流体力学的微分の変分公式とその応用について

Variational formulas for L_s-canonical differentials and applications

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1. Modulus of the closing of (R, χ)

Let R be an open Riemann surface of genus one and $\chi = \{A, B\}$ be a fixed canonical homology basis of R modulo dividing cycles. Consider a triplet $(\hat{R}, \hat{\chi}, \hat{i})$ consisting of a (closed) torus $\hat{R}$, a canonical homology basis $\hat{\chi} = \{\hat{A}, \hat{B}\}$ of $\hat{R}$, and a conformal embedding $\hat{i}$ of R into $\hat{R}$ such that $\hat{i}(A)$ (resp. $\hat{i}(B)$) is homologous to $\hat{A}$ (resp. $\hat{B}$) in $\hat{R}$. We say that two such triplets $(\hat{R}, \hat{\chi}, \hat{i})$ and $(\hat{R}', \hat{\chi}', \hat{i}')$ are equivalent if there is a conformal mapping f of $\hat{R}$ onto $\hat{R}'$ with $f \circ \hat{i} = \hat{i}'$ on R. Each equivalence class is called a *closing* of (R, χ) and is denoted by $[\hat{R}, \hat{\chi}, \hat{i}]$. The closing $[\hat{R}, \hat{\chi}, \hat{i}]$ carries a unique holomorphic differential $\phi^{\hat{R}}$ with $\int_{\hat{A}} \phi^{\hat{R}} = 1$, which is called the *normal differential* for $(\hat{R}, \hat{\chi})$. We call

$$\tau[\hat{R}, \hat{\chi}, \hat{i}] := \int_{\hat{B}} \phi^{\hat{R}}$$

the *modulus* of $[\hat{R}, \hat{\chi}, \hat{i}]$, and remark that two different closings may have the same modulus. We denote by $\mathcal{C}(R, \chi)$ the set of closings of (R, χ) and put

$$\mathfrak{M}(R, \chi) = \{\tau = \tau[\hat{R}, \hat{\chi}, \hat{i}] \in \mathbb{C} \mid [\hat{R}, \hat{\chi}, \hat{i}] \in \mathcal{C}(R, \chi)\}.$$

The moduli set $\mathfrak{M}(R, \chi)$ obviously lies in the upper half plane $\mathbb{H}$.

2. Canonical differentials and Moduli disk $\mathfrak{M}(R, \chi)$

Let R be an open Riemann surface of genus one in a Riemann surface $\widetilde{R}$ such that $R \Subset \widetilde{R}$ and ∂R consists of C^ω smooth contours; $\partial R = C_1 + \cdots + C_\nu$. Let ϕ be a holomorphic differential on $\overline{R} = R \cup \partial R$, precisely, on a neighborhood of $\overline{R}$ in $\widetilde{R}$. We say that ϕ is a *canonical differential* on R in the sense of Kusunoki if ϕ satisfies $\int_{C_j} \phi = 0$ and $\operatorname{Im} \phi = 0$ on C_j $(j = 1, \ldots, \nu)$. In other words, for a thin tubular neighborhood V_j of C_j, the branch on V_j of the abelian integral $\Phi(z) = \int^z \phi$ is a single-valued holomorphic function on V_j such that $\operatorname{Im} \Phi(z) = const.$ on C_j.

Theorem 1 (Kusunoki [1]). *Let R be as above, and let $\chi = \{A, B\}$ be a canonical homology basis of R modulo dividing cycles. For every $-1 < s \leq 1$ there uniquely exists a holomorphic differential ϕ_s on R such that*
(i) $e^{-\frac{\pi i}{2}s}\phi_s$ is a canonical differential on R,　(ii) $\int_A \phi_s = 1$.

We call ϕ_s the L_s-*canonical differential* for (R, χ). Shiba showed that there uniquely exists a closing $[\hat{R}_s, \hat{\chi}_s, \hat{i}_s]$ of (R, χ) such that the transplant of ϕ_s extends to the normal differential $\phi^{\hat{R}_s}$ on $\hat{R}_s$, where $\hat{\chi}_s = \{\hat{A}_s, \hat{B}_s\}$. Thus, $\tau_s = \int_B \phi_s$ is equal to $\tau[\hat{R}_s, \hat{\chi}_s, \hat{i}_s] = \int_{\hat{B}_s} \phi^{\hat{R}_s}$. Therefore, for $-1 < s \leq 1$, the L_s-canonical differential ϕ_s for (R, χ) satisfies the following properties on precisely C_j:

$$\int_{C_j} \phi_s = 0 \quad \text{and} \quad \operatorname{Im}[e^{-\frac{\pi i}{2}s}\phi_s] = 0 \text{ on } C_j \quad (j = 1, \ldots, \nu).$$

