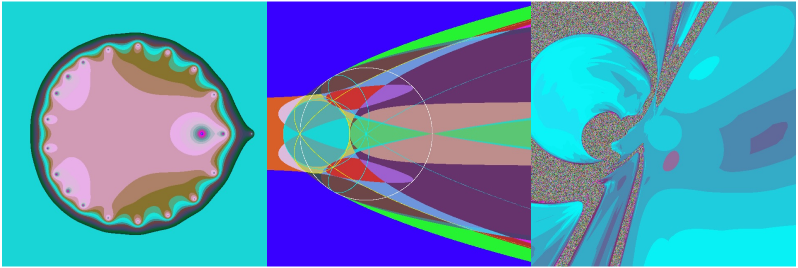
Affluence of Siegel balls in surface automorphisms



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Abstract

There are many automorphisms of rational surfaces which have Siegel balls or cycles of Siegel balls.

Such surface automorphisms are studied by E.Bedford, KH.Kim and C.McMullen.

We prove that the standard Coxeter element $w \in W_n$ can be realized by a surface automorphism with a cycle of Siegel balls, for all n sufficiently large.

Coxeter element and Siegel ball

THEOREM. For all n sufficiently large, the standard Coxeter element $w \in W_n$ can be realized by a surface automorphism with a cycle of Siegel balls.

REM. McMullen proved the following.

THEOREM(McMullen, 2005). For all n sufficiently large with $n \neq 2, 4 \bmod 6$, the standard Coxeter element $w \in W_n$ can be realized by a surface automorphism with a Siegel disk.

Minkowski lattice

Let $\mathbb{Z}^{1,n}$ denote the lattice $\mathbb{Z}^{n+1}$ equipped with the Minkowski inner product

$$x \cdot y = x_0 y_0 - x_1 y_1 - x_2 y_2 - \cdots - x_n y_n,$$

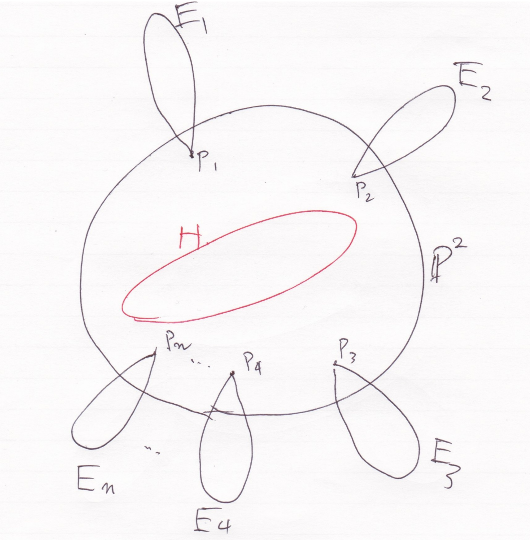
for basis $e_0, e_1, \cdots, e_n$.

Blowup and (co)homology

Let $\pi : S \to \mathbb{CP}^2$ be a blowup of $\mathbb{CP}^2$ at n distinct points $p_1, p_2, \cdots, p_n$.

$E_i = \pi^{-1}(p_i) \subset S$, exceptional fiber,
$H \subset S$, generic line.

A basis of $H^2(S, \mathbb{Z})$ is given by $[H]$ and $[E_i]$, $i = 1, \cdots, n$.



E1
E2
P1
P2
H
P2
Pn
P4
P3
E3
En
E4

Marking isomorphism

Define $\phi : \mathbb{Z}^{1,n} \to H^2(S, \mathbb{Z})$ by

$$\phi(e_0) = [H], \qquad \phi(e_i) = [E_i], \quad i = 1, \cdots, n.$$

Minkowski inner product	intersection pairing
$e_0 \cdot e_0 = 1$	$H \cdot H = 1$
$e_i \cdot e_j = -\delta_{ij}$	$E_i \cdot E_j = -\delta_{ij}$
$e_0 \cdot e_i = 0$	$H \cdot E_i = 0$

ϕ is called a **marking isomorphism**, and (S, ϕ) is called a **marked blowup**.

