# Abundance of Siegel balls
# in a family of complex Hénon maps

Shigehiro Ushiki

Graduate School of Human and Environmental Studies

Kyoto University

Karuizawa, January, 2016

ABSTRACT

There exists a real two-parameter family of complex Hénon maps such that the number of coexisting cycles of Siegel balls is unbounded in the subfamily of any open set of parameters.

## 1.1 Complex Hénon map

For parameters $(\alpha, \beta) \in \mathbb{C}^2$, define our Hénon map $H_{\alpha,\beta} : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$H_{\alpha,\beta} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ \beta(y^2 + \alpha) - \beta^2 x \end{pmatrix}.$$

It is conjugate to the classical complex Hénon map

$$h_{b,c} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + c + by \\ x \end{pmatrix}$$

by conjugacy map and correspondence of parameters given by

$$\phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \beta y \\ \beta x \end{pmatrix}, \quad b = -\beta^2, \quad c = -\alpha b,$$

$$\phi \circ H_{\alpha,\beta} \circ \phi^{-1} = h_{b,c}$$

## 1.2 Fixed point

The fixed point $P_* = \begin{pmatrix} y_* \\ y_* \end{pmatrix}$ of Hénon map is given by quadratic equation

$$y_* = \beta(y_*^2 + \alpha) - \beta^2 y_*,$$

or
$$y_*^2 - (\beta + \beta^{-1})y_* + \alpha = 0.$$
The Jacobian matrix at the fixed point is as follows.
$$DH_{\alpha,\beta}|_{P_*} = \begin{pmatrix} 0 & 1 \\ -\beta^2 & 2\beta y_* \end{pmatrix} = \beta \begin{pmatrix} 0 & \beta^{-1} \\ -\beta & 2y_* \end{pmatrix}.$$
$$\text{trace } DH_{\alpha,\beta} = 2\beta y_*, \quad \det DH_{\alpha,\beta} = \beta^2.$$

## 1.3 Our family

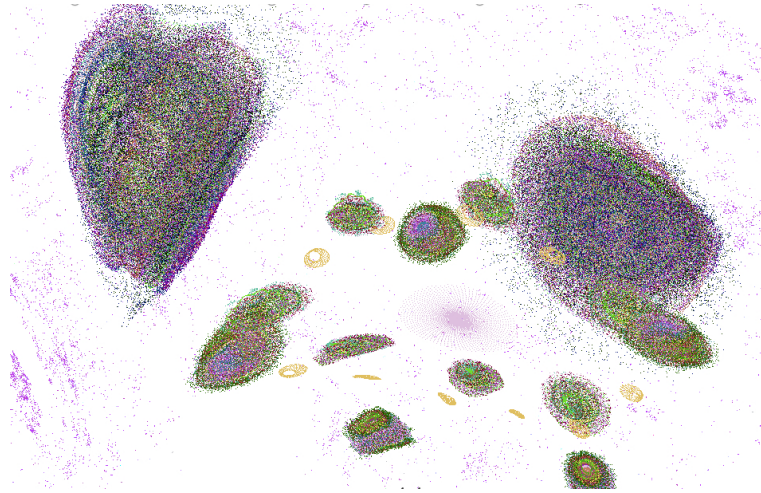We specify the eigenvalues at the fixed point. For eigenvalues $\beta\mu, \beta\mu^{-1}$, we have
$$y_* = \frac{\mu + \mu^{-1}}{2} \quad \text{and} \quad \alpha = (\beta + \beta^{-1})y_* - y_*^2.$$
Our family is given by
$$\beta = \cos\theta + i\sin\theta, \quad \mu = \cos\varphi + i\sin\varphi.$$
with
$$y_* = \cos\varphi, \quad \alpha = 2\cos\theta\cos\varphi - \cos^2\varphi.$$



Co-existing cycles of Siegel balls

## 2.1 Abundance of Siegel balls

THEOREM For any open set $U \subset (\mathbb{R}/2\pi\mathbb{Z})^2$ and any integer $N > 1$, there exists a point $(\theta_N, \varphi_N) \in U$ such that the Hénon map $H_{\alpha,\beta}$ for this parameter has more than $N$ cycles of Siegel balls.

The proof of this theorem is based on the following theorem of parabolic bifurcations.

