Workshop on Arithmetic Geometry in Kanazawa (Nov, 2009)

On the history of the Sato-Tate conjecture

Tetsushi Ito (Kyoto Univ, Math Dept)

November 24(Tue), 10:50-12:00

Ishikawa prefecture bunkyo hall, Kanazawa, Japan

This talk

- No serious mathematics
- Just relax, smile, have tea & snacks

Are you ready for this? Go ahead!

Welcome to Japan,

and
welcome to Kanazawa!

By the way,

Where is Kanazawa?


Here!!

Kanazawa

- is the biggest city in the Hokuriku region,
- has a population of 450,000,
- is a castletown ruled over by the Maeda family for three centuries after the first lord Toshiie Maeda.
- In the Edo Era (1603-1868), Maeda family was the biggest feudal lord (daimyo) except for the Tokugawa Shogunate. “Kaga Hyakuman-goku” (cf. Banquet : Kanazawa Daimyo Jaya)

Who is Maeda?

Toshiie Maeda (1537-99, first lord)

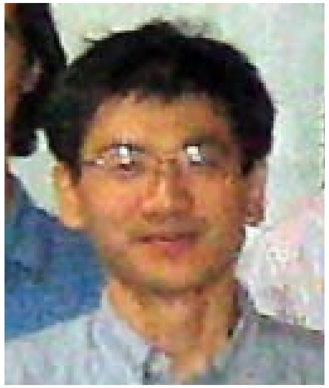




And then,

Why Math in Kanazawa?

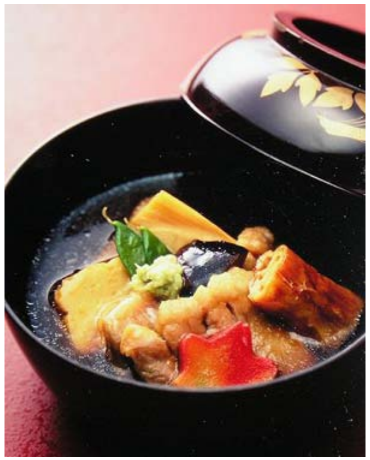
Takeshi Saito was born in Kanazawa.



Necessary conditions for Good Math

- Historical and beautiful city. “small Kyoto”

  Kenrokuen is one of Japan’s “three most beautiful landscape gardens”
- Delicious seafoods and local speciality
- At least 4 Starbucks in the city



First of all

Thanks to all the speakers, participants, especially Teruyoshi and Kai-Wen for helping me a lot.

Five-year program (2009-2012)

Supp by JSPS Grant-in-Aid for Young Scientists (S)

“Comprehensive studies on Shimura varieties, arithmetic geometry, Galois representations, and automorphic representations”

Kyoto (Nov 2008), Ehime (Feb 2009), Kyoto (Apr 2009), Kesennuma (July 2009), Kanazawa (Nov 2009)...

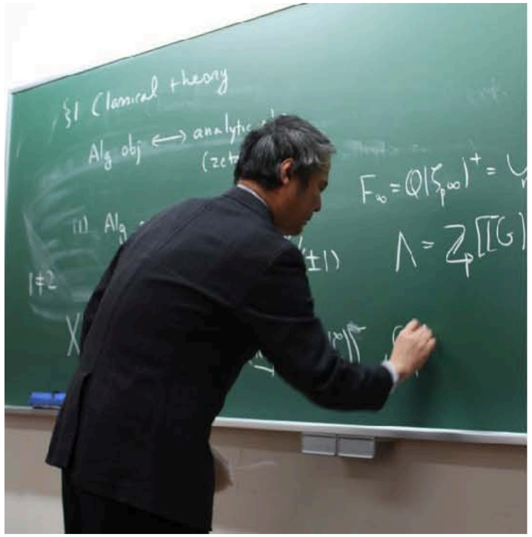
Kyoto (Nov 2008)

Kyoto (Nov 2008) continued

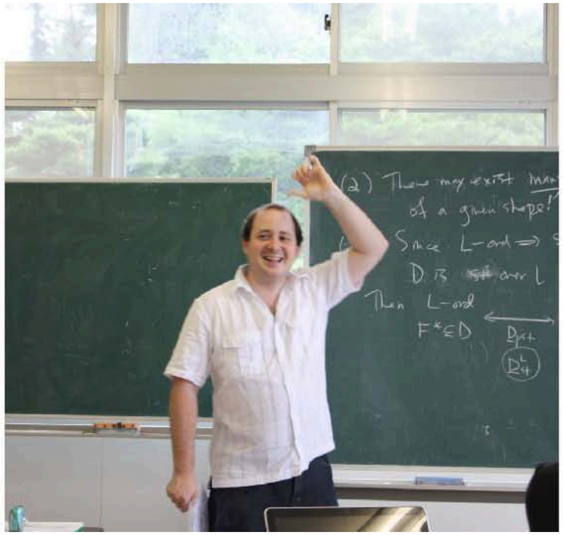
Ehime (Feb 2009)


Kyoto (Apr 2009)

Mini-workshop on Iwasawa theory



Kesennuma (July 2009)



Bad news...

Unfortunate effect of economic crisis and change of administration party

- **JSPS called off application to "Young Scientists (S)" after next year (Oct 16).**
- **The Japanese government decided to reduce the scientific research funds (Kakenhi) (Nov 13).**
- **Situation of our grant after next year is unclear...**

Good news!

Finally, I can gather excellent young researchers in arithmetic geometry and related area.

Thank you very much for coming. I'm very looking forward to attending the lectures!

Plan of this talk

1. General introduction — finished
2. History of "Sato's conjecture"

Tea & Coffee Break (10min)

4. Birth of "Sato's conjecture"
5. Outline of the proof of Sato-Tate conj
   (oversimplified exposition)

I'm sorry...

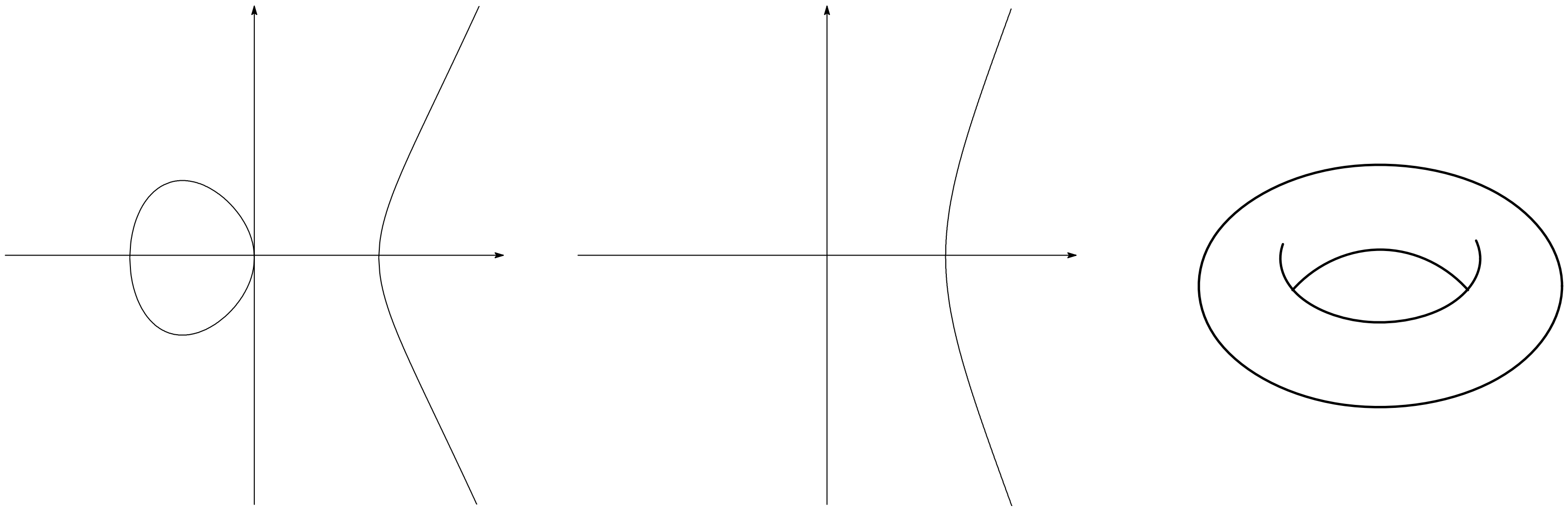
Big overlap between this talk and my talks at

- RIMS Workshop (Dec 2006)
- Workshop on the Sato-Tate conjecture
  (Jan 2007, Tokyo Inst Tech)
- Ehime workshop (Feb 2009)
- Annual meeting of the Math Soc of Japan
  (Mar 2009) ($\exists$ ashamed video streaming)

History of the Sato-Tate conjecture

Recall: An elliptic curve E over $\mathbb{Q}$ is defined by a cubic equation of the form:

$$E : y^2 = x^3 + ax + b \quad (a, b \in \mathbb{Z},\ 4a^3 + 27b^2 \neq 0)$$



We would like to study the arithmetic of E.