Reflections

For $\alpha \in \mathbb{Z}^{1,n}$ with $\alpha \cdot \alpha = -2$,

reflection $\rho_\alpha : \mathbb{Z}^{1,n} \to \mathbb{Z}^{1,n}$ is defined by

$$\rho_\alpha(x) = x + (x \cdot \alpha)\alpha.$$

As $\rho_\alpha(x) \cdot \rho_\alpha(y) = x \cdot y$, $\rho_\alpha \in O(\mathbb{Z}^{1,n})$.

$$\begin{aligned} \rho_\alpha(x) \cdot \rho_\alpha(y) &= (x + (x \cdot \alpha)\alpha) \cdot (y + (y \cdot \alpha)\alpha) \\ &= x \cdot y + (x \cdot \alpha)(\alpha \cdot y) + (y \cdot \alpha)(x \cdot \alpha) + (x \cdot \alpha)(y \cdot \alpha)(\alpha \cdot \alpha) \\ &= x \cdot y. \end{aligned}$$

Simple roots and canonical vector

Let $\alpha_0 = e_0 - e_1 - e_2 - e_3$,
and $\alpha_i = e_i - e_{i+1}, \quad i = 1, \cdots, n-1.$
$\alpha_0, \alpha_1, \cdots, \alpha_{n-1}$ are called **simple roots**. They define reflections.

$$s_i = \rho_{\alpha_i} \in O(\mathbb{Z}^{1,n}).$$

Vector

$$k_n = (-3, 1, 1, \cdots, 1) \in \mathbb{Z}^{1,n}$$

is called the **canonical vector**.

$$k_n \cdot \alpha_i = 0, \qquad i = 0, 1, \cdots, n-1.$$

$$s_i(k_n) = k_n, \qquad i = 0, 1, \cdots, n-1.$$

k_n is orthogonal to simple roots. And k_n is fixed by reflections s_i.

Weyl group

Let

$$L_n = \{x \in \mathbb{Z}^{1,n} \mid x \cdot k_n = 0\}$$

be the set of vectors orthogonal to the canonical vector k_n.

The orthogonal group $O(L_n)$ can be regarded as a subgroup of $O(\mathbb{Z}^{1,n})$ and contains $s_i, i = 0, \cdots, n-1$.

The **Weyl group** W_n is the subgroup of $O(L_n)$ generated by $s_0, s_1, \cdots, s_{n-1}$.

Realization of elements of W_n

Theorem(Nagata,1961)

If $F : S \to S$ is an automorphism, then there exists a unique element $w \in W_n$, such that

$$\begin{array}{ccc} \mathbb{Z}^{1,n} & \xrightarrow{w} & \mathbb{Z}^{1,n} \\ \downarrow \phi & & \downarrow \phi \\ H^2(S,\mathbb{Z}) & \xrightarrow{F_*} & H^2(S,\mathbb{Z}) \end{array}$$

commutes.

We say w is **realized** by F.

Coxeter element

Element $w \in W_n$ is called a **Coxeter element** if it is a product of generators $s_0, \cdots, s_{n-1}$, taken one at a time in any order.

All Coxeter elements are conjugate.

If $w : L_n \to L_n$, $w \in W_n$ is a Coxeter element, the characteristic polynomial is as follows.

$$P_n(t) = \det(tI - w) = \frac{t^{n-2}(t^3 - t - 1) + (t^3 + t^2 - 1)}{t - 1}.$$

Standard Coxeter element

The element

$$w = s_1 s_2 \cdots s_{n-1} s_0 \in W_n$$

is called the **standard Coxeter element**. We have

$$\begin{aligned}
w(e_0) &= 2e_0 - e_2 - e_3 - e_4 \\
w(e_1) &= e_0 - e_3 - e_4 \\
w(e_2) &= e_0 - e_2 - e_4 \\
w(e_3) &= e_0 - e_2 - e_3 \\
w(e_i) &= e_{i+1} \qquad (4 \leq i \leq n-1) \\
w(e_n) &= e_1
\end{aligned}$$

	H = e_0	P_1^* = e_1	P_1 = e_2	P_2 = e_3	Q = e_4	fQ = e_5	f^2Q = e_6	f^3Q = e_7	f^4Q = e_8	f^5Q = e_9	f^6Q = e_{10}
H: e_0	$2-t$	1	1	1							
P_* e_1		$-t$									1
e_2	-1		$-1-t$	-1							
e_3	-1	-1		$-1-t$							
e_4	-1	-1	-1		$-t$						
e_5					1	$-t$					
e_6						1	$-t$				
e_7							1	$-t$			
e_8								1	$-t$		
e_9									1	$-t$	
e_{10}										1	$-t$

Linear fractional recurrences

Birational map studied in [BK1],[BK2],

$$f_{a,b}(x,y) = (y, \frac{y+a}{x+b})$$

Birational map studied in [M2],

$$f_{a,b}(x,y) = (a,b) + (y, \frac{y}{x})$$

Self-anti-conjugate map

$$f_{\alpha,\beta}(x,y) = (y, \frac{y+\alpha}{x+i\beta} + i\beta),$$

To avoid confusions, we set

$$n = \nu + 3,$$

$$a = \alpha + i\beta, \qquad b = 2i\beta.$$

Here, n is the number of blowup points. And ν is the number of iterations to have a surface automorphism.