## 2.2 Elliptic-parabolic bifurcation

Let $p > 1$ be an integer, and let $\nu$ be a prime $p$-th root of unity,

$$\nu^k \neq 1 \quad (k = 1, \cdots, p-1), \qquad \nu^p = 1.$$

If $\nu$ is an eigenvalue of $DH_{\alpha,\beta}|_{P_*}$, then

$$\text{trace } DH_{\alpha,\beta}|_{P_*} = \nu + \beta^2 \bar{\nu}, \qquad y_* = \frac{\beta\bar{\nu} + \beta^{-1}\nu}{2},$$

$$\alpha_0 = (\beta + \beta^{-1})y_* - y_*^2.$$

Suppose $\{P_n\} = \left\{ \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right\}$ is $p$-periodic. From

$$x_{n+1} = y_n, \qquad y_{n+1} = \beta(y_n^2 + \alpha) - \beta^2 x_n,$$

we have

$$\beta^{-1} y_{n+1} = y_n^2 + \alpha - \beta y_{n-1}.$$

THEOREM     For prime $p$-th root $\nu$ of 1, and for all $\beta \in \mathbb{C}$, except for a finite number of values, there exists a constant $\alpha_1 \neq 0$ and a family $y_n(\varepsilon)$ of $p$-cycles for parameter $\alpha = \alpha(\varepsilon^p)$, such that

$$\alpha(\varepsilon^p) = \alpha_0 + \alpha_1 \varepsilon^p + O(\varepsilon^{2p}), \qquad y_n(\varepsilon) = y_* + \nu^n \varepsilon + O(\varepsilon^2),$$

holds for $\varepsilon \in \mathbb{C}$ near $\varepsilon = 0$.

PROPOSITION     The constant $\alpha_1$ in the previous theorem depends upon $\beta$ and is a non trivial rational function of $\beta$.

## 2.3 Self-anti-conjugate cycles

THEOREM     In the previous theorem, if $|\beta| = 1$ then $\alpha_0 \in \mathbb{R}$ and $\alpha_1 \in \mathbb{R}$. Moreover, if $\alpha \in \mathbb{R}$, near $\alpha_0$, then

$$y_n = \overline{y_{-n}} \qquad or \qquad y_n = \overline{y_{1-n}}$$

for some $\varepsilon$.

THEOREM    If $p$-periodic orbit is self-anti-conjugate, *i.e.*

$$y_n = \overline{y_{-n}} \qquad or \qquad y_n = \overline{y_{1-n}},$$

then the jacobian matrix along the orbit is of the form

$$D(H^{\circ p}_{\alpha,\beta})\big|_{P_0} = \beta^p A,$$

with $\det A = 1$ and trace $A \in \mathbb{R}$.

## 2.4 Trace function

Let
$$\tau(\beta, \alpha) = \beta^{-p} \text{ trace } D(H^{\circ p}_{\alpha,\beta})\big|_{P_0}.$$

PROPOSITION    $\tau(\beta, \alpha)$ is holomorphic near $(\beta_0, \alpha_0)$, and non-constant with respect to $\alpha$.

PROPOSITION    If $|\beta| = 1$ and $\alpha \in \mathbb{R}$, then $\tau(\beta, \alpha) \in \mathbb{R}$ near $(\beta, \alpha_0(\beta))$, and $-2 < \tau(\beta, \alpha_0(\beta)) < 2$. (Except for finitely many values of $\beta$.)

## 3.1 Discrete Fourier expansion

Recall the equation of $p$-periodic point. $(y_{n+p} = y_n)$

$$\beta^{-1} y_{n+1} = y_n^2 + \alpha - \beta y_{n-1}, \qquad n = 0, \cdots, p-1.$$

Discrete Fourier expansion

$$y_n = u_0 + \nu^n u_1 + \nu^{2n} u_2 + \cdots + \nu^{kn} u_k + \cdots + \nu^{(p-1)n} u_{p-1}$$

$$= \sum_{k=0}^{p-1} \nu^{kn} u_k$$

gives rise to the following equation.

$$(F_0) \quad (\beta + \beta^{-1}) u_0 = u_0^2 + \sum_{\ell=1}^{p-1} u_\ell u_{p-\ell} + \alpha,$$

$$(F_1) \quad (\beta\bar{\nu} + \beta^{-1}\nu)u_1 \;=\; 2u_0u_1 + \sum_{\ell=2}^{p-1} u_\ell u_{p+1-\ell},$$

$$(F_k) \quad (\beta\bar{\nu}^k + \beta^{-1}\nu^k)u_k \;=\; 2u_0u_k + \sum_{\ell=1}^{k-1} u_\ell u_{k-\ell} + \sum_{\ell=k+1}^{p-1} u_\ell u_{p+k-\ell}.$$

$$k = 2, \cdots, p-1.$$

This equation has a solution, corresponding to the elliptic-parabolic fixed point $P_*$.