$\rightarrow$ **Key : Reciprocity Law**

The Problem of Reciprocity Law

Problem: For a scheme X of finite type over Spec $\mathbb{Z}$, count $\#X(\mathbb{F}_p)$ and vary p.

Very difficult problem even if X is finite over Spec $\mathbb{Z}$. The answer for

$$X = \text{Spec } \mathbb{Z}[T]/(T^2 - a)$$

is given by the Law of Quadratic Reciprocity. Striking fact is that the answer depends only on $p \pmod{4a}$.

Class Field Theory

Class Field Theory (Takagi, Artin, 1920's) tells us the answer depends only on $p \pmod{N}$ (for some N) if and only if $X \otimes_{\mathbb{Z}} \mathbb{Q}$ is a "abelian extension" of $\mathbb{Q}$.

"Sato-Tate Conj"

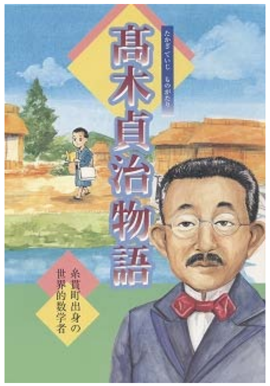
= "Non-abelian Reciprocity Law for elliptic curves"

Class Field Theory (2)

A surprising discovery of Takagi is Class Field Theory exists for all number fields K. It was expected only for $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{-D})$ (Kronecker's Jugendtraum) or unramified extensions of K (Hilbert Class Field). The fact

"All abel ext of all number fields are class fields"

was not even expected before.



Birth of "Langlands' Functoriality"

(cf. R. P. Langlands (1936–))

The proof of CFT was complete by Takagi after considering CFT of all number fields simultaneously.

"Langlands' Functoriality" = relation between CFT's
for different groups/number fields

Inevitable tool to understand many thms & phenomena in number theory & rep theory

(Non-abel CFT(= Gal v.s. Autom), BC/AI, Endoscopy, Sato-Tate, Formal deg conj (cf. Ichino's talk))

Prove thms for all K simultaneously

The proof of CFT (for K) doesn't seem to work only with abel ext of K.

- We need to consider CFT for <u>all L</u> ($[L : K] < \infty$),
- We also need non-abel ext L/K at some point.

(unless we have "Kronecker-Weber thm")

Proof of CFT was buried in the forest of many Functorialities. (eg. Proof of LLC for $GL(n)$ (Harris-Taylor, Henniart), "Potential Automorphy")

Role of "Ramification"

— seems ubiquitous in number theory

The proof of unramified CFT doesn't seem to work only with unramified extensions.

- Iwasawa Theory, Exp Rec Law, (φ, Γ)-modules

  — Very ramified extensions simplify the problem

- Euler systems, Taylor-Wiles machinery

  — role of "slightly ramified auxiliary ext"

  $q_1, \ldots, q_r \equiv 1 \pmod{\ell^N}$

- Khare-Wintenberger's proof of Serre's conj

  "Killing ramification"

Even now, CFT tells us many things

[1] Taylor, “Reciprocity laws and density theorems”, Shaw Prize Lecture (2007).

[2] Yoshida, “Enigma of Teiji Takagi and Class Field Theory (高木貞治と類体論の謎)”, Gendai Shiso (現代思想, Modern Thought), Dec 2009.
(to appear in bookstores on Nov 27(Fri)).

Let's go back to the Sato–Tate conj.

Sato-Tate Conj (Mikio Sato, Mar 1963)

$E : y^2 = x^3 + ax + b \ (a, b \in \mathbb{Z}, 4a^3 + 27b^2 \neq 0)$

Elliptic Curve without Complex Multiplication

Define $\theta_p \ (0 \leqq \theta_p \leqq \pi)$ **by**

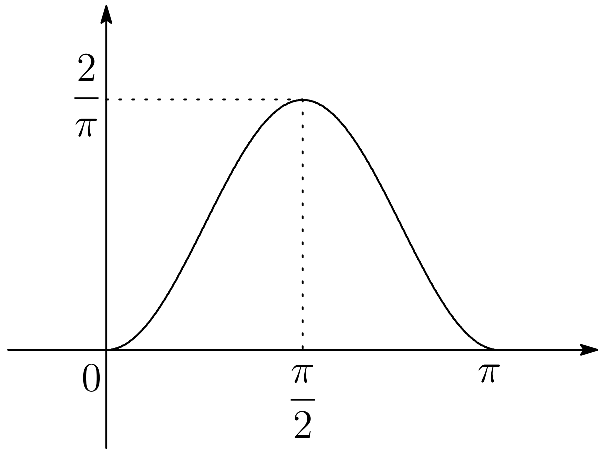
$$p + 1 - \#E(\mathbb{F}_p) = 2\sqrt{p}\cos\theta_p.$$

Then, for $0 \leqq \alpha < \beta \leqq \pi$,

$$\lim_{N\to\infty} \frac{\#\{p \mid p\text{:prime},\ p \leqq N,\ \alpha \leqq \theta_p \leqq \beta\}}{\#\{p \mid p\text{:prime},\ p \leqq N\}} = \frac{2}{\pi}\int_\alpha^\beta \sin^2\theta\, d\theta$$

The distribution of $\{\theta_p\}_p$

is the graph of $y = \frac{2}{\pi}\sin^2\theta$



Brief History

1920's Class Field Theory (Takagi, Artin)

1933 Hasse's thm $\left|p+1-\#E(\mathbb{F}_p)\right| \leq 2\sqrt{p}$

1955 "Taniyama's conj" $L(s,E) = L(s,f)$

1957 Comp syst of ℓ-adic Gal rep (Taniyama)

1963 "Sato's conj", "Tate's conj"

1968 Serre's book "Abel ℓ-adic Rep and Ell Curves",

Symmetric power L-functions $L(s, E, \mathrm{Sym}^n)$

1970– Automorphic Rep/L-Fct,

Langlands' principle of functoriality,

Langlands conj (Non-abelian CFT)

Weil conj/Ramanujan conj

Brief History (continued)

1980– Development of Shimura var (LNM 900)

Iwasawa Main Conj

Deform of Galois rep, p-adic Hodge theory

1990– Shimura-Taniyama conj (semistable case),

($\Rightarrow$ Fermat's Last Thm)

LLC for $GL(n)$

2001– Full proof of Shimura-Taniyama conj

p-adic LLC for $GL(2, \mathbb{Q}_p)$ & (φ, Γ)-mod

Framed deformation ($\rightarrow$ number theory is reduced

to geometry of Gal deform sp (nilp orbits).)

and...

April 2006

Clozel, Harris, Shepherd-Barron, Taylor announced

a proof of the Sato-Tate conj for elliptic curves

$$E : y^2 = x^3 + ax + b \quad (a, b \in \mathbb{Z}, 4a^3 + 27b^2 \neq 0)$$

when $j(E) = \frac{1728 \cdot 4a^3}{4a^3+27b^2}$ is not an integer.

They necessarily proved similar results for elliptic curves

over <u>all totally real fields simultaneously</u>

(whose j-invariants are not algebraic integers).




and...

July 2009

Barnet-Lamb, Geraghty, Harris, Taylor announced a proof of the Sato-Tate conj for

- all non-CM elliptic curves over tot real fields, and
- all non-CM holom elliptic modular forms ($/\mathbb{Q}$).

Key to the proof, among others

- Fund Lemma (Laumon, Ngô, Waldspurger)
- Geom of Shimura var and constr of Gal rep (Mantovan, Shin, "Book Project" (Harris))
- Modularity Lifting Thm over Hida families (G)
- Cohomology of Calabi-Yau families (Katz, B-L)

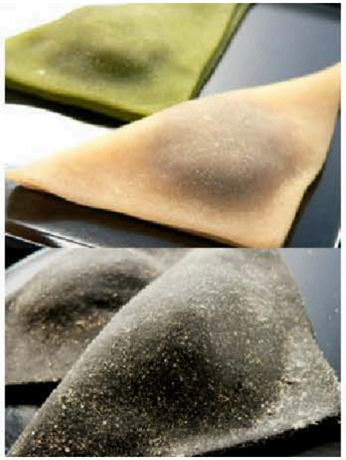
and...

Very very Recently (October 2009),

Barnet-Lamb, Gee, Geraghty obtained a proof of the Sato-Tate conj for all non-CM Hilbert modular forms on $GL(2)$ over totally real fields.

Tea & Coffee Break

10min

Birth of “Sato’s conjecture”

1963 (March) Sato made the conj based on computer experiments. (Sato received Ph.D from Univ of Tokyo in 1963, then moved to Osaka.)