Parameters a, b are for birational maps studied by Bedford and Kim, which is convenient for maps related to real eigenvalues.

Parameters α, β are for self-anti-conjugate maps, which is convenient for maps related to non-real eigenvalues.

Linear fractional recurrences

For parameter $(\alpha, \beta) \in \mathbb{C}^2$, let

$$f_{\alpha,\beta}(x, y) = (y, \frac{y+\alpha}{x+i\beta} + i\beta)$$

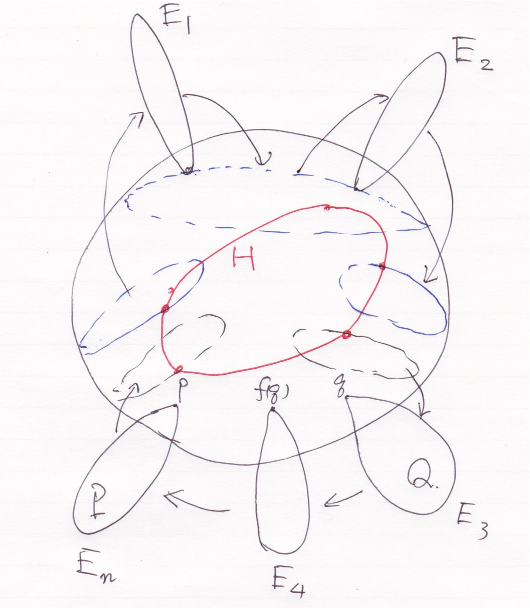
be a birational map.

The indeterminate point p_* of $f_{\alpha,\beta}$ and the indeterminate point q_* of $f^{-1}_{\alpha,\beta}$ are

$$p_* = (-i\beta, -\alpha), \qquad q_* = (-\alpha, i\beta).$$

Let $\mathcal{V}_\nu$ denote the set of parameters $(\alpha, \beta) \in \mathbb{C}^2$ satisfying

$$f^k_{\alpha,\beta}(q_*) \neq p_*, k = 0, 1, \cdots, \nu - 1, \quad \text{and} \quad f^\nu_{\alpha,\beta}(q_*) = p_*$$


E1
E2
H
p
f(q)
q
P
Q.
E3
En
E4

Surface automorphism

For $(\alpha, \beta) \in \mathcal{V}_\nu$, let $f = f_{\alpha,\beta}$ and let

$$\pi : S \to \mathbb{CP}^2$$

be the blowup of $\mathbb{CP}^2$ at $n = \nu + 3$ points

$$q_*, f(q_*), \cdots, f^\nu(q_*) = p_*, p_1 = [0:1:0], p_2 = [0:0:1].$$

THEOREM(Bedford and Kim, 2007)
If $(\alpha, \beta) \in \mathcal{V}_\nu$, then $f_{\alpha,\beta}$ realizes the standard Coxeter element $w \in W_{\nu+3}$.

$$\begin{array}{cccccccc} H & P_* & P_1 & P_2 & Q & fQ & \cdots & f^{\nu-1}Q \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\ & p_* & p_1 & p_2 & q_* & f(q_*) & \cdots & f^{\nu-1}(q_*) \end{array}$$

$$\begin{array}{rcl} f_*(H) & = & 2H - P_1 - P_2 - Q \\ f_*(P_*) & = & H - P_2 - Q \\ f_*(P_1) & = & H - P_1 - Q \\ f_*(P_2) & = & H - P_1 - P_2 \\ f_*(f^i Q) & = & f^{i+1}Q \qquad i = 0, \cdots, \nu - 2, \\ f_*(f^{\nu-1}Q) & = & P_* \end{array}$$

$$\chi_\nu(t) = t^{\nu+1}(t^3 - t - 1) + (t^3 + t^2 - 1) = (t-1)P_n(t).$$

$$\begin{array}{cccccccc} e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & \cdots & e_n \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\ H & P_* & P_1 & P_2 & Q & fQ & \cdots & f^{\nu-1}Q \end{array}$$

$$\begin{aligned} w(e_0) &= 2e_0 - e_2 - e_3 - e_4 \\ w(e_1) &= e_0 - e_3 - e_4 \\ w(e_2) &= e_0 - e_2 - e_4 \\ w(e_3) &= e_0 - e_2 - e_3 \\ w(e_i) &= e_{i+1} \qquad (4 \le i \le n-1) \\ w(e_n) &= e_1 \end{aligned}$$