$$u_0 = y_*, \quad u_1 = u_2 = \cdots = u_{p-1} = 0, \quad \alpha = \alpha_0.$$

Rewrite these equations as follows.

$$(F_0) \quad (\beta + \beta^{-1})u_0 \;=\; u_0^2 + \sum_{\ell=1}^{p-1} u_\ell u_{p-\ell} + \alpha,$$

$$(F_1') \quad (\beta\bar{\nu} + \beta^{-1}\nu - 2u_0)u_1 \;=\; \sum_{\ell=2}^{p-1} u_\ell u_{p+1-\ell},$$

$$(F_k') \quad (\beta\bar{\nu}^k + \beta^{-1}\nu^k - 2u_0)u_k \;=\; \sum_{\ell=1}^{k-1} u_\ell u_{k-\ell} + \sum_{\ell=k+1}^{p-1} u_\ell u_{p+k-\ell}.$$

$$k = 2, \cdots, p-1.$$

## 3.2 Emanating branch of periodic points

Parameter $\alpha$ appears only in $(F_0)$. For each $\beta$, let $\varepsilon \in \mathbb{C}$ be a small parameter and let $\delta \in \mathbb{C}$ be a constant to be determined. Dependence upon $\beta$ will be considered later. Suppose $u_0 = y_* - \dfrac{\delta}{2}\varepsilon^p$, and $u_1 = \varepsilon v_1$. From equation $(F')$, we may suppose, inductively, $u_k = \varepsilon^k v_k$, $(k = 2, \cdots, p-1)$. Here, $v_1, v_2, \cdots, v_{p-1}$ are functions of $\varepsilon$. (Later, we see they are functions of $\varepsilon^p$.)

$$(E_0) \quad u_0 \;=\; y_* - \frac{\delta}{2}\varepsilon^p,$$

$$(E_k) \quad u_k \;=\; \varepsilon^k v_k, \quad k = 1, \cdots, p-1.$$

Rewrite equation $(F)$ using $(E)$, with $y_* = \dfrac{\beta\bar{\nu} + \beta^{-1}\nu}{2}$, to get the following equation.

$$(G_1) \qquad \delta v_1 \;=\; \sum_{\ell=2}^{p-1} v_\ell v_{p+1-\ell},$$

$$(G_k) \qquad (\beta(\bar{\nu}^k - \bar{\nu}) + \beta^{-1}(\nu^k - \nu) + \delta\varepsilon^p)v_k \;=\; \sum_{\ell=1}^{k-1} v_\ell v_{k-\ell} \;+\; \varepsilon^p \sum_{\ell=k+1}^{p-1} v_\ell v_{p+k-\ell},$$

$$k = 2, \cdots, p-1.$$

Note that $\alpha$ will be computed by $(F_0)$, afterwards.

## 3.3 Principal part equation $(L)$

Let $v_k = a_k + O(\varepsilon)$, $k = 1, \cdots, p-1$. And let $\gamma_k = \beta(\bar{\nu}^k - \bar{\nu}) + \beta^{-1}(\nu^k - \nu)$ for $k = 2, \cdots, p-1$. Equation $(G)$, as $\varepsilon \to 0$, yields the following equation.

$$(L_1) \qquad \delta a_1 \;=\; \sum_{\ell=2}^{p-1} a_\ell a_{p+1-\ell},$$

$$(L_k) \qquad \gamma_k a_k \;=\; \sum_{\ell=1}^{k-1} a_\ell a_{k-\ell}, \qquad k = 2, \cdots, p-1.$$

Inductively from $(L_2), \cdots, (L_{p-1})$, we have

$$a_k \;=\; \frac{1}{\gamma_k} \sum_{\ell=1}^{k-1} a_\ell a_{k-\ell} \;=\; \eta_k a_1^k,$$

with $\eta_k = \eta_k(\beta)$ rational function of $\beta$, for $k = 2, \cdots, p-1$.
Or

$$\eta_1 = 1, \qquad \eta_k = \frac{1}{\gamma_k} \sum_{\ell=1}^{k-1} \eta_\ell \eta_{k-\ell}.$$

From $(L_1)$,

$$\delta a_1 \;=\; \Phi(\beta) a_1^{p+1}$$

with a rational function $\Phi(\beta)$.

PROPOSITION. $\Phi(\beta)$ is a non-trivial rational function of $\beta$.