1963-68 Tate and Serre gave theoretical explanation using L-functions (“Tate’s conj $\Rightarrow$ Sato’s conj”)

Computer experiments by Kanji Namba (logician).

Ref. Namba, “Dedekind η functions and Sato’s $\sin^2$-conjecture” (in Japanese), 16th Symposium on the history of mathematics, Tsuda Univ, 2006.

Fig 1.

第二表

									) 0	0.5
0°- 5°	0	0	0	0	0	0	0	0		
5°- 10°	0	0	0	0	1	0	0	1		
10°- 15°	1	0	2	0	1	0	2	6	) 7	6.0
15°- 20°	0	2	2	1	1	1	0	7		
20°- 25°	2	1	2	3	2	0	0	10	) 17	21.8
25°- 30°	3	3	5	2	4	2	0	19		
									) 44	46.0
30°- 35°	5	5	4	1	4	2	4	25		
35°- 40°	9	3	8	6	4	3	6	39	) 77	75.8
40°- 45°	3	9	3	6	5	8	4	[illegible]		
45°- 50°	10	3	6	8	11	14	6	58	) [illegible]	107.5
50°- 55°	11	8	6	7	10	3	7	52		
55°- 60°	5	12	11	10	10	13	[illegible]	70		
									) 142	137.3
60°- 65°	11	16	10	10	11	11	3	72		
65°- 70°	11	9	15	10	11	13	8	77	) 162	161.5
70°- 75°	15	10	14	16	14	10	6	85		
75°- 80°	16	15	16	17	11	9	5	89	) 180	177.4
80°- 85°	16	15	16	7	15	12	10	91		
85°- 90°	12	21.5	8.5	22.5	16.5	18.5	8	107.5		
									) 192	182.4
90°- 95°	12	9.5	13.5	16.5	12.5	14.5	6	84.5		
95°-100°	13	8	13	12	12	16	6	80	) 163	177.4
100°-105°	7	10	16	9	14	18	9	83		
105°-110°	17	12	8	17	7	10	8	79	) 167	161.5
110°-115°	16	14	15	9	15	8	11	88		
115°-120°	12	11	15	11	3	16	8	76		
									) 142	137.3
120°-125°	7	10	8	9	15	10	7	66		
125°-130°	8	11	9	12	7	13	3	63	) 111	107.5
130°-135°	7	8	3	12	9	7	2	48		
135°-140°	8	7	4	6	9	3	5	42	) 70	75.8
140°-145°	5	6	4	2	6	3	2	28		
145°-150°	3	1	6	3	6	5	1	25		
									) 43	46.0
150°-155°	3	4	3	3	0	2	3	18		
155°-160°	1	3	2	0	1	3	1	11	) 19	21.8
160°-165°	1	1	1	2	1	2	0	8		
165°-170°	0	1	1	0	1	0	0	3	) 4	6.0
170°-175°	0	1	0	0	0	0	0	1		
175°-180°	0	0	0	0	0	0	0	0	) 0	0.5
TOTAL	250	250	250	250	250	250	150	1650		

（$\eta(\tau)^2\eta(5\tau)^2$ の q 展開の係数 a_p についての $\alpha_p = \frac{1}{2}(a_p \pm \sqrt{a_p^2 - 4p})$ の角分布。—○—○— は 10°間隔の度数分布，—×—×— は 5°間隔のそれ。）

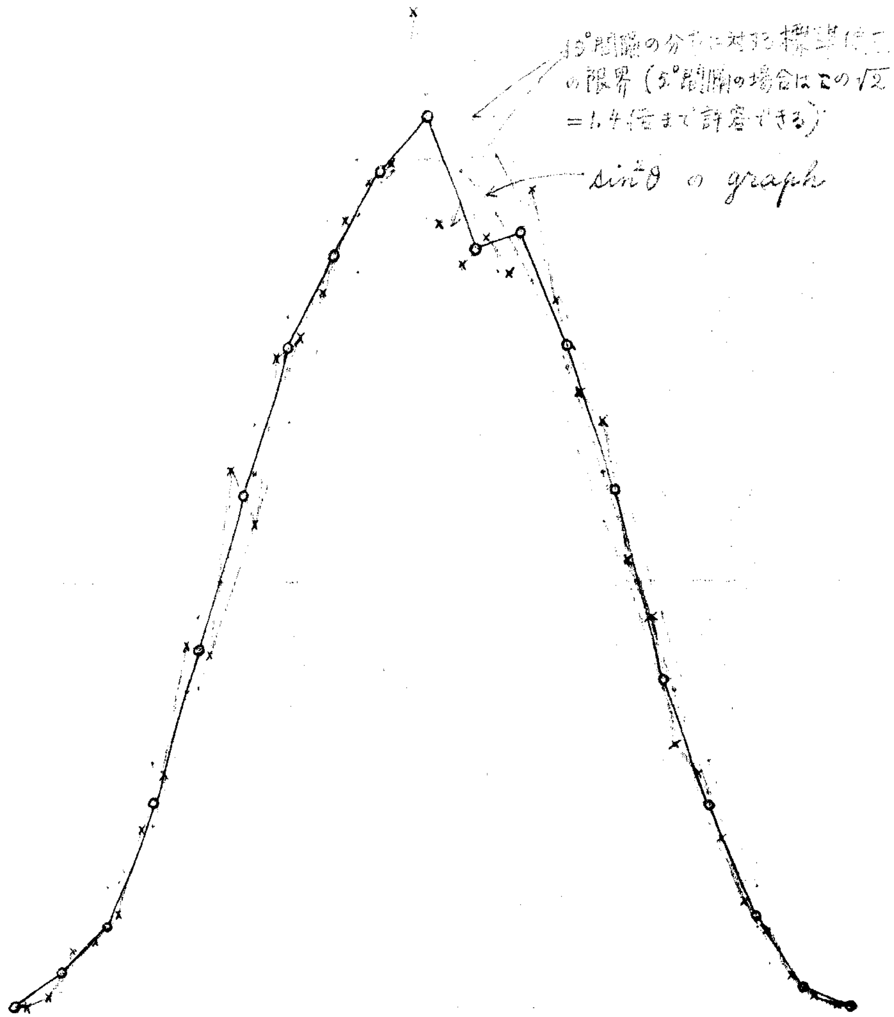
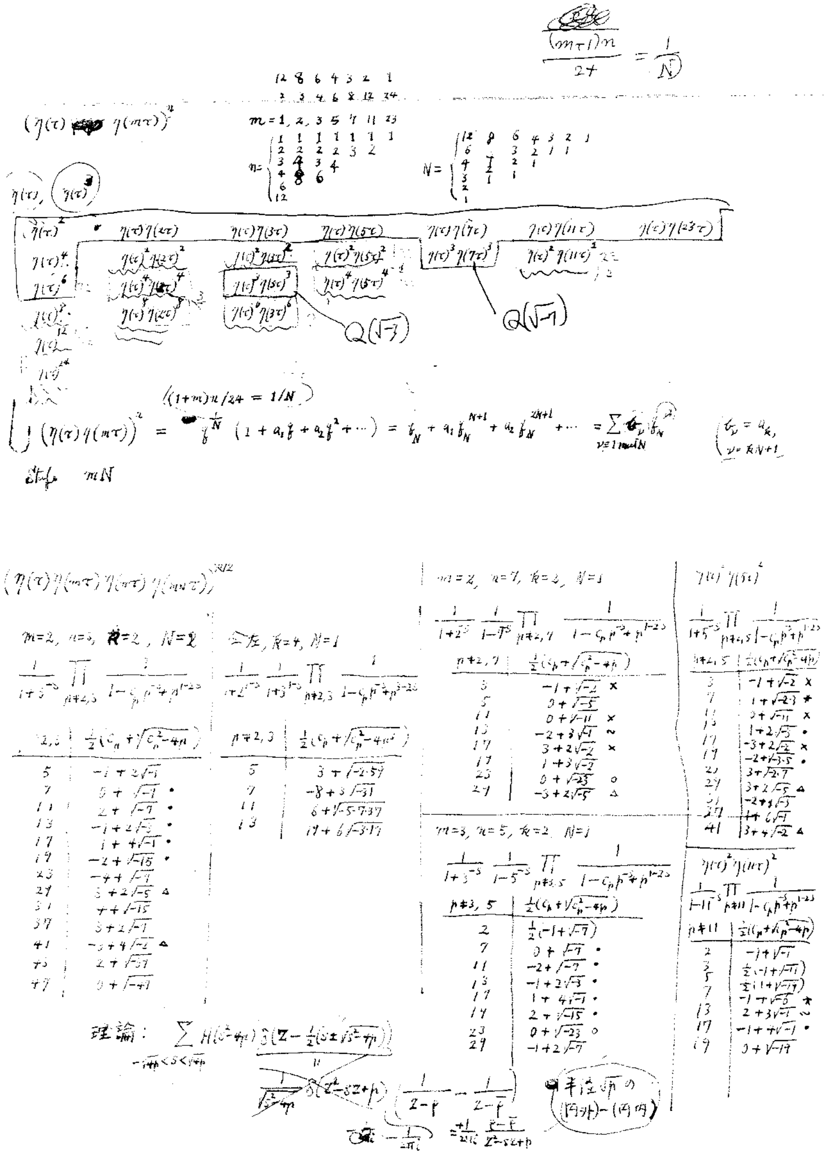
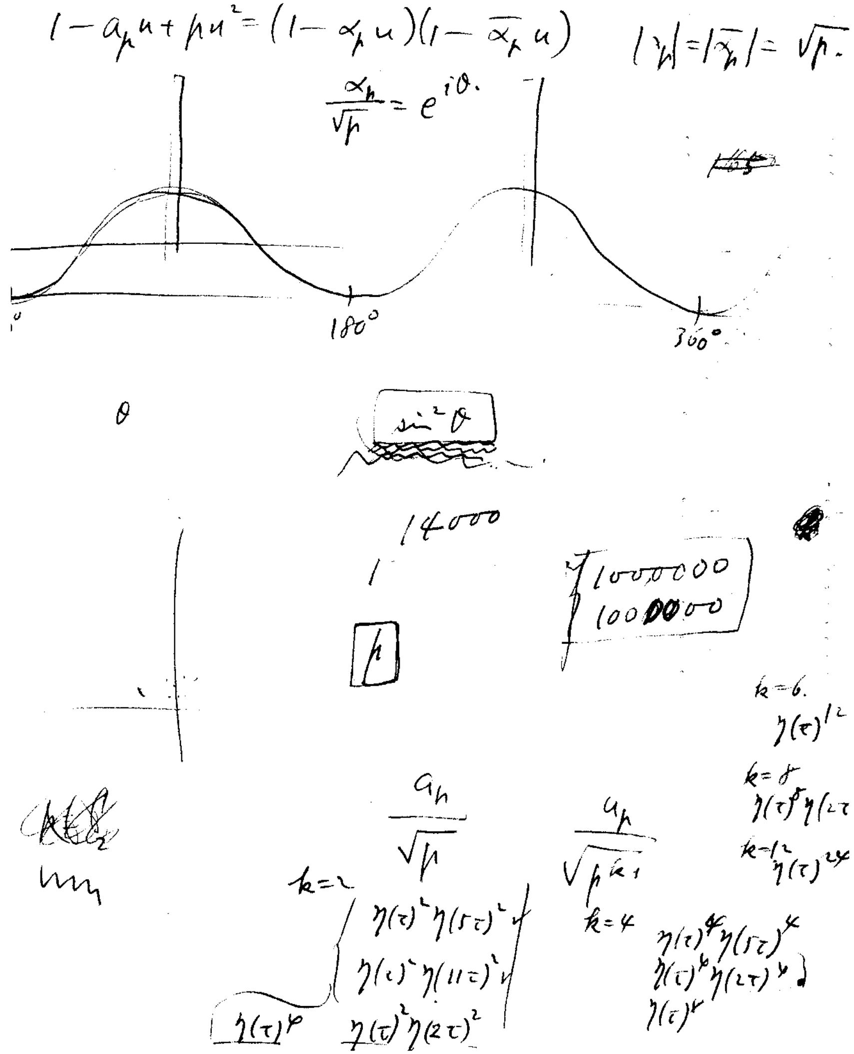