A linear automorphism $R(x, y) = (\alpha x, \beta y)$ of $\mathbb{C}^2$ is an **irrational rotation** if $|\alpha| = |\beta| = 1$ and R has dense orbits on $S^1 \times S^1$. Let $F : X \to X$ be a holomorphic endomorphism of complex surface X. A domain $U \subset X$ is a **Siegel ball** for F if $F(U) = U$ and $F|U$ is analytically conjugate to $R|\Delta^2$ for some irrational rotation R. Here S is a complex surface and Δ is the unit disk in $\mathbb{C}$.

Siegel balls

We look for $f_{\alpha,\beta}$, $(\alpha,\beta) \in \mathcal{V}_n$, such that $f_{\alpha,\beta}$ has a Siegel ball or a cycle of Siegel balls. (Siegel ball stands for rotation domain of rank 2)

THEOREM(McMullen, 2005). For all n sufficiently large with $n \neq 2, 4 \mod 6$, the standard Coxeter element $w \in W_n$ can be realized by a surface automorphism with a Siegel disk.

THEOREM(Bedford and Kim, 2009). Suppose $\nu \geq 8$, $j = 2$ or 3, and j divides ν. Then there exists $(\alpha,\beta) \in \Gamma_j \cap V_\nu$ such that $f_{\alpha,\beta}$ has a rank 2 rotation domain centered at FP_r, as well as a rank 1 rotation domain centered at FP_s.

THEOREM(U.). For all n sufficiently large, the standard Coxeter element $w \in W_n$ can be realized by a surface automorphism with a cycle of Siegel balls.

We look for a cycle of Siegel balls when f does not have a Siegel ball(of rank 2) around fixed points.

First, observe the priodic points of period 2.

Linear fractional recurrence map has a unique cycle of period 2. In order to compute the determinant δ_2 and the trace τ_2 of the Jacobian matrix along the cycle, our family of self-anti-conjugate maps is convenient.

δ_2 and τ_2

PROPOSITION. For self-anti-conjugate map

$$f_{\alpha,\beta}(x,y) = (y, \frac{y+\alpha}{x+i\beta} + i\beta),$$

we have

$$\delta_2 = \frac{1-\alpha+i\beta}{1-\alpha-i\beta} \quad \text{and} \quad \tau_2 = \frac{2\alpha-4\beta^2-1}{1-\alpha-i\beta}.$$

Proof.

If (x, y) and (y, x) is a cycle of period 2, $(x \neq y)$, then

$$x - i\beta = \frac{y+\alpha}{x+i\beta}, \qquad y - i\beta = \frac{x+\alpha}{y+i\beta},$$

and $x + y = -1$, $xy = 1 + \beta^2 - \alpha$.

Along the 2-cycle, as

$$Df^2 = \begin{pmatrix} 0 & 1 \\ -\frac{x+\alpha}{(y+i\beta)^2} & \frac{1}{y+i\beta} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\frac{y+\alpha}{(x+i\beta)^2} & \frac{1}{x+i\beta} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -\frac{y-i\beta}{y+i\beta} & \frac{1}{y+i\beta} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\frac{x-i\beta}{x+i\beta} & \frac{1}{x+i\beta} \end{pmatrix}$$

$$= \frac{1}{(y+i\beta)(x+i\beta)} \begin{pmatrix} -(x-i\beta)(y+i\beta) & y+i\beta \\ -(x-i\beta) & -(x+i\beta)(y-i\beta)+1 \end{pmatrix}$$

Hence we have

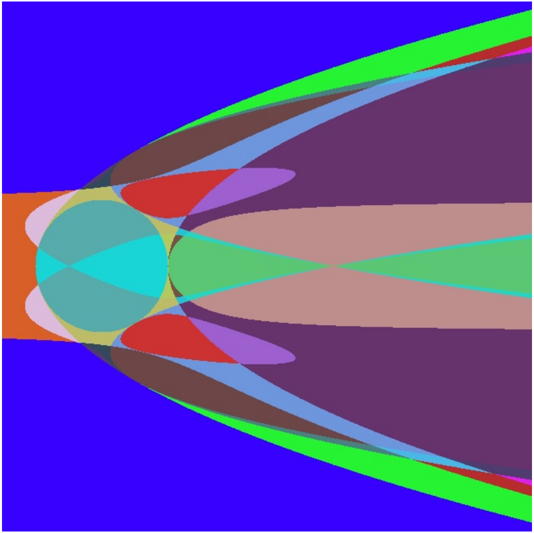
$$\delta_2 = \det(Df^2) = \frac{y - i\beta}{y + i\beta}\frac{x - i\beta}{x + i\beta} = \frac{1 - \alpha + i\beta}{1 - \alpha - i\beta},$$

