Thom class in complex analytic geometry

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Čech の方法で局所化された de Rham, Dolbeault, Bott-Chern コホモロジー各々に Thom 類を定め, Riemann-Roch 等の複素解析幾何の諸問題との関わりについて話したい.

以下に概観と Thom 類に関する予備的事項を記す.

1 Overview

	I	II	III
Cohomology	de Rham	Dolbeault	Bott-Chern
Manifold M	C^∞ oriented/complex	complex	complex
Subset S in M	subcomplex	subvariety	subvariety
Vector bundle E	real oriented/complex	holomorphic	holomorphic
Connection ∇	C^∞	of type $(1,0)$	Hermitian & $(1,0)$
Characteristic class	Euler/Chern	Atiyah	Bott-Chern
Core of Thom class	angular/BM form	Bochner-Martinelli	“BM potential”

2 Thom class

In the following the homology is that of locally finite chains (Borel-Moore homology) and the cohomology is that of cochains on finite chains, both with coefficients in $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{C}$.

We list [15] as a general reference for the Thom isomorphism and the Thom class of a real vector bundle. In general they are defined in cohomology with $\mathbb{Z}_2$ coefficients, while for an oriented vector bundle, they can be defined in cohomology with $\mathbb{Z}$ coefficients. They can also be described in terms of differential forms, in which case the cohomology involved is with $\mathbb{C}$ coefficients. This is done in [9] using cohomology with compact support in the vertical direction. Here we use Čech-de Rham cohomology instead as in [16]. This way we can express relevant local informations more explicitly.

The Poincaré, Alexander and Thom isomorphisms can be defined from combinatorial viewpoint in homology and cohomology with $\mathbb{Z}$ coefficient (cf. [10], also [18, 20]). They can also be defined in terms of Čech-de Rham cohomology (cf. [16, 18, 20]).

In the sequel, we denote by M an oriented C^∞ manifold of dimension m.

A. Thom class of a submanifold : Let S be a closed oriented submanifold of M of dimension d'. Set $k' = m - d'$. We have the following commutative diagram :

$$\begin{array}{ccc} H^p(S) & \xrightarrow[T]{\sim} & H^{p+k'}(M, M \smallsetminus S) \\ \wr\downarrow P & \swarrow_{A}^{\sim} & \\ H_{d'-p}(S), & & \end{array} \tag{2.1}$$

where P, A and T denotes the Poincaré, Alexander and Thom isomorphisms.

Definition 2.2 The *Thom class* Ψ_S of S is defined so that it corresponds to the other classes by :

$$\begin{array}{ccc} [1] \in H^0(S) & \xrightarrow[T]{\sim} & H^{k'}(M, M \smallsetminus S) \ni \Psi_S \\ \wr\downarrow P & \swarrow_{A}^{\sim} & \\ [S] \in H_{d'}(S). & & \end{array}$$

B. Thom class of an oriented real vector bundle : Let $\pi : E \to M$ be an oriented real vector bundle of rank k'. We denote by Z the image of the zero section, which is diffeomorphic with M. We replace M and S in the case A above by E and Z, respectively.

Definition 2.3 The *Thom class* Ψ_E of E is defined by $\Psi_E = \Psi_Z$ so that there are correspondences :

$$\begin{array}{ccc} [1] \in H^0(Z) & \xrightarrow[T]{\sim} & H^{k'}(E, E \smallsetminus Z) \ni \Psi_E \\ \wr\downarrow P & \swarrow_{A}^{\sim} & \\ [Z] \in H_m(Z). & & \end{array}$$

Let S be as in the case A above and N_S the normal bundle of S in M. By the tubular neighborhood theorem, we have an isomorphism

$$\Psi_S \in H^{k'}(M, M \smallsetminus S) \simeq H^{k'}(N_S, N_S \smallsetminus Z) \ni \Psi_{N_S}$$

and in the above isomorphism, Ψ_S corresponds to Ψ_{N_S}.

In the case of complex vector bundles, we have :

Theorem 2.4 ([16]) *For a complex vector bundle $\pi : E \to M$ of rank k,*

$$\Psi_E = c^k(\pi^* E, s_\Delta) \qquad \text{in} \quad H^{2k}(E, E \smallsetminus Z),$$

where $c^k(\pi^* E, s_\Delta)$ denotes the localization of the top Chern class $c^k(\pi^* E)$ of the pull-back bundle $\pi^* E$ by the diagonal section s_Δ.

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