Fig 2.



Q(√−7)
Q(√−1)



1 − a_p u + p u² = (1 − α_p u)(1 − ᾱ_p u)
|α_p| = |ᾱ_p| = √p
α_p / √p = e^{iθ}
180°
360°
θ
14000
1000000
k=6.

Fig 3. (Letter from Sato to Namba, May 13, 1963)

難波 完二 様　　　　5月13日

その后も元気のことと思います。　こちらも、大阪へ移って ひと月あまり経ち だいぶ当地にも慣れて来ました。大学の研究仲間についても、いま住んでいる下宿の環境についても、申分ないと思いますが、研究以外の雑用がむやみと多いのには些か消耗しています。大学での勤務時間から言っても実働量から言っても、教育大の先生達の確かに2倍は大学へ service していると思います。

それはさておき、同封するのは3月に君が計算した例の楕円母数型式

$$\eta(\tau)^2\eta(5\tau)^2 = \sum_{l\equiv 1 (\mathrm{mod}\ 2)} a_n q^{l}$$

の係数 a_p $(p \neq 2, 5;\ p < 14000)$

について、$1 - a_p u + p u^2 = (1-\alpha_p u)(1-\overline{\alpha_p} u)$ と因数分解したときの

$$\alpha_p = \frac{1}{2}\left(a_p + \sqrt{a_p^2 - 4p}\right) = \sqrt{p}\cdot e^{i\theta_p} \quad (0<\theta_p<180^\circ)$$

の偏角 θ_p の分布を、永島君に実行して貰った結果に、少し整理を加えたものです。

第一表 i, ii. は 永島君に作ってもらった data のコピー。

第二表は、第1列が $p=3, 7, 11, \cdots$ 以下250番目まで。第2列が $p=1607$(251番目)から500番目まで、最右の列が $p=12583$(1501番目)から13777(1650番目)まで。

Total の欄の右側は、角度を10°区切りにあらくした場合の度数分布。

一番右は、分布が $\sin^2\theta$ に比例するものと仮定して、各々の section における期待値を

$$\int_{-5^\circ}^{+5^\circ}\sin^2\theta\, d\theta,\quad \int_{5^\circ}^{15^\circ}\sin^2\theta\, d\theta,\quad \cdots,\quad \int_{175^\circ}^{185^\circ}\sin^2\theta\, d\theta$$

に比例するように計算したもの。

グラフの方も、見れば意味は解るでしょう。この場合、実際の度数分布を「確率変数」と見るとき、期待値からのズレの大きさを示す標準偏差を点線で示しておきました。(これは、正しくは二項分布を使って算出すべきだが、近似的に Poisson 分布で代用できる。そうすると、図中の平均値を $N\ (= \frac{1650}{18} = 91.7)$ とすると、期待値の曲線は $2N\cdot\sin^2\theta$、標準偏差は $\sqrt{2N\cdot\sin^2\theta} = \sqrt{2N}\cdot\sin\theta$ 従って点線の曲線は $2N\cdot\sin^2\theta \pm \sqrt{2N}\cdot\sin\theta$ となる)

図と表とから推定される様に、

α_p の角分布が $\sin^2\theta$ に比例する。

と言う仮説は極めて確からしいと言えます。このことは、落着いて充分時間をかけて考えれば、現在の我々の能力でもたぶん理論的に説明できるだろうと思うのですが、いま差当っては そのような theory な頭脳労働は気が重いので、それはしばらく後廻しにして、もう少しいろいろと、筋肉労働で実行できる資料収集をしたいと考えています。そのために、阪大の計算機も利用出来るなら利用したいと思って少しそちらの方にさぐりを入れている所です。土井君はじめ、阪大の若い人達もこのプランには相当乗気でいます。

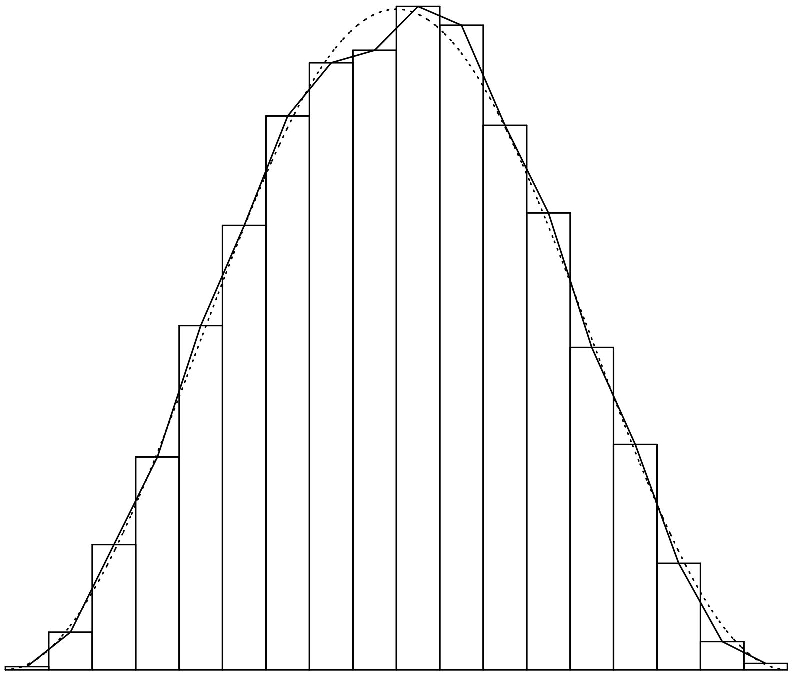
「数理科学」に載せるための原稿もそのうち書くつもりです。書き上げたら、あと、計算のプログラムに関することを君に書き足してもらおうと思います。いずれにせよ、なるべく学会のときまでには或程度目鼻をつけたいと思います。