and

$$\tau_2 = \mathrm{trace}(Df^2) = \frac{1 - (x - i\beta)(y + i\beta) - (x + i\beta)(y - i\beta)}{(y + i\beta)(x + i\beta)}$$

$$= \frac{2\alpha - 4\beta^2 - 1}{1 - \alpha - i\beta}.$$

Parameter space of linear fractional recurrences



Γ_1-family of linear fractional recurrences

As proved in [BK2](and adapted to our family), if $f_{\alpha,\beta}$ has an invariant cubic curve with a cuspidal singularity, $f_{\alpha,\beta}$ is parametrized by $t \in \mathbb{C}$ as follows.

$$\tau = t^2 + t^3, \qquad \delta = t^5,$$

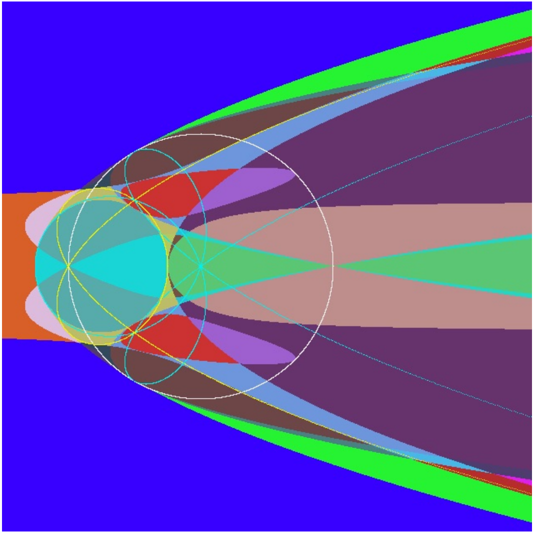
$$\alpha = \frac{2\delta - \tau - \tau\delta}{2\tau^2}, \qquad i\beta = \frac{1-\delta}{2\tau}.$$

These give

$$\delta_2 = \frac{1}{t^2},$$

$$\tau_2 = \frac{(1+t^2)(1-2t^2-t^3+t^4-t^5-2t^6+t^8)}{t^4(1+t+t^2+t^3+t^4)}.$$

Γ_j curves



$\Gamma_1 \cap \mathcal{V}_n$

Moreover, if t satisfies $\chi_\nu(t) = 0$ for some ν, then the birational automorphism $f_{\alpha,\beta}$ prescribed above defines an automorphism of a surface which is the blowup of $\mathbb{CP}^2$ in $\nu + 3$ points. So, $f_{\alpha,\beta}$ realizes the standard Coxeter element $w \in W_{\nu+3}$.

Let $\frac{\lambda}{t}$ and $\frac{\mu}{t}$ be the eigenvalues of Df^2 along the 2-cycle.

Then we have

$$\lambda\mu = 1, \qquad \lambda + \mu = t\tau_2.$$

Let $T = t\tau_2$. T is a rational function of t.

Proposition. $|\lambda| = |\mu| = 1, \lambda = \bar{\mu}$ if and only if $T \in [-2, 2]$.

PROPOSITION. When $|t| = 1$, T is real. The range of T, for $|t| = 1$, includes an open interval in $[-2, 2]$ and another open interval in $\mathbb{R} \setminus [-2, 2]$.

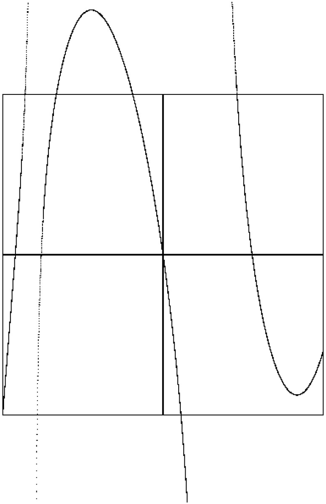
PROOF. Let $c = t + \frac{1}{t}$. Then

$$T(t) = \hat{T}(c) = \frac{c(c^4 - 6c^2 - c + 7)}{c^2 + c - 1}$$

is a real valued rational function of $c \in [-2, 2]$, with poles at $c = \frac{-1 \pm \sqrt{5}}{2}$, and vanishes at $c = 0$. Hence the range of T^2 on $[-2, 2]$ is $[0, \infty)$.