PROOF.      Obviously, $\Phi(\beta)$ is a rational function of $\beta$. We show that $\Phi(-\nu) > 0$. If $\beta = -\nu$, then

$$\gamma_k = 2 - (\bar{\nu}^{k-1} + \nu^{k-1}), \quad k = 2, \cdots, p-1.$$

Therefore, $0 < \gamma_k(-\nu) \le 4$, for $k = 2, \cdots, p-1$. These imply $\eta_k(-\nu) > 0$, for $k = 2, \cdots, p-1$. And we have $\Phi(-\nu) > 0$, since

$$\Phi(\beta) = \sum_{\ell=2}^{p-1} \eta_\ell(\beta)\eta_{p+1-\ell}(\beta).$$

If $|\beta| = 1$, then $\gamma_k \in \mathbb{R}$, $(k = 2, \cdots, p-1)$, $\eta_k \in \mathbb{R}$, $(k = 1, \cdots, p-1)$, and $\Phi(\beta) \in \mathbb{R}$. These are non-zero except for a finite number of values of $\beta$.

Now, we determine the constant $\delta$ by $\delta = \Phi(\beta)$. Equation $(L)$ has a solution

$$a_1 = 1, \quad a_k = \eta_k(\beta), \quad k = 2, \cdots, p-1.$$

If $|\beta| = 1$, then $a_1, \cdots, a_{p-1}$ are all real. Other solutions give the same periodic orbit ( for different choice of initial point).

Now, we go back to equation $(G)$.

$$(G_1) \qquad \delta v_1 = \sum_{\ell=2}^{p-1} v_\ell v_{p+1-\ell},$$

$$(G_k) \qquad (\gamma_k + \delta\varepsilon^p)v_k = \sum_{\ell=1}^{k-1} v_\ell v_{k-\ell} + \varepsilon^p \sum_{\ell=k+1}^{p-1} v_\ell v_{p+k-\ell},$$
$$k = 2, \cdots, p-1.$$

Observe that $\varepsilon$ appears only as $\varepsilon^p$ in this equation. Let $\kappa = \varepsilon^p$ and rewrite the equation as follows.

$$(\Gamma_1) \qquad w_1 = \sum_{\ell=2}^{p-1} v_\ell v_{p+1-\ell} - \delta v_1,$$

$$(\Gamma_k) \qquad w_k = \sum_{\ell=1}^{k-1} v_\ell v_{k-\ell} - \gamma_k v_k + \kappa \left( \sum_{\ell=k+1}^{p-1} v_\ell v_{p+k-\ell} - \delta v_k \right),$$
$$k = 2, \cdots, p-1.$$

These define a map
$$\Gamma : \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{p-1} \to \mathbb{C}^{p-1}$$
by
$$\Gamma(\beta, \kappa, v_1, \cdots, v_{p-1}) = (w_1, \cdots, w_{p-1}).$$

$\Gamma$ is a quadratic polynomial in $v_1, \cdots, v_{p-1}$, with coefficients rational in $\beta$, and affine in $\kappa$.

## 3.4 Matrix $\Lambda_\beta$

Except for a finite number of values of $\beta$,
$$\Gamma(\beta, 0, a_1, \cdots, a_{p-1}) = (0, \cdots, 0)$$

holds. The jacobian matrix $\Lambda_\beta = \left( \dfrac{\partial w_i}{\partial v_j} \right)$ at $(\beta, 0, a_1, \cdots, a_{p-1})$ is as follows.

$$
\Lambda_\beta =
\begin{pmatrix}
-\delta & 2a_{p-1} & 2a_{p-2} & \cdots & \cdots & 2a_2 \\
2a_1 & -\gamma_2 & 0 & \cdots & \cdots & 0 \\
2a_2 & 2a_1 & -\gamma_3 & 0 & \cdots & 0 \\
2a_3 & 2a_2 & 2a_1 & -\gamma_4 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
2a_{p-2} & 2a_{p-3} & \cdots & 2a_2 & 2a_1 & -\gamma_{p-1}
\end{pmatrix}.
$$

All components of $\Lambda_\beta$ are rational functions of $\beta$.

PROPOSITION    If $p \geq 3$, the matrix $\Lambda_\beta$ is regular except for a finite number of values of $\beta$.

PROOF    Obviously, the determinant of $\Lambda_\beta$ is a rational function of $\beta$. We show that it is non-trivial. Let

$$
M_\beta =
\begin{pmatrix}
-\delta & a_{p-1} & a_{p-2} & \cdots & \cdots & a_2 \\
a_1 & -\gamma_2 & 0 & \cdots & \cdots & 0 \\
a_2 & a_1 & -\gamma_3 & 0 & \cdots & 0 \\
a_3 & a_2 & a_1 & -\gamma_4 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
a_{p-2} & a_{p-3} & \cdots & a_2 & a_1 & -\gamma_{p-1}
\end{pmatrix}.
$$

Equation $(L)$ is equivalent to

$$M_\beta \begin{pmatrix} a_1 \\ \vdots \\ a_{p-1} \end{pmatrix} = \mathbf{0}.$$

As we have non-trivial solutions for all values of $\beta$, except for a finite number of values,

$$\det M_\beta = 0$$

holds as a rational function of $\beta$.