$\eta(\tau)^2\eta(5\tau)^2$ 以外の分、つまり $\eta(\tau)^2\eta(11\tau)^2$, $\eta(\tau)^4\eta(5\tau)^4$, $\eta(\tau)^4\eta(2\tau)^4$, $\eta(\tau)^4\eta(4\tau)^4$, $\eta(\tau)^{12}$, $\eta(\tau)^{24}$ など、についても、永島君の Program を使って、θ_p の分布表を作ってくれませんか。僕の予想では、これらはすべて上と同じ $\sin^2\theta$ 則に従うだろうと、期待しているのですが。

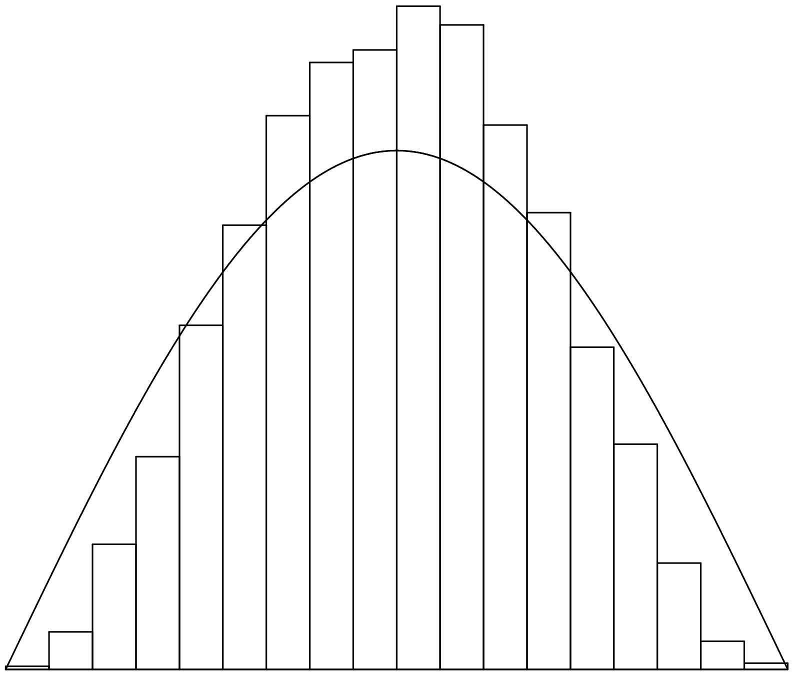
$p \gg 14000$ なる p について、全然別の方法で a_p を求められることを、[illegible]に ちょっと会ったとき話しましたが、あれは memory 容量が(命令の分以外は)殆んど不要なので、試作中の HITAC 5020 を利用させて貰うことが出来れば、おあつらえ向きゆえ、又そのうちに出来れば plan を準備したいと思っています。

Fig 4. (Calculation for $X_0(11)$, 1900 primes except 11)

$0^\circ \leq \theta < 10^\circ$	1	$60^\circ \leq \theta < 70^\circ$	177	$120^\circ \leq \theta < 130^\circ$	146
$10^\circ \leq \theta < 20^\circ$	12	$70^\circ \leq \theta < 80^\circ$	194	$130^\circ \leq \theta < 140^\circ$	103
$20^\circ \leq \theta < 30^\circ$	40	$80^\circ \leq \theta < 90^\circ$	198	$140^\circ \leq \theta < 150^\circ$	72
$30^\circ \leq \theta < 40^\circ$	68	$90^\circ \leq \theta < 100^\circ$	212	$150^\circ \leq \theta < 160^\circ$	34
$40^\circ \leq \theta < 50^\circ$	110	$100^\circ \leq \theta < 110^\circ$	206	$160^\circ \leq \theta < 170^\circ$	9
$50^\circ \leq \theta < 60^\circ$	142	$110^\circ \leq \theta < 120^\circ$	174	$170^\circ \leq \theta \leq 180^\circ$	2

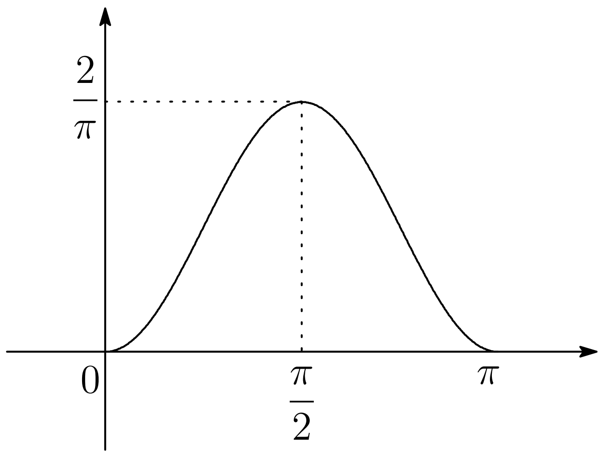


v.s. $\sin^2\theta$



v.s. $\sin\theta$

Fig 5. (Kato's comment)



looks like

Ref.

- Tate, "Algebraic Cohomology Classes", Woodshole (July 1964). (Published as "Algebraic cycles and poles of zeta functions")
- Serre, "Abelian ℓ-adic Representations and Elliptic Curves", Benjamin 1968.
  He gave theoretical explanation in terms of Gal rep.
  "Sato-Tate" seemed to appear for the first time.

Tate wrote

Assuming $f(t) = f(\pi - t)$ we conclude that $c_\nu = 0$ for ν odd, and consequently

$$f(t) = \frac{1}{\pi}(1 - \cos 2t) = \frac{2}{\pi}\sin^2 t$$

I understand that M. Sato has found this $\sin^2$ distribution law experimentally with machine computations. Conjecture 2 seems to offer an explanation for it!

Conj 2 = Tate conj (on poles of Hasse-Weil zeta)

Unfortunate misunderstanding

Sato's contribution was sometimes overlooked in number theory.

e.g. Wrong explanation seemed to appear in

> Birch, "How the number of points of an elliptic curve over a fixed prime field varies", JLMS 43 (1968)

Possible reason : Serre was not aware of "which Sato" made such a striking conjecture.

Q. How many Sato's are there?

A. So many

Sato families

In fact, Sato is the most famous family name in Japan.

1. Sato : 4.7 million families
2. Suzuki : 4.2 million
3. Takahashi : 3.5 million
4. Tanaka : 3.2 million
5. Watanabe : 2.68 million
6. Ito : 2.65 million

   ...

17. Saito : 1.47 million

   ...

1947. Hida : $\exists$ 1953 Hida families

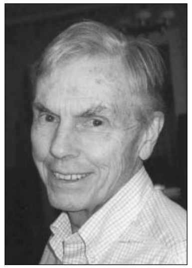
Sato and Tate

Mikio Sato and John Tate received the Wolf prize.

Sato and Tate Receive 2002-2003 Wolf Prize

The 2002–2003 Wolf Prize in Mathematics has been awarded to Mikio Sato, of the Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan; and to John T. Tate, Department of Mathematics, University of Texas, Austin. Sato was honored "for his creation of 'algebraic analysis', including hyperfunction and microfunction theory, holonomic quantum field theory, and a unified theory of soliton equations." Tate was honored "for his creation of fundamental concepts in algebraic number theory." The two share the $100,000 prize.

Mikio Sato



John T. Tate

Mikio Sato

Mikio Sato's vision of "algebraic analysis" and mathematical physics initiated several fundamental branches of mathematics. He created the theory of hyperfunctions and invented microlocal analysis, which allowed for a description of the structure of singularities of (hyper)functions on cotangent bundles. Hyperfunctions, together with integral Fourier operators, have become a major tool in linear partial differential equations. Along with his students, Sato developed holonomic quantum field theory, providing a far-reaching extension of the mathematical formalism underlying the two-dimensional Ising model, and introduced along the way the famous tau functions. Sato provided a unified geometric description of soliton equations in the context of tau functions and infinite-dimensional Grassmann manifolds. This was extended by his followers to other classes of equations, including self-dual Yang-Mills and Einstein equations. Sato has generously shared his ideas with young mathematicians and has created a flourishing school of algebraic analysis in Japan.

Mikio Sato was born in 1928 in Tokyo. He received his B.Sc. (1952) and his Ph.D. (1963) from the University of Tokyo. He was a professor at Osaka University and at the University of Tokyo before moving to the Research Institute for Mathematical Sciences at Kyoto University in 1970. He served as director of that institute from 1987 to 1991. He is now a professor emeritus at Kyoto University. He received the Asahi Prize of Science (1969), the Japan Academy Prize (1976), the Person of Cultural Merits award of the Japanese Education Ministry (1984), the Fujiwara Prize (1987), and the Schock Prize of the Royal Swedish Academy of Sciences (1997). In 1993 he was elected to foreign membership in the U.S. National Academy of Sciences.