Note that $\hat{T}(-1) = 3$, $\hat{T}(2) = -\frac{6}{5}$.

Graph of T



Salem polynomial

A complex number is an **algebraic integer** if it is the zero of a polynomial with integer coefficients and leading coefficient 1.

Its **minimal polynomial** is the lowest degree polynomial of that type it satisfies.

Its (Galois) **conjugates** are the zeros of its minimal polynomial.

A **Salem number** is an algebraic integer $\tau > 1$ conjugate to τ^{-1}, all of whose conjugates, excluding τ and τ^{-1} lie on $|z| = 1$.

A **Salem polynomial** is the minimal polynomial of a Salem number.

Multiplicative independence and Diophantine condition

DEFINITION. Nonzero complex numbers $\lambda_1, \cdots, \lambda_n$ are **multiplicatively independent** if

$$\lambda_1^{k_1} \cdots \lambda_n^{k_n} = 1 \quad \Longrightarrow \quad k_1 = \cdots = k_n = 0.$$

DEFINITION. $\lambda_1, \cdots, \lambda_n$ are **jointly Diophantine** if there exist positive constants C and M, such that

$$|\lambda_1^{k_1} \cdots \lambda_n^{k_n} - 1| > C(\max|k_i|)^{-M} > 0 \quad \text{for} \quad \forall (k_1, \cdots, k_n) \in \mathbb{Z}^n \setminus \{\mathbf{o}\}.$$

THEOREM If the derivative $DF_p : T_pX \to T_pX$ has jointly Diophantine eigenvalues $(\lambda_1, \cdots, \lambda_n) \in (S^1)^n$, then F has a Siegel ball at p.

The proof is due to Siegel for $n = 1$ and to Sternberg for $n > 1$.

THEOREM(McMullen) If $\lambda_1, \cdots, \lambda_n$ are multiplicatively independent algebraic numbers on $S^1 \subset \mathbb{C}$, then they are jointly Diophantine.

PROOF By Fel'dman, using Gel'fond-Baker method, for algebraic numbers $k_0, \cdots, k_n$, not all zero,

$$|k_0 2\pi i + k_1 \log \lambda_1 + \cdots + k_n \log \lambda_n| > \exp(-M(d + \log H)),$$

where d is the degree of the field $\mathbb{Q}[k_0, \cdots, k_n, \lambda_1, \cdots, \lambda_n]$, $M = M(\lambda_1, \cdots, \lambda_n, d)$ is a constant depending only on $\lambda_1, \cdots, \lambda_n$ and d, $H = \max H(k_i)$. Here, height $H(k) = \Sigma|a_j|$, if $p(k) = 0$, $p(x) = \sum_{j=0}^{s} a_j x^j$ is an irreducible polynomial with relatively prime integer coefficients a_j.

For $k_i \in \mathbb{Z}$, $H = \max|k_i|$, and M depends only on (λ_j). Hence,

$$\exp(-M(d + \log H)) = (e^d H)^{-M} = e^{-dM}(\max|k_i|)^{-M}.$$

PROPOSITION. Let $U(t)$ be a rational function which satisfies $U(t) \in \mathbb{R}$ if $|t| = 1$. Let t be a root of a Salem polynomial with $|t| = 1$, and let t' be a conjugate of t with $|t'| = 1$.

Suppose we have $0 \leq |U(t)| < 2 < |U(t')|$. Let λ, μ be the roots of simultaneous equations

$$\lambda\mu = 1, \qquad \lambda + \mu = U(t).$$

Then for $k \in \mathbb{Z}, k \neq 0$, values λt^k and μt^k are multiplicatively independent and $|\lambda t^k| = |\mu t^k| = 1$.

PROOF. Let λ', μ' be the conjugates of λ, μ corresponding to t'. As $|U(t)| < 2$ and $|U(t')| > 2$, we have $|\lambda| = |\mu| = 1$, and $|\lambda'| = |\mu'|^{-1} \neq 1$. Now, suppose $(\lambda t^k)^i (\mu t^k)^j = 1$. Then $(\lambda'(t')^k)^i (\mu'(t')^k)^j = 1$ as well. And thus $i = j$. Therefore $((\lambda t^k)(\mu t^k))^i = t^{2ki} = 1$. Since t satisfies a Salem polynomial, it is not a root of unity, and thus $i = 0$.