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Theorem 2 (Shiba [6]). *(1) The moduli set $\mathfrak{M}(R,\chi)$ is a closed disk; there exists $\tau^* \in \mathbb{H}$ and $\rho \in \mathbb{R}$ such that $0 < \rho < \operatorname{Im}\tau^*$ and $\mathfrak{M}(R,\chi) = \{\tau \in \mathbb{H} \mid |\tau - \tau^*| \le \rho\}$.*

(2) Each boundary point of the moduli disk $\mathfrak{M}(R,\chi)$ corresponds to the single element of $\mathcal{C}(R,\chi)$ as follows: $\tau_s = \tau^* + \rho e^{(s-\frac{1}{2})\pi i}, -1 < s \le 1$.

We call $\mathfrak{M}(R,\chi)$ the *moduli disk* for (R,χ) and 2ρ the *Euclidean diameter* for R.

3. Results

Let $(\widetilde{\mathcal{R}},\pi,\Delta)$ be a holomorphic family such that $\widetilde{\mathcal{R}}$ is a two-dimensional complex manifold; $\Delta = \{t \in \mathbb{C} \mid |t| < r\}$; and π is a holomorphic projection from $\widetilde{\mathcal{R}}$ onto Δ. We assume that each fiber $\widetilde{R}(t) = \pi^{-1}(t)$, $t \in \Delta$ is non-compact, irreducible and non-singular in $\widetilde{\mathcal{R}}$, so that $\widetilde{R}(t)$ is an open Riemann surface. Let $(\mathcal{R},\pi|_{\mathcal{R}},\Delta)$ be a sub-holomorphic family of $(\widetilde{\mathcal{R}},\pi,\Delta)$ such that $\mathcal{R} \subset \widetilde{\mathcal{R}}$; $\partial\mathcal{R}$ is C^ω smooth in $\widetilde{\mathcal{R}}$ of real 3-dimensional surface; $R(t) = (\pi|_{\mathcal{R}})^{-1}(t) \Subset \widetilde{R}(t)$, $t \in \Delta$; and $R(t)$ is a bordered Riemann surface of genus one with C^ω smooth boundary $\partial R(t)$ in $\widetilde{R}(t)$: $\partial R(t) = C_1(t) + \cdots + C_\nu(t)$ (ν does not depend on $t \in \Delta$).

For $t \in \Delta$, we fix a canonical homology basis $\chi(t) = \{A(t), B(t)\}$ of $R(t)$ modulo dividing cycles $C_j(t)$ ($j = 1,\ldots,\nu$) such that $A(t)$ and $B(t)$ move continuously in $\mathcal{R}$ with t. Each $R(t)$, $t \in \Delta$ carries the L_s-canonical differential $\phi_s(t,z)$ for $(R(t),\chi(t))$, so that $\int_{A(t)} \phi_s(t,z) = 1$. We set $\tau_s(t) = \int_{B(t)} \phi_s(t,z)$. As usual we write $\phi_s(t,z) = f_s(t,z)dz$ by use of the local parameter of $\overline{R(t)}$.

Theorem 3 ([3]). *For $-1 < s \le 1$, we have*

$$\frac{\partial^2 \operatorname{Im}[e^{-\pi i s}\tau_s(t)]}{\partial t\partial\bar{t}} = -\frac{1}{2}\int_{\partial R(t)} k_2(t,z)|f_s(t,z)|^2|dz| - \left\| \frac{\partial\phi_s(t,z)}{\partial\bar{t}} \right\|^2_{R(t)}.$$

Here

$$k_2(t,z) = \left(\frac{\partial^2\varphi}{\partial t\partial\bar{t}}\left|\frac{\partial\varphi}{\partial z}\right|^2 - 2\operatorname{Re}\left\{ \frac{\partial^2\varphi}{\partial\bar{t}\partial z}\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial\bar{z}} \right\} + \left|\frac{\partial\varphi}{\partial t}\right|^2 \frac{\partial^2\varphi}{\partial z\partial\bar{z}} \right) \left|\frac{\partial\varphi}{\partial z}\right|^{-3} \tag{1}$$

on $\partial\mathcal{R}$ where $\varphi(t,z)$ is a C^2 smooth defining function of $\partial\mathcal{R}$ in $\widetilde{\mathcal{R}}$ (see [5, (1.2)]).