John T. Tate

For over a quarter of a century, John Tate's ideas have dominated the development of arithmetic algebraic geometry. Tate has introduced path-breaking techniques and concepts that initiated many theories that are very much alive today. These include Fourier analysis on local fields and adele rings, Galois cohomology, the theory of rigid analytic varieties, and p-divisible groups and p-adic Hodge decompositions, to name but a few. Tate has been an inspiration to all those working in number theory. Numerous notions bear his name: Tate cohomology of a finite group, Tate module of an abelian variety, Tate-Shafarevich group, Lubin-Tate groups, Neron-Tate heights, Tate motives, the Sato-Tate conjecture, Tate twist, Tate elliptic curve, and others. John Tate is a revered name in algebraic number theory.

John Tate was born in 1925 in Minneapolis. He received his A.B. from Harvard College (1946) and his Ph.D. from Princeton University (1950). He was a research assistant and instructor at Princeton (1950–53) and a visiting professor at Columbia University (1953–54) before moving to Harvard University. He was a professor at Harvard until 1990, when he accepted his present position as professor and Sid W. Richardson Chair in Mathematics at the University of Texas at Austin. Tate received the AMS Cole Prize (1956), a Sloan Fellowship (1959–61), and a Guggenheim Fellowship (1965–66). He was elected to the U.S. National Academy of Sciences (1969) and was named a foreign member of the French Academy of Sciences (1992) and an honorary member of the London Mathematical Society (1999).

About the Wolf Prize

The Israel-based Wolf Foundation was established by the late German-born inventor, diplomat, and philanthropist Ricardo Wolf. A resident of Cuba for many years, Wolf became Fidel Castro's ambassador to Israel, where Wolf lived until his death in 1981. The Wolf Prizes have been awarded since 1978 to outstanding scientists and artists "for achievements in the interest of mankind and friendly relations among peoples, irrespective of nationality, race, color, religion, sex, or political view." The prizes of $100,000 are given each year in four out of five scientific fields, in rotation: agriculture, chemistry, mathematics, medicine, and physics. In the arts the prize rotates among architecture, music, painting, and sculpture. The 2002–2003 prizes will be conferred by the president of Israel at a ceremony at the Knesset (the Israeli parliament) in Jerusalem on May 11, 2003.

—Allyn Jackson

It seemed the first time for them to meet each other.

But, "Sato-Tate" was mentioned only for Tate's work.

Yoshihiko Yamamoto (1941-2004)

a student of Sato, wrote Master's thesis (Feb 1966, Osaka Univ) on the "Sato conjecture" (in Japanese), but it was not published.

He wrote

> **'Sato conjecture': "If E is not of CM type, the distribution of θ_p is proportional to $\sin^2\theta$."**

Yamamoto's Master's Thesis (Feb 1966, Osaka Univ)

修 士 学 位 論 文

文 題 目

Sato予想について

昭和41年2月16日

専攻名　数学

氏　名　山本芳彦

大阪大学大学院理学研究科

Sato予想について

山本芳彦

§1 Introduction

E を有理数体 $\mathbb{Q}$ 上で方程式

$$y^2 = f(x)$$

$$f(x) = a_1x^3 + a_2x^2 + a_3x + a_4 \in \mathbb{Z}[x]$$

（$\mathbb{Z}$ は有理整数環）で定義された elliptic curve とする。p を素数として，E を mod p で reduction して得られる elliptic curve を $\tilde{E}_p$（E は p について きず をもたないとする）とする。$\tilde{E}_p$ は $\mathbb{Z}/(p)$ 上定義される。

$\tilde{E}_p$ の合同zeta函数を $Z_p(u)$ とすると

$$Z_p(u) = \frac{1 + a_pu + pu^2}{(1-u)(1-pu)}$$

ここで a_p は $|a_p| < 2\sqrt{p}$ なる有理整数で

$$a_p = \sum_{u=0}^{p-1} \left(\frac{f(u)}{p}\right)$$

で与えられる。（$\left(\frac{a}{p}\right)$ は Legendre の記号）

$$1 + a_pu + pu^2 = (1-\alpha_pu)(1-\bar{\alpha}_pu)$$

$$\begin{cases} \alpha_p = \sqrt{p}\, e^{i\theta_p} \\ \bar{\alpha}_p = \sqrt{p}\, e^{-i\theta_p} \end{cases}$$

とする。θ_p は合同zeta函数 $Z_p(u)$ の零点の偏角をあらわしている。

$$\zeta(s; E) = \prod_p{}' Z_p(p^{-s})$$

を E の Hasse zeta 函数という。（積は きず をもたない素数 p 全体を動く。）

$\zeta(s; E)$ は有限個の きず をもつ素数 p に対する $Z_p(p^{-s})$ なる因子を除いて

$$\zeta(s)\,\zeta(s+1)\,L(s+\tfrac{1}{2}; E)^{-1}$$

と一致する。ここで $\zeta(s)$ は Riemann の zeta 函数

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$$

で，また

$$L(s+\tfrac{1}{2}; E) = \prod_p{}' \frac{1}{1 + a_pp^{-s} + p^{1-2s}}$$

である。

E が虚数乗法をもつとき（以下 CM型という）には $L(s; E)$ は $A_0(E) = \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}(\sqrt{-d})$ の量指標（Größencharakter）を持った L函数と一致することが知られており，このとき θ_p は（mod 2π で考えて）$[0, 2\pi)$ の中で一様に分布する。

E が CM型でないとき特殊な場合を除いて一般に $L(s; E)$ の性質は不明である。しかし実験によると，このときの θ_p の分布の型は一定であると思われる。それが 'Sato予想' である。その内容は次の通りである。

"E が CM型でないならば θ_p の分布は $\sin^2\theta$ に比例する。"

Interview with Sato (Notices AMS 54 (2007))

In the interview, elliptic curves/modular forms were never mentioned.

Interview with Mikio Sato

Mikio Sato is a mathematician of great depth and originality. He was born in Japan in 1928 and received his Ph.D. from the University of Tokyo in 1963. He was a professor at Osaka University and the University of Tokyo before moving to the Research Institute for Mathematical Sciences (RIMS) at Kyoto University in 1970. He served as the director of RIMS from 1987 to 1991. He is now a professor emeritus at Kyoto University. Among Sato's many honors are the Asahi Prize of Science (1969), the Japan Academy Prize (1976), the Person of Cultural Merit Award of the Japanese Education Ministry (1984), the Fujiwara Prize (1987), the Schock Prize of the Royal Swedish Academy of Sciences (1997), and the Wolf Prize (2003).

This interview was conducted in August 1990 by the late Emmanuel Andronikof; a brief account of his life appears in the sidebar. Sato's contributions to mathematics are described in the article "Mikio Sato, a visionary of mathematics" by Pierre Schapira, in this issue of the *Notices*.

Andronikof prepared the interview transcript, which was edited by Andrea D'Agnolo of the Università degli Studi di Padova. Masaki Kashiwara of RIMS and Tetsuji Miwa of Kyoto University helped in various ways, including checking the interview text and assembling the list of papers by Sato. The *Notices* gratefully acknowledges all of these contributions.

—*Allyn Jackson*

Learning Mathematics in Post-War Japan

Andronikof: *What was it like, learning mathematics in post-war Japan?*

Sato: You know, there is a saying that goes like this: in happy times lives are all the same, but sorrows bring each individual a different story. In other words, I can tell of my hardships, but this will not answer your general question. Besides, I think the reader's interest should lie in the formation of the ideas of hyperfunctions, microlocal analysis, and so forth. It is true that in my young age I encountered some difficulties, but I don't think I should put emphasis on such personal matters.

Andronikof: *Still, I think we could start from a personal level. We could mix up journalism with mathematics, and go from one to the other. After all, you might not have become a—I would say, such a—mathematician without the experience of these hard times.*

Sato: Let me tell you this. In pre-war Japan, school was organized like the old German system. Elementary school ranged from the age of six to twelve, then followed middle school from twelve to seventeen, then three years of high school before entering university, where you graduated after three years. After the War, the system was changed to the American one: the five years of middle school were replaced by three years of junior high school and three years of high school. In order to become a graduate student, one then has to attend university for four years.