Cycle of Siegel balls

PROPOSITION. Let t be a root of characteristic equation $\chi_\nu(t) = 0$. Assume t is not a root of unity, $|t| = 1$, and $|T(t)| < 2$, where $T(t) = t\tau_2$, defined for Γ_1-family of linear fractional recurrences. If there exists a conjugate t' of t with $|T(t')| > 2$, then the linear fractional recurrence $f_{\alpha,\beta}$ given by

$$\alpha = \frac{-1 - t + 2t^3 - t^5 - t^6}{2t^2(1+t)^2}, \qquad \beta = \frac{1 - t^5}{2it^2(1+t)}$$

defines an automorphism of a blowup of $\mathbb{CP}^2$ at $\nu + 3$ points which has a cycle of Siegel balls of period 2.

PROOF. The determinant of the unique 2-cycle of the linear fractional recurrence $f_{\alpha,\beta}$ is $\delta_2 = \frac{1}{t^2}$, and the trace of the 2-cycle is given by

$$\tau_2 = \frac{(1+t^2)(1-2t^2-t^3+t^4-t^5-2t^6+t^8)}{t^4(1+t+t^2+t^3+t^4+t^5)}.$$

The eigenvalues $\frac{\lambda}{t}$ and $\frac{\mu}{t}$ are given by

$$\lambda\mu = 1, \qquad \lambda + \mu = t\tau_2 = T(t).$$

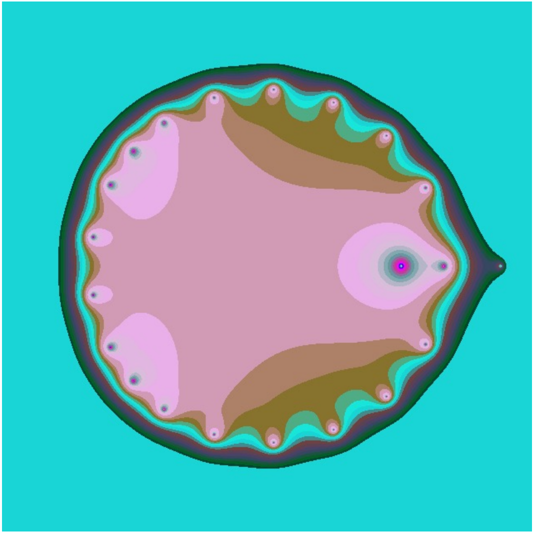
Apply the preceding proposition to prove the existence of a cycle of Siegel balls. The algebraic and multiplicatively independent eigenvalues satisfy the Diophantian condition.

PROPOSITION. For $\nu \in \mathbb{N}$ and $k \in \mathbb{Z}$, the characteristic equation

$$\chi_\nu(x) = x^{\nu+1}(x^3 - x - 1) + (x^3 + x^2 - 1) = 0$$

has at lest one solution t such that $|t| = 1$ and $t = e^{i\theta}$ with $\frac{2k-1}{\nu+1}\pi \leq \theta \leq \frac{2k+1}{\nu+1}\pi$.

$\chi_{17}(t)$



PROOF. Equation $\chi_\nu(x) = 0$ is equivalent to (if $1 + x - x^3 \neq 0$)

$$x^{\nu+1} = \frac{x^3 + x^2 - 1}{1 + x - x^3}.$$

We look for a solution t with $|t| = 1$. Let $t = s^2$ and $s = e^{i\vartheta}$ with $-\frac{\pi}{2} \leq \vartheta < \frac{\pi}{2}$. Let

$$\psi = \frac{t^3 + t^2 - 1}{1 + t - t^3} = \frac{s^3 + s - s^{-3}}{s^{-3} + s^{-1} - s^3}.$$

As $s + s^3 - s^{-3} = \cos\vartheta + i(7\sin\vartheta - 8\sin^3\vartheta)$, we have

$$\tan(\frac{\arg\psi}{2}) = \tan\vartheta(7 - 8\sin^2\vartheta)$$

which is continuous for $-\frac{\pi}{2} < \vartheta < \frac{\pi}{2}$.

So, we see that $-\pi < \arg\psi < \pi$.

As

$$\lim_{\vartheta \to \pm \frac{\pi}{2}} \frac{\text{arg } \psi}{2} = \mp \frac{\pi}{2},$$

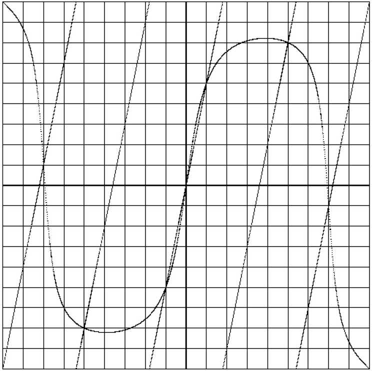
function $\theta \mapsto \vartheta \mapsto \text{arg } \psi$ defines a continuous function.