Observe the sweeping-out process of $M_\beta$. To sweep out the off-diagonal components of the first line, other lines of $M_\beta$ are used with diagonal components $\gamma_2, \cdots, \gamma_{p-1}$ as pivots. These pivots are non-trivial rational functions of $\beta$. To suppress a term in $(1, k)$-component of $M_\beta$, say $t_k$, we add the $k$-th line multiplied by $t_k/\gamma_k$ to the first line. Then the $(1, j)$-component, say $c_j$, becomes

$$c_j + \frac{a_{k-j}}{\gamma_k} t_k, \qquad j = 1, \cdots, k-1.$$

In this process, all the components of the first line, except for $-\delta$, are sums of terms of the form

$$\frac{1}{\gamma_{k_1}^{m_1} \cdots \gamma_{k_\ell}^{m_\ell}}.$$

When all the off-diagonal components of the first line are swept out, the first line vanishes.

Next, let us compute $\det \Lambda_\beta$ in a similar way. To compare the sweeping-out process, let $b_k = a_k$ and rewrite the off-diagonal components of $\Lambda_\beta$ as

$$2a_k = a_k + b_k, \qquad k = 1, \cdots, p-1.$$

Sweep-out the off-diagonal components of the first line of $\Lambda_\beta$ to get a lower triangle matrix. The terms without $b_k$'s are exactly same as in the sweep-out procedure of $M_\beta$. The terms without $a_k$'s contribute exactly same. There are other terms consisting of $a_k$'s and $b_k$'s. Anyway, all the terms are always sums of terms of the form

$$\frac{1}{\gamma_{k_1}^{m_1} \cdots \gamma_{k_\ell}^{m_\ell}}.$$

After the sweeping-out, in the $(1,1)$-component of the triangle matrix, $-\delta$ cancels the $a_k$-only terms. And $b_k$-only terms gives $\delta$. The remaining terms are a sum of terms of the form

$$\frac{1}{\gamma_{k_1}^{m_1} \cdots \gamma_{k_\ell}^{m_\ell}},$$

with positive coefficients.

The $(1,1)$-component of the triangle matrix is obviously a rational function of $\beta$. To prove the non-triviality of the rational function, we show that it does not vanish for $\beta = -\nu$. As

$$\gamma_k(\beta) \;=\; \beta(\bar{\nu}^k - \bar{\nu}) + \beta^{-1}(\nu^k - \nu),$$

$$\gamma_k(-\nu) \;=\; 2 - (\nu^{k-1} + \bar{\nu}^{k-1}) > 0, \qquad k = 2, \cdots, p-1.$$

These imply the positivity of $a_1, \cdots, a_{p-1}$ and $\delta$. Furthermore, terms of the form

$$\frac{1}{\gamma_{k_1}^{m_1} \cdots \gamma_{k_\ell}^{m_\ell}}$$

are all positive. Hence the $(1,1)$-component of the triangle matrix is strictly positive and greater than $\delta$. We conclude that

$$\det \Lambda_{-\nu} \neq 0.$$

## 3.5 Solution of Equation $(\Gamma)$

Now, we go back to equation $(\Gamma)$.

$$(\Gamma_1) \qquad w_1 \;=\; \sum_{\ell=2}^{p-1} v_\ell v_{p+1-\ell} - \delta v_1,$$

$$(\Gamma_k) \qquad w_k \;=\; \sum_{\ell=1}^{k-1} v_\ell v_{k-\ell} - \gamma_k v_k \;+\; \kappa \left( \sum_{\ell=k+1}^{p-1} v_\ell v_{p+k-\ell} - \delta v_k \right),$$

$$k = 2, \cdots, p-1.$$

$$\Gamma(\beta, \kappa, v_1, \cdots, v_{p-1}) \;=\; (w_1, \cdots, w_{p-1}).$$

PROPOSITION    For any $\beta_0 \in \mathbb{C}$, except for a finite number of values, there exists a neighborhood, $U$, of $(\beta_0, 0) \in \mathbb{C}^2$, such that implicit functions

$$v_1(\beta, \kappa), v_2(\beta, \kappa), \cdots, v_{p-1}(\beta, \kappa)$$

defined by

$$\Gamma(\beta, \kappa, v_1, \cdots, v_{p-1}) \; = \; (0, \cdots, 0)$$

with

$$v_k(\beta_0, 0) \; = \; a_k(\beta_0), \qquad k = 1, \cdots, p-1,$$

exist and holomorphic in $U$.

