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<u>History</u>

① Iwasawa Theory of ideal class gp (over $\mathcal{O}[[\Gamma]]$) $\rightsquigarrow$ ② Iwasawa theory of ordinary motives (over $\mathcal{O}[[\Gamma]]$)

<u>Ref</u> $\left\{\begin{array}{l}\text{[Coates - Perrin-Riou]} \\ \text{[Greenberg]}\end{array}\right.$

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Iwasawa volume

$\rightarrow$ ③ Iwasawa Theory for nearly ordinary deformation (over a deformation ring R) $\longleftarrow$ [Kato's]

<u>Ref</u> [Greenberg /94]

(articles by Hida - O, Panchishkin)

<u>Today</u> We explain about ③

(setting, known results, difficulties, new phenomena applications ?)

<u>General setting</u> (influenced by [Greenberg /94] but modified & improved).

$\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$ $\quad \{\zeta_{p^n}\}_{n\geq 1}$ $\quad p \geq 3$ prime fixed

Introduce a triple $\{R, \mathcal{T}, \mathcal{S}\}$

R : complete Noeth. local domain with fin. res. field of char $= p$

$\mathcal{T} \cong R^{\oplus d} \circlearrowleft G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$
cont
unram. outside finite set of primes

$\mathcal{S} \subset \mathrm{Hom}_{\mathrm{cont}}(R, \overline{\mathbb{Q}_p})$: Zariski dense in $R \otimes \overline{\mathbb{Q}_p}$

<u>Rem</u> R could be finite over $\mathcal{O}[[X_1, \cdots, X_n]]$

$\mathcal{S} \subset \{\text{arithmetic specializations of } R\}$ $(1+X_i) \mapsto \zeta \cdot (1+p)^{n_i}$

Assume 4 conditions for $\{R, \mathcal{T}, \mathcal{S}\}$

(Geom) For $\forall \kappa \in \mathcal{S}$, the specialization $V_\kappa = \mathcal{T} \otimes_{R, \kappa} \overline{\mathbb{Q}_p}$ is geometric.
(<u>i.e.</u> $\exists$ pure motive $M_\kappa/\mathbb{Q}$ s.t. V_κ = p-adic realization of M_κ)

(Pan) $\exists$ a $G_{\mathbb{Q}_p}$-stable filtration :

$$0 \to \mathcal{T}^+ \to \mathcal{T} \to \mathcal{T}^- \to 0$$

by free R-modules.

s.t. $V_\kappa^+ := \mathcal{T}^+ \otimes_{R, \kappa} \overline{\mathbb{Q}_p}$ has H-T wt > 0
(resp. V_κ^-) (resp. ≤ 0)

(Crit) For each $\kappa \in \mathcal{S}$, a motive M_κ is critical (*) in the sense of Deligne.

(NV) $\exists \kappa \in \mathcal{S}$, $L(V_\kappa, 0) \neq 0$

(※) By Deligne, a motive M is called <u>critical</u> if Γ-factors of $L(M,s)$ & $L(M^*(1),s)$ have no poles at $s=0$.

<u>Conj</u> (Deligne)
Let $C^+_{M,\infty} \in \mathbb{C}$ complex period (a period integral) of M

$$\Rightarrow \quad \frac{L(M,0)}{C^+_{M,\infty}} \in \overline{\mathbb{Q}}$$

<u>Expectations</u>

<u>Conj. A</u> Let $\mathcal{A} = \mathcal{T} \otimes_R R^{PD}$ ← Pontryagin dual

The Pontrjagin dual $\mathrm{Sel}^{PD}_{\mathcal{A}}$ of the $\mathrm{Sel}_{\mathcal{A}} \underset{\text{local cond}}{\cap} H^1(\mathbb{Q},\mathcal{A})$ is torsion $/R$.