Corollary 4 ([3]). *Assume that $\mathcal{R}$ is a Stein manifold. Then*

(1) the Euclidean diameter $2\rho(t)$ of $\mathfrak{M}(R(t),\chi(t))$ is subharmonic on Δ;

(2) if $\rho(t)$ is harmonic on Δ, then the center $\tau^*(t)$ of $\mathfrak{M}(R(t),\chi(t))$ holomorphically moves on Δ and $\rho(t) = \rho(0)$ for every $t \in \Delta$.

If time allows, we shall discuss the relation between Theorem 3 and the following variational formulas for L_1- and L_0-principal functions:

Let $R(t)$ be a bordered Riemann surface of genus g (≥ 0) with C^ω smooth boundary $\partial R(t)$ in $\widetilde{R}(t)$. Let $\{A_l(t), B_l(t)\}_{l=1}^g$ be a canonical homology basis of $R(t)$ modulo dividing cycles $C_j(t)$ ($j = 1,\ldots,\nu$) such that $A_l(t)$ and $B_l(t)$ vary continuously with $t \in \Delta$ in $\mathcal{R}$. Assume that there exists a section $\mathbf{a} := \{a(t) \in R(t) \mid t \in \Delta\}$ of $\mathcal{R}$ over Δ. Let $\mathcal{V} := \Delta \times \{|z-\zeta| < r\}$ be a π-local coordinate of a neighborhood $\mathcal{U}$ of $\mathbf{a}$ in $\mathcal{R}$ such that $\mathbf{a}$ corresponds to $\Delta \times \{\zeta\}$. Let $t \in \Delta$ be fixed. Then among all harmonic functions $u(t,z)$ on $R(t) \setminus \{a(t)\}$ with singularity $\operatorname{Re}\frac{1}{z-\zeta}$ at $a(t)$ normalized so that $\lim_{z\to\zeta}(u(t,z)-\mathrm{Re}\,\frac{1}{z-\zeta})=0$, we have two uniquely determined functions $p_i(t,z)$ $(i=1,0)$ with the following boundary conditions (L_i): for $j=1,\dots,\nu$,

$$(L_1)\quad p_1(t,z)=c_j(t)\ \text{(constant) on } C_j(t)\quad\text{and}\quad \int_{C_j(t)}\frac{\partial p_1(t,z)}{\partial n_z}ds_z=0;$$

$$(L_0)\quad \frac{\partial p_0(t,z)}{\partial n_z}=0\quad \text{on } C_j(t).$$

Then $p_i(t,z)=\mathrm{Re}\left\{\frac{1}{z-\zeta}+\sum_{n=1}^{\infty}A_n^i(t)(z-\zeta)^n\right\}$ at ζ $(i=1,0)$. Each $R(t)$, $t\in\Delta$ carries the L_i-*principal function* $p_i(t,z)$ and the L_i-*constant* $\alpha_i(t):=\mathrm{Re}\,\{A_1^i(t)\}$ for $(R(t),a(t))$.

Lemma 5 ([2]).

$$\frac{\partial^2\alpha_1(t)}{\partial t\partial\bar t}=-\left(\frac{1}{\pi}\int_{\partial R(t)}k_2(t,z)\left|\frac{\partial p_1(t,z)}{\partial z}\right|^2 ds_z+\frac{4}{\pi}\iint_{R(t)}\left|\frac{\partial^2 p_1(t,z)}{\partial\bar t\partial z}\right|^2 dxdy\right). \tag{2}$$

$$\begin{aligned}\frac{\partial^2\alpha_0(t)}{\partial t\partial\bar t}&=\frac{1}{\pi}\int_{\partial R(t)}k_2(t,z)\left|\frac{\partial p_0(t,z)}{\partial z}\right|^2 ds_z+\frac{4}{\pi}\iint_{R(t)}\left|\frac{\partial^2 p_0(t,z)}{\partial\bar t\partial z}\right|^2 dxdy\\&\quad-\frac{2}{\pi}\mathrm{Im}\left\{\sum_{l=1}^{g}\frac{\partial}{\partial t}\left(\int_{A_l(t)}*dp_0(t,z)\right)\frac{\partial}{\partial\bar t}\left(\int_{B_l(t)}*dp_0(t,z)\right)\right\}.\end{aligned} \tag{3}$$

Here $k_2(t,z)$ is the same as (1).

If $\mathcal{R}$ is pseudoconvex and each $R(t)$ is **planar**, the contrast between the superharmonicity of $\alpha_1(t)$ and the subharmonicity of $\alpha_0(t)$ is unified with the Schiffer span $s(t):=\alpha_0(t)-\alpha_1(t)$. In [2] we applied the *logarithmically* subharmonicity of $s(t)$ to show the simultaneous uniformization of moving *planar* Riemann surfaces of class O_{AD}.

References

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