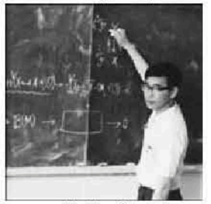
When I entered the middle school in Tokyo in 1941, I was already lagging behind: in Japan, the school year starts in early April, and I was born in late April 1928. The system was rigid, and thus I had to wait one year before getting in. Actually, it did not really matter, since I was not a quick boy. On the contrary, when I was a child, say, four like my son is now[1], I was called *bonchan*, which means a boy who is very slow in responding, very inadequate. I think I am very much the same now, ha! ha! Anyway, I turned thirteen right after entering middle school. In December of that year, Japan entered the war against the allied forces: U.S., UK, Holland, and China.

Andronikof: *Hectic times?*

Sato: Not so much in the beginning, as Japan was in a winning position. After Pearl Harbor, the British fleet was destroyed in the Far East, Singapore was occupied, and so on. Things looked favorable for Japan. But soon after, a year or so later, things started changing.

This was the beginning of my hard experiences. My regular courses in middle school lasted for only two years, and the rest of my school life was total chaos. The war in the Pacific ended on May 15, 1945. The first atomic bomb was dropped on August 6, 1945, after which the USSR declared war on Japan in order to secure the Kurilsk and Sachalin islands. At that time I was fifteen. Being a teenager, I had to work in factories. From 1943 to 1945, I had to carry coal. Very hard work… bad food… In late 1944, the systematic bombings of memorizing names and years and so on is important, and in this field my performance was extremely low. That gave me the feeling that studying at school was a kind of unpleasant job. Doing my own mathematics—that is, not school mathematics, but reading those books I mentioned—was like watching television would be for a present-day boy. See, I was probably indulging in such things to forget the unpleasant school courses. A way to escape the school system. During and after the war things became harder and harder and I went deeper and deeper into mathematics, so to speak, like another would dive into alcohol.

[1] *That is, in August 1990.*


Sato at blackboard, around 1972.

After middle school—though we had not completed it…I was admitted to high school. At that time, high school was rather elitist, more like École Polytechnique or École Normale Supérieure in France. The high school that I entered was called the First High School, and was closely attached to Tōdai[2] (Imperial University at the time). Both were national, i.e., non-private. The First High School is considered to be the top of elite schools, and I was lucky enough to skip the entrance examinations, because of the war. Well, there was a kind of test, but just to check some ability in mathematics: if they had tested my knowledge, then I couldn't have entered. Today, there are entrance examinations at many universities, including Tokyo or Kyoto: a bad test… But this is not interesting.

After the war, chaos occurred again—or rather, persisted. As I said, because of the devaluation of the yen, my family was starving. My father was sick, and I had a younger sister (by nine years) and a younger brother (by five years). I had to support them, so, in 1948, after three years of the First High School, I immediately started to work as a full-time teacher at the *new* high school, just when the school system was changed and middle school was cut by half. Housing and food conditions were extremely bad at the time, as you can imagine: like in Eastern Europe or Southeast Asia now. I entered Tōdai in 1949, having failed to enter in 1948. I had very little time to get prepared, then.

These hard times as schoolteacher lasted ten years, from 1948 to 1958. In 1958 I published the theory of hyperfunctions, in order to get a job at the university. I was an old student at the time, but it was like today: finishing university is sort of automatic, provided you succeed in getting in, where the competition is very tough.

[2] *University of Tokyo.*

Andronikof: *I read in your CV that you got a BSc in physics after your BSc in mathematics at Tōdai.*

Sato: You see, in Japan teaching depends on each professor, and one of my professors was very strict. At the time I was to graduate, he called me up, and told me that my term paper was very good but I had not attended the mandatory exercise sessions—not even once. This was an obligation that I didn't know of. Remember, that's why I was called *bonchan* when I was a little boy, and I'm still very much that way now. So, he said: "I cannot give you the points, so you cannot graduate". Then, he remained silent and watched me for a good minute. He opened his mouth again and said: "Okay, I'll give you the lowest points, so you can just graduate. Your paper is the top one". But this barred me from getting a position at the university as an assistant, which is customary for top students. Being assistant in Japan is a tenured position. The second-best student may also get some special position, and hence is assured of some top financial support. Anyway, I lost that kind of chance then. Since at the time I had also become interested in theoretical physics, I just moved to physics for two years under the new, American-style university system. I was still teaching full time in high school, so in physics I ran into the same academic problems as in mathematics. After two years at the Tōdai Physics Department, I moved to the graduate school of another university, Tokyo School of Education, where Professor Tomonaga taught theoretical physics. I stayed there until 1958.

This was the end of my twenties. At the time I was undergoing some kind of crisis in physical strength. Since by then my younger brother and sister were able to support themselves, my duty to them was sort of accomplished. I was able to return to my own life, so to speak, and go back to mathematics.

The Birth of Hyperfunctions and Microfunctions

Andronikof: *So you decided to go back to mathematics, rather than physics?*

Sato: Yes, and it was a good decision since competition in physics seemed stiffer. See, after these tough years I was beginning to feel physically tired, and my youth was leaving me. Even if I wasn't a man of quick response, I nevertheless understood that I had to face real life, so to speak, and to try to show what I could do in mathematics.

an official seminar in Tōdai, but rather a kind of "Jacobin Club". Among the participants, there were many very eager young students, including Kawai and Kashiwara. I met them there for the first time, and the group of Kawai, Kashiwara, and myself was formed that year.

In spring 1969 some old friends of mine in Komaba, which is part of the Faculty of General Education of the University of Tokyo, arranged to have me go there as a professor. I stayed in Komaba for two years.

Andronikof: *And when did you come to RIMS?*


Sato and Emmanuel Andronikof, 1990.

Sato: It was in June 1970. Actually, in Tokyo I had a great understanding with Komatsu and other seniors, as well as with many young mathematicians who gathered at our seminar. Among the participants, besides Kashiwara and Kawai who were extremely active, there were Morimoto, Kaneko, Fujiwara, Shintani, Uchiyama, and some others. So, I could supervise a lot of people who were very eager to study mathematics with me, and I thought I should better stay at Tōdai than come to RIMS. Anyway, Professor Kôsaku Yosida, who was director of the Institute from 1969 to 1972, and of whom I was once an assistant, put great pressure on me to come to RIMS. He had already asked on the occasion of the seminar he had organized in 1969. But since I had a position at Komaba, that was delayed until 1970. I was unhappy when I had to move to Kyoto because it meant I would be separated from this group: I could bring Kawai and Kashiwara to Kyoto, but I had to leave others behind.

The Katata Conference and S-K-K

Andronikof: *As for the "milestones" in the birth of hyperfunction and microfunction theory, can you comment on the famous Katata conference in fall 1971?*

Sato: Actually, what I said at that conference was sort of completed quite early, just after 1969. I have already told you how microfunctions originated in preparing the talk I gave at the international symposium at RIMS in April 1969. I had planned to present some of the things I had in mind, like the cotangential decomposition of hyperfunctions, so I had to check whether my ideas were working or not. I started to check this in the three-hour *shinkansen*[8] trip, commuting from Tokyo to Kyoto (or vice versa, I don't remember) to attend a pre-symposium meeting at RIMS. You could say that the basic part of the theory was conceived during these three hours. But later I checked it in detail, and it was completed at the international symposium. The final touch was a proof of microlocal regularity for elliptic systems[9]. At the time, I employed Fritz John's method of plane wave decomposition. Of course, the idea went back to 1960, when I attended Professor Hitotumatu's talk on the edge-of-the-wedge theorem.

[8] *A high-speed Japanese train.*

Andronikof: *When were the famous S-K-K[10] proceedings written?*

Sato: The basic structure of the paper hinges on my talk at the Katata conference, but the manuscript was completely prepared by Kawai and Kashiwara. Let us say I presented the whole story, but did not prove every detail. For example, concerning the notion of microdifferential operators, I worked out some cohomological constructions, but then Kawai and Kashiwara gave a better, more direct presentation, by which the proof of the invertibility for microelliptic operators, instead of using Fritz John's plane wave method, reduced to a kind of abstract nonsense. Kawai and Kashiwara must have taken a lot of effort to complete every detail.

The work was done between 1969 and 1971: surely the golden age of microfunctions. At the time, the three of us were working together, in the same places. In 1969 we were in Tokyo, then we moved to RIMS in 1970. Kawai came here as an assistant, while Kashiwara had only a kind of grant since he was very young at the time. He became assistant in 1971. I think the main part of the job was finished prior to the Katata conference, and was already presented in my talk at the Nice Congress [International Congress of Mathematicians] in 1970. To be precise, in the Nice talk the structure theorem for microdifferential systems was not yet finished. It was presented at the summer school on partial differential equations at Berkeley in 1971. I also prepared a kind of preprint, which did not appear in the proceedings of the Berkeley summer school, though it was distributed. There, I stated the structure theorem, asserting that all microdifferential systems are—at least generically—classified into three categories, the most important being what we called Lewy-Mizohata type system. The proof of this reduced to some simple nonlinear equations

[9] *Now known as Sato's theorem.*

[10] *M. Sato, T. Kawai, and M. Kashiwara,* Microfunctions and pseudo-differential equations, *In Komatsu (ed.),* Hyperfunctions and pseudo-differential equations, *Proceedings Katata 1971, Lecture Notes in Mathematics, no. 287, Springer, 1973, pp. 265–529.*

More crucially,

Sato didn't publish papers on "Sato's conjecture". His colleagues (e.g. Kuga) wrote few expository articles.