We see that $\psi = -1$ if and only if $t = -1$ (under the condition $|t| = 1$).

The graph of arg ψ and the graph of $(\nu + 1)\theta$ are shown in the following figure.

For each integer k, they intersect at lest once in interval $\frac{2k-1}{\nu+1}\pi \leq \theta \leq \frac{2k+1}{\nu+1}\pi$.

Graph of arg ψ



Proof of the main theorem

THEOREM(U.). For all n sufficiently large, the standard Coxeter element $w \in W_n$ can be realized by a surface automorphism with a cycle of Siegel balls.

PROOF. Let us consider the solutions of the characteristic equation $\chi_\nu(x) = 0$. The maximum of the distance, measured in argument, between two nearest solutions in $S^1 \subset \mathbb{C}$ is smaller than $\frac{4}{\nu+1}\pi$, since each interval of size $\frac{2}{\nu+1}\pi$ in the previous proposition contains at least one solution.

The number of possible root of unity for solutions of the characteristic equation is finite and uniformly bounded.

There are open intervals I_1, I_2 of arguments of t, such that

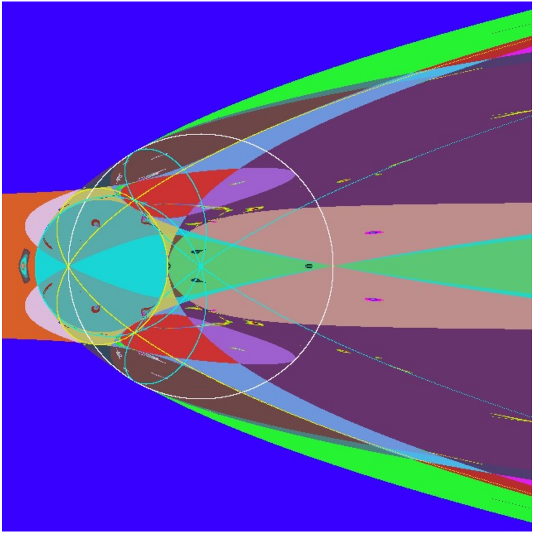
$$T(t) \in [-2.2], \quad \text{if } \arg t \in I_1,$$

$$T(t) \in \mathbb{R} \setminus [-2,2], \quad \text{if } \arg t \in I_2.$$

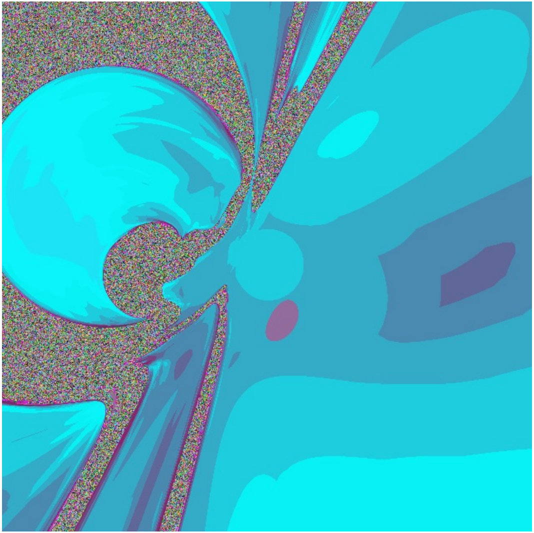
So, for sufficiently large ν, we can find Galois conjugates t and t' of solutions of $\chi_\nu(x) = 0$, which are not a root of unity, with $T(t) \in [-2,2]$ and $T(t') \in \mathbb{R} \setminus [-2,2]$.

The linear fractional recurrence $f_{\alpha,\beta}$ specified by t in the Γ_1 family gives an automorphism of blowup of the projective plane with a period-two cycle of Siegel balls.

parameter space for V_{17}



Anti-conjugate-axis slice for V_{17} map



References

[BK1] E.Bedford and KH. Kim, Periodicities in Linear Fractional Recurrences: Degree growth of birational surface maps, Mich. Math. J. **54**(2006), 647-670.

[BK2] Eric Bedford and Kyounghee Kim, Dynamics of Rational Surface Automorphisms: Linear Fractional Recurrences. J. Geomet. Anal. **19**, 553-583(2009).

[M1] Curtis T. McMullen, Dynamics on K3 surfaces Salem numbers and Siegel disks, J. reine angew. Math. 545(2002),201-233.

[M2] Curtis T. McMullen, Dynamics on blowups of the projective plane., Publ. Sci. IHES, **105**, 49-89(2007).