PROOF     Except for a finite number of values of $\beta$,

$$\Gamma(\beta, 0, a_1, \cdots, a_{p-1}) \; = \; (0, \cdots, 0)$$

holds. As we verified, the jacobian matrix $\Lambda_\beta \; = \; \left( \dfrac{\partial w_i}{\partial v_j} \right)$ at $(\beta, 0, a_1, \cdots, a_{p-1})$ is regular. Apply the implicit function theorem to get the implicit function.

## 3.6 Parameter $\alpha$

Functions $v_1, \cdots, v_{p-1}$, with $\kappa = \varepsilon^p$ give solutions of equation $(G)$. Next, let us go back to equation $(F)$. We introduced redundant parameters $\varepsilon$ and $\delta$. Parameter $\delta$ was determined as a rational function of $\beta$. The redundant parameter $\kappa = \varepsilon^p$ is related to the remaining parameter $\alpha$ by equation $(F_0)$. From equation $(F_0)$, we have

$$(K) \qquad \alpha \; = \; (\beta + \beta^{-1})(y_* - \frac{\delta}{2}\kappa) - (y_* - \frac{\delta}{2}\kappa)^2 - \kappa \sum_{\ell=1}^{p-1} v_\ell v_{p-\ell},$$

which is a function of $\beta$ and $\kappa$.

As

$$y_* \; = \; \frac{\beta\bar{\nu} + \beta^{-1}\nu}{2}, \qquad \alpha_0 = (\beta + \beta^{-1})y_* - y_*^2,$$

$$\alpha - \alpha_0 \; = \; -\kappa\left( \sum_{\ell=1}^{p-1} v_\ell v_{p-\ell} - y_*\delta + \frac{\delta}{2}(\beta + \beta^{-1}) \right) - \frac{\delta^2}{4}\kappa^2$$

$$= \; -\kappa\left( \sum_{\ell=1}^{p-1} a_\ell a_{p-\ell} + \frac{\delta}{2}(\beta(1 - \bar{\nu}) + \beta^{-1}(1 - \nu)) \right) + o(\kappa).$$

By setting

$$\alpha_1 \; = \; -\left( \sum_{\ell=1}^{p-1} a_\ell a_{p-\ell} + \frac{\delta}{2}(\beta(1 - \bar{\nu}) + \beta^{-1}(1 - \nu)) \right),$$

we have
$$\alpha = \alpha_0 + \alpha_1 \kappa + o(\kappa).$$

PROPOSITION $\alpha_1$ is a non-trivial rational function of $\beta$, and takes real value if $|\beta| = 1$.

PROOF If $\beta = -\nu$, then, as in the preceding proposition, $a_1, \cdots, a_{p-1}$ and $\delta$ are real and positive. Moreover, $\beta(1-\bar{\nu}) + \beta^{-1}(1-\nu) = 2 - (\nu + \bar{\nu}) > 0$. Hence, $\alpha_1(-\nu) < 0$, which shows $\alpha_1$ is non-trivial. If $|\beta| = 1$, then $\beta^{-1} = \bar{\beta}$. The reality of $\gamma_1, \cdots, \gamma_{p-1}, a_1, \cdots, a_{p-1}$, and $\delta$ is obvious.

PROPOSITION For all $\beta_0 \in \mathbb{C}$, except for a finite number of values, there exists a neighborhood, $U$, of $(\beta_0, \alpha_0(\beta_0)) \in \mathbb{C}^2$, such that the implicit function $\kappa = \kappa(\beta, \alpha)$ satisfying $\kappa(\beta_0, \alpha_0(\beta_0)) = 0$ defined by equation $(K)$ exists and holomorphic in $U$.

## 4.1 Choice of initial point

Now, fix $\beta_0$ with $|\beta_0| = 1$, and set $\alpha_0 = \alpha_0(\beta_0) \in \mathbb{R}$. If $\alpha \in \mathbb{R}$, and $|\alpha - \alpha_0|$ is sufficiently small, then $\kappa(\beta_0, \alpha)$ is real, since the preceding procedure keeps the realities.

Solutions $v_1, \cdots, v_{p-1}$ of equation $(G)$ are real, since $\kappa(\beta_0, \alpha) \in \mathbb{R}$. We choose a $p$-th root, $\varepsilon$, of $\kappa(\beta_0, \alpha)$ and a self-anti-conjugate periodic orbit as follows. Choice of $\varepsilon$ determines the choice of the initial point of the periodic orbit.