Sato seemed to dislike number theory although he made significant contributions to it (e.g. Kuga-Sato, Sato-Tate, etc).

There is a big "Sato school" at RIMS on algebraic analysis, microlocalization, $\mathcal{D}$-modules, hyperfunctions, soliton equations,... etc. Nobody was interested in number theory as far as I know.

My Conclusion

- According to the scratch papers, Sato seemed to consider elliptic modular forms (η-products) also.
- Usually, "Sato-Tate" is a conj for ell curves. It's not precise. "Sato-Tate for modular forms" makes perfect sense (theoretically and historically).
- It is not clear (to me) whether Sato conjectured

  "Sato-Tate Conj is true for some ell curves" or

  "Sato-Tate Conj is true for all ell curves".

  (We need Gal rep to believe it for all ell curves.)
- Anyway, "Sato-Tate" seems an appropriate name. Both contributions are big (as well as Serre's).

Outline of Proof of Sato-Tate conj

(Oversimplified Exposition)

Idea

- Control the arithmetic of an elliptic curve E via analytic properties of zeta functions and L-functions associated to E.
- Analytic properties of L-functions are established via the Reciprocity Law (= automorphy of Gal rep).

Ex. (Hadamard, de la Vallée-Poisson, 1896)

The Riemann zeta function $\zeta(s)$ is holomorphic and non-vanishing on Re$(s) \geq 1$ (except $s = 1$)

$\Rightarrow$ Prime Number Theorem

$$\lim_{N \to \infty} \left(\#\{p \,|\, \textbf{prime}\} \cdot \frac{\log N}{N} \right) = 1$$

Tate and Serre (1963-68)

defined symmetric power L-functions $L(s, E, \mathrm{Sym}^n)$ and proved if $L(s, E, \mathrm{Sym}^n)$ is holom and non-van on Re$(s) \geq 1 + \frac{n}{2}$ (for $n \geq 1$), then "Sato-Tate conj" is true.

Symmetric power L-functions

$E : y^2 = x^3 + ax + b \ (a, b \in \mathbb{Z}, 4a^3 + 27b^2 \neq 0)$

non-CM elliptic curve

$p + 1 - \#E(\mathbb{F}_p) = 2\sqrt{p}\cos\theta_p, \ \ \alpha_p = \sqrt{p}e^{i\theta_p}, \ \ \beta_p = \sqrt{p}e^{-i\theta_p}$

Def.

$$L(s, E) := \prod_p \frac{1}{(1 - \alpha_p\, p^{-s})(1 - \beta_p\, p^{-s})}$$

$$L(s, E, \mathbf{Sym}^n) := \prod_p \prod_{k=0}^{n} \frac{1}{1 - \alpha_p^k\, \beta_p^{n-k}\, p^{-s}}$$

(We ignore bad factors, although I like them.

I'm a bad person.)

Non-abelian CFT $\Rightarrow$ Sato-Tate

(1) $L(s, E, \text{Sym}^n)$ is the L-function ass to

$$\text{Sym}^n \rho_{E,\ell}\colon \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \text{GL}_2(\overline{\mathbb{Q}}_\ell) \longrightarrow \text{GL}_{n+1}(\overline{\mathbb{Q}}_\ell).$$

(2) $\text{Sym}^n \rho_{E,\ell}$ is irreducible (Serre, 1972).

(3) Non-abelian CFT predicts $\text{Sym}^n \rho_{E,\ell}$ is automorphic.

Namely,

$$L(s + \tfrac{n}{2}, \text{Sym}^n \rho_{E,\ell}) = L(s + \tfrac{n}{2}, E, \text{Sym}^n) = L(s, \pi)$$

for a cusp autom rep π of $\text{GL}_{n+1}(\mathbb{A}_{\mathbb{Q}})$.

(4) $L(s, \pi)$ is holom and non-van on $\text{Re}(s) \geq 1$

(Jacquet-Shalika, 1976/77).

Breakthrough after 2006

Clozel, Harris, Shepherd-Barron, Taylor.... established "Potential Automorphy" of Sym$^n\rho_{E,\ell}$.

(Automorphy was not yet established.)

Thm (CHSBT, BLGHT,...)

For $n \geq 1$, there exists a tot real Gal ext $F/\mathbb{Q}$ s.t. (Sym$^n\rho_{E,\ell})|_F$ is automorphic.

Proof of Potential Automorphy

- Taylor-Wiles-Kisin method ($R^{\text{red}} = T$),

  Modularity (Automorphy) Lifting Thm

  "ρ (mod ℓ) : Automorphic $\Rightarrow$ ρ : Automorphic"

- Cohom of Calabi-Yau var (Dwork families)

  Find suff many comp syst of ℓ-adic Gal rep,

  and connect "chain of automorphy".

Potential Automorphy $\Rightarrow$ Sato-Tate — "classical"

- Cyclic base change (Arthur-Clozel)
- Brauer's induction thm (1946) — He established

  meromorphy of Artin L-functions.

Question

What can we say and can't we say about

the Sato-Tate conj over K (K is not nec tot real)?

What we can't say...

Descent of Automorphy (?)

L/K : cyclic ext (not nec tot real)

ρ : ℓ-adic Gal rep of K s.t. $\rho|_L$ is abs irred

$\rho|_L$: Automorphic $\overset{???}{\Rightarrow}$ ρ : Automorphic

cf. [CHT], Lemma 4.2.2 uses [AC], Thm 4.2 (descent of autom rep) and existence of Gal rep in a crucial way.

What we can't say... (continued)

K : number field (not nec tot real)

E : non-CM ell curve/K

Potential automorphy of $\mathrm{Sym}^n \rho_{E,\ell}$ does not seem to imply the Sato-Tate conj.

According to "Brauer ind argument", we need "Potential autom & Descent of autom":

For $n \geq 1$, there exists a tot real Gal ext $F/\mathbb{Q}$

s.t. $(\mathrm{Sym}^n \rho_{E,\ell})|_L$ is autom for all $F/L/\mathbb{Q}$

(F/L : solvable)

then, the Sato-Tate conj is true for E.

What we can say...

<u>Thm</u> K : number field, E : non-CM ell curve/K s.t.

(1) $K/\mathbb{Q}$ is cyclic or $[K:\mathbb{Q}]=3$,

(2) $j(E)\in\mathbb{Q}$ (but E might not be defined over $\mathbb{Q}$).

Then, the Sato-Tate conj is true for E/K.

Proof

$\exists$ E' over $\mathbb{Q}$ s.t. $\rho_{E,\ell}$ is a quad twist of $\rho_{E',\ell}|_K$

$$L(s,\mathbf{Sym}^n\rho_{E,\ell}) = L(s,(\mathbf{Sym}^n\rho_{E',\ell})|_K\otimes\chi)$$
$$= L(s,(\mathbf{Sym}^n\rho_{E',\ell})\otimes\mathbf{Ind}_{K/\mathbb{Q}}\chi)$$

Use Arthur-Clozel AI ($K/\mathbb{Q}$:cyclic) or JPSS ($[K:\mathbb{Q}]=3$)

+ [BLGHT] (for $E'/\mathbb{Q}$) + Rankin-Selberg

Problem Give an example of ell curve E over $\mathbb{Q}(2^{1/5})$ s.t. the Sato-Tate conj is known to hold for $E/\mathbb{Q}(2^{1/5})$.

Rem.

Galois closure of $\mathbb{Q}(2^{1/5})/\mathbb{Q}$ is solvable. (can use BC)

$\mathbb{Q}(2^{1/5})/\mathbb{Q}$ is not Galois. (BC is not available)

$\mathbb{Q}(2^{1/5})$ is neither tot real nor CM. (Constr of Gal rep is not available)

Outline of the Kanazawa workshop

- Sato-Tate conj and beyond (Barnet-Lamb, Gee)
- p-adic modular forms ('Hida families')

  and arith applications (Geraghty, Loeffler, Sasaki)
- Autom Rep and applications (Ichino, Abe, Shin)
- Cohom, Arith Geom, Shim var (Lan, Harashita, Mieda)
- Iwasawa theory and applications (Pottharst, Ohshita)
- p-adic Langlands (Imai), Bhargava's work (Taniguchi)

  ($\rightarrow$ New density thm, Counting wt 1 mod forms)

  Top modular forms (Zakharevich)
- Hyperspecial talk (Yoshida – TBA)

Finally,

Thank you very much for your attention!

Enjoy Kanazawa and the Kanazawa workshop!