<u>Conj B</u> (to be more precise later)

$\exists!$ p-adic L-function $L_{p,\mathcal{T}} \in R \otimes \mathcal{O}_{\mathbb{C}_p}$ which interpolates special values $L(V_\kappa, 0)$ $\forall \kappa \in S$

<u>Conj C</u> (I.M.C.)

$$\underset{\substack{\uparrow \\ \left(\begin{array}{c}\text{modification factor} \\ = 1 \text{ in most cases}\end{array}\right)}}{e_{\mathcal{A}}} \cdot (L_{p,\mathcal{T}}) = \frac{\mathrm{char}(\mathrm{Sel}^{PD}_{\mathcal{A}})}{\mathrm{char}(\mathcal{A}^{G_\mathbb{Q}})^{PD} \cdot \mathrm{char}(\mathcal{A}^{*\,G_\mathbb{Q}})^{PD}}$$

Conjectures A, B, C $\rightsquigarrow$ various new phenomena
$\rightsquigarrow$ need various new tools

For "Two-variable Hida deformation"

this part

Conj A O.K.

Conj B (if Galois image is large)

Conj C $(L_{p,\mathcal{T}}) \subset \mathrm{char}(\mathrm{Sel}_{\mathcal{A}}^{p\mathfrak{p}})$
if Galois image is large

★ cyclotomic deformation (① + ②) of Intro

$T \cong \mathcal{O}^{\oplus d} \circlearrowleft G_{\mathbb{Q}}$
geom, p-adic Galois rep.
(critical & ordinary)

$\Lambda_{cyc} = \mathcal{O}[[\Gamma]]$

$\tilde{\chi} : G_{\mathbb{Q}} \twoheadrightarrow \Gamma \hookrightarrow \Lambda_{cyc}^{\times}$ tautological char

$\Lambda_{cyc}(\tilde{\chi})$: free Λ_{cyc}-module of rank 1 on which $G_{\mathbb{Q}}$ acts via $\tilde{\chi}$

$(\chi : G_{\mathbb{Q}} \longrightarrow 1 + p\mathbb{Z}_p$ cyclo char$)$

We put

$$R := \Lambda_{cyc}$$

$$\mathcal{T} := T \otimes \Lambda(\tilde{\chi})$$

$$\mathcal{S} \subseteq \left\{ \chi^j \underset{\substack{\uparrow \\ \text{finite char of } \Gamma}}{\phi} : \Lambda_{cyc} \longrightarrow \overline{\mathbb{Q}}_p \;\middle|\; T \otimes \chi^j \text{ is critical} \right\}$$

(Geom), (Pan), (Crit) is O.K. (almost) by definition
(NV) is always conjectured to be true.

Example 1 $\quad T = \mathcal{O}(\omega\psi)$ $\quad \psi$: even Dirichlet char

ω : Teichmüller char

Theorem

$$\exists!\ L_{p,T} \in \begin{cases} \frac{1}{\gamma-1}\Lambda_{cyc} & \text{if } \psi = \mathbb{1} \\ \Lambda_{cyc} & \text{if } \psi \neq \mathbb{1} \end{cases}$$

s.t. $\forall r \geq 1$, $\forall \phi$: finite char of Γ

$$\chi^{1-r}\phi\,(L_{p,T}) = \left(1 - (\psi\phi\omega^{r-1})(p)\cdot p^{r-1}\right) L(\psi\phi\omega^{r-1}, 1-r)$$

Example 2

f : eigen cuspform of $GL(2)/\mathbb{Q}$, weight $k \geq 2$

Assume f is ordinary ($a_p(f)$ is a p-unit)

$T = T_f$: Constructed by Deligne (- Shimura)

$S = \{ \chi^j\phi : \Lambda_{cyc} \to \overline{\mathbb{Q}}_p \mid 1 \leq j \leq k-1 \}$

(Geom) OK. by Scholl.

(Crit) by definition

(Pan) is proved by Deligne, Mazur-Wiles

(NV) OK (if $k \geq 3$ ← Euler product; proved by Rohrlich if $k = 2$)

<u>Thm</u> (Mazur - Tate - Teitelbaum)

$\underset{\sim\sim}{C_{f,\infty}} \in \mathbb{C}^\times / (r_{f,(p)})^\times$: (p-optimal) complex period.