CASE I If $p$ is odd or $\kappa(\beta_0, \alpha) > 0$, then the equation $\varepsilon^p = \kappa(\beta_0, \alpha)$ has a real root. Real $\varepsilon$ gives real solutions $u_0, \cdots, u_{p-1}$ of equations $(F)$ and $(E)$. These give rise to a periodic orbit $y_n = u_0 + \nu^n u_1 + \cdots + \nu^{(p-1)n} u_{p-1}$, with real $y_0$. We see that $y_{-n} = \overline{y_n}$.

CASE II If $p$ is even and $\kappa(\beta_0.\alpha) < 0$, then take a solution of $\varepsilon^p = \kappa(\beta_0, \alpha)$ satisfying $\varepsilon = \bar{\nu}\bar{\varepsilon}$. The solutions of equation $(F)$ are as follows.

$$u_0 = y_* - \frac{\delta}{2}\varepsilon^p, \qquad u_k = \varepsilon^k v_k, \quad (k = 1, \cdots, p-1).$$

As $u_0$ and $v_1, \cdots, v_{p-1}$ are real, we have

$$y_n = \sum_{k=0}^{p-1} \nu^{kn} u_k = u_0 + \sum_{k=1}^{p-1} \nu^{kn} \varepsilon^k v_k = u_0 + \sum_{k=1}^{p-1} \nu^{k(n-1)} \bar{\varepsilon}^k v_k$$

$$= u_0 + \overline{\sum_{k=1}^{p-1} \nu^{-k(n-1)} \varepsilon^k v_k} = \overline{\sum_{k=0}^{p-1} \nu^{k(1-n)} u_k} = \overline{y_{1-n}}.$$

In both cases, the obtained periodic orbit is self-anti-conjugate. The jacobian matrix along the orbit is of the form

$$D(H_{\alpha,\beta}^{\circ p})\big|_{P_0} = \beta^p A,$$

with $\det A = 1$ and trace $A \in \mathbb{R}$.

## 4.2 Trace function

Let

$$\tau = \tau(\beta, \alpha) = \beta^{-p} \text{ trace } D(H_{\alpha,\beta}^{\circ p})\big|_{P_0}.$$

$\tau$ is an algebraic function of complex parameters $(\beta, \alpha)$. Note that $\tau$ does not depend on the choice of $\varepsilon$ among the $p$-th root of $\kappa$, since the choice of $\varepsilon$ corresponds to the choice of the initial point in the periodic orbit. $\tau$ is locally univalent and continuous near $(\beta_0, \alpha_0)$. Hence $\tau$ is holomorphic in $(\beta, \alpha)$ near $(\beta_0, \alpha_0)$. As we saw, if $|\beta| = 1$ and $\alpha \in \mathbb{R}$, then $\tau(\beta, \alpha) \in \mathbb{R}$ near $(\beta_0, \alpha_0)$, and $-2 < \tau(\beta_0, \alpha_0) < 2$.

PROPOSITION    $\tau(\beta, \alpha)$ is holomorphic in $(\beta, \alpha)$ and non-constant with respect to $\alpha$ near $(\beta_0, \alpha_0)$.

PROOF    Consider the analytic continuation of $\tau$. As $\tau$ is algebraic, continuation along the real axis of $\alpha$ exists by avoiding branch points choosing some branch. If $|\alpha|$ is sufficiently large, then all the periodic points of the Hénon map are hyperbolic, and $|\tau(\beta_0, \alpha)| > 2$ there.

Trace function can be computed as follows.

$$\text{Let} \quad B = \begin{pmatrix} 0 & \beta^{-1} \\ -\beta & 0 \end{pmatrix} \quad \text{and} \quad Y_k = \begin{pmatrix} 0 & 0 \\ 0 & 2y_k \end{pmatrix}.$$

$$\tau = \beta^{-p} \text{ trace } D(H_{\alpha,\beta}^{\circ p})\big|_{P_0}$$

$$= \text{trace} \left( \left( \begin{array}{cc} 0 & \beta^{-1} \\ -\beta & 2y_{p-1} \end{array} \right) \cdots \left( \begin{array}{cc} 0 & \beta^{-1} \\ -\beta & 2y_0 \end{array} \right) \right)$$

$$= \text{trace} \ \left( (B + Y_{p-1}) \cdots (B + Y_0) \right).$$

$$\tau = \sum_{k=0}^{[\frac{p}{2}]} (-1)^k 2^{p-2k} \left( \sum_{0 \le i_1 < i_2 < \cdots < i_{p-2k} < p(\text{with some extra condition})} y_{i_1} y_{i_2} \cdots y_{i_{p-2k}} \right).$$

Here, the extra condition can be described as follows. The complement of the indices $\{0, 1, \cdots, p-1\} \setminus \{i_1, \cdots, i_{p-2k}\}$ consists of $k$ pairs of consecutive indices $\{j_1, j_1+1, j_2, j_2+1, \cdots, j_k, j_k+1\}$. Coefficients of $\tau(\kappa)$ are rational functions of $\beta$.

## 5.1 Abundance of Siegel balls

For $(\theta, \varphi) \in (\mathbb{R}/2\pi\mathbb{Z})^2$, let

$$\beta = \cos\theta + i\sin\theta, \quad \alpha = 2\cos\theta\cos\varphi - \cos^2\varphi,$$

which defines a family of self-anti-conjugate Hénon maps

$$H_{\alpha,\beta} \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} y \\ \beta(y^2 + \alpha) - \beta^2 x \end{array} \right),$$

with $|\beta| = 1$ and $\alpha \in \mathbb{R}$.

THEOREM    For any open set $U \subset (\mathbb{R}/2\pi\mathbb{Z})^2$ and any integer $N > 1$, there exists a point $(\theta_N, \varphi_N) \in U$ such that the Hénon map $H_{\alpha,\beta}$ for this parameter has more than $N$ cycles of Siegel balls.

PROOF    Fixed points of $H_{\alpha,\beta}$ are given by

$$y_* = \cos\theta \pm (\cos\theta - \cos\varphi).$$

We choose

$$y_* = \cos\varphi$$

and set

$$\mu = \cos\varphi + i\sin\varphi.$$

Eigenvalues of jacobian matrix at the fixed point

$$DH_{\alpha,\beta} = \beta \begin{pmatrix} 0 & \beta^{-1} \\ -\beta & 2\cos\varphi \end{pmatrix}$$

are $\beta\mu$ and $\beta\bar{\mu}$. The arguements of these eigenvalues are

$$\theta + \varphi \quad \text{and} \quad \theta - \varphi \quad (\text{mod } 2\pi).$$

By Siegel's theorem (or Brjuno's theorem), there is a subset $W_1 \subset (\mathbb{R}/2\pi\mathbb{Z})^2$ of full measure, such that the fixed point of the corresponding Hénon map has a Siegel ball.

We set

$$U_1 = U, \quad p_1 = 1, \quad \text{and} \quad V_1 = U_1 \cap W_{p_1}.$$

Inductively, we assume $U_m$ is an open subset of $U$ and $V_m$ is a full measure subset of $U_m$, such that $H_{\alpha,\beta}$ for any $(\theta,\varphi) \in V_m$ has $m$ cycles of Siegel balls of periods $p_1, \cdots, p_m$.

In open set $U_m$, there is a point $(\theta,\varphi) \in U_m$, such that $\frac{1}{2\pi}(\theta + \varphi) = \frac{q}{p}$ is rational with $p > p_m$ and, $p$ and $q$ are mutually prime. Then perturb $(\theta,\varphi)$ keeping $\theta + \varphi = \frac{2\pi q}{p}$, so that $\beta = \cos\theta + i\sin\theta$ avoids the values of $\beta$ forbidden in the preceding propositions.

There is an open set of parameters containing such a parameter, such that the Hénon map $H_{\alpha,\beta}$ has a neutral cycle of period $p$, which is self-anti-conjugate with eigenvalues of the form $\beta^p \lambda$ and $\beta^p \bar{\lambda}$, with $-2 < \tau(\beta,\alpha) < 2$. The trace function $\tau(\beta,\alpha)$ is a non-trivial analytic function with respect to $\alpha$. Determinant of $H_{\alpha,\beta}^p$ is $\beta^{2p}$. Hence, the eigenvalues of the neutral $p$ cycle varies effectively.

Note that $\tau(\beta,\alpha) \in \mathbb{R}$, if $\alpha \in \mathbb{R}$, and $-2 < \tau(\beta,\alpha_0(\beta)) < 2$. This implies that there is an open subset $U_{m+1} \subset U_m$ and a full measure set $W_p \subset U_{m+1}$ of parameters, such that the Hénon map has a Siegel ball of period $p$.

Set $V_{m+1} = V_m \cap W_p \subset U_{m+1}$, and $p_{m+1} = p$. $V_{m+1}$ is a full measure subset of $U_{m+1}$. Continue this procedure until $m = N$.

$V_N$ is a set of positive measure. Hence, we can find a parameter $(\theta_N, \varphi_N) \in V_N \subset U$.