(r_f : the ring of integers of the Hecke field of f)

Then, $\exists!\ L_{p,\mathcal{J}} \left(= L_{p,\mathcal{J}}(C^+_{f,\infty}) \right) \in \Lambda_{cyc} \otimes_{\mathbb{Z}} \mathbb{Q}$

s.t. $$\chi^j \phi (L_{p,j}) = (-1)^{j-1} (j-1)! \left(1 - \frac{p^{j-1} \phi\omega^{-j}(p)}{a_p(f)}\right)$$

$$\times \begin{pmatrix} \text{Gauss} \\ \text{sum} \end{pmatrix} \times \left(\frac{p^{j-1}}{a_p(f)}\right)^{\mathrm{ord}_p \mathrm{Cond}(\phi\omega^{-j})}$$

$$\times \frac{L(f, \phi\omega^{-j}, j)}{(2\pi\sqrt{-1})^j \cdot \underset{\sim\sim}{C^+_{f,\infty}}}$$

<u>Conj</u> $L_{p,\mathcal{J}} \in \Lambda_{cyc}$

(If f is not $\equiv$ Eisenstein series mod p

$\Rightarrow$ Conj is o.k.)

(see [Greenberg - Vatsal, Inv. 2000] $\Rightarrow$ known results for the integrality)

<u>Rem</u> Some constructions for cyclo. p-adic L-functions

- $GL(2)_{/\mathbb{Q}} \times GL(2)$
- $GL(2n)_{/F}$ F : any number field
- standard rep of $GSp_4(\mathbb{Q})$ (partial / incomplete / conditional!)

★★ More (non-cyclotomic) deformations $\mathcal{T}$

Note that

$\left\{ \begin{array}{l} \text{- "}L_{p,\mathcal{T}}\text{" (if exists) depends on the choice} \\ \qquad\qquad \text{of complex periods of motive } M. \\ \text{- periods are not canonical!} \end{array} \right.$

Remark

let $L \in R$

- the values $\{ \kappa(L) \in \overline{\mathbb{Q}}_p \}_{\kappa \in S}$ characterises the element L
- $\{ \underset{\uparrow}{|\kappa(L)|_p} \}_{\kappa \in S}$ does not characterise L

  p-adic value

$\left(\text{i.e. } \exists L, L' \in R \text{ s.t. } |\kappa(L)|_p = |\kappa(L')|_p \ \forall \kappa \in S, \quad L/L' \notin R^\times \right)$

Question How to formulate "interpolation property" of conjectural p-adic L-function?

Idea = Introduce p-adic periods

M_κ : motive at $\kappa \in \mathcal{S}$
defined over $\mathbb{Q}$ with coefficients F ($\uparrow$ #-field)
of rank $= n$

$H_B(M_\kappa)$: Betti realization of $rk = n/_F$

$H_{dR}(M_\kappa)$: de Rham realization of $rk = n$

(Then, after choosing basis of
$H_B(M_K)^{C=+}$ & $\mathrm{Fil}^0 H_{dR}(M_K)$)

$$C^+_{K,\infty} \in \mathbb{C} \quad \left(\text{resp. } C^+_{K,p} \in \mathbb{C}_p\right)$$

is defined as a determinant of $(*_\infty)$, $(*_p)$

$$(*_\infty) : \quad H_B(M_\kappa)^{C=+} \hookrightarrow H_B \otimes_F \mathbb{C} \xrightarrow{\sim} H_{dR}(M_K) \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow \left(H_{dR}/_{\mathrm{Fil}^0}\right) \otimes \mathbb{C}$$

$$(*_p) : \quad H_B(M_\kappa)^{C=+} \hookrightarrow H_B \otimes B_{HT} \xrightarrow{\sim} H_{dR} \otimes B_{HT} \longrightarrow \left(H_{dR}/_{\mathrm{Fil}^0}\right) \otimes B_{HT}$$

$$(\mathrm{Crit}) \Rightarrow \quad C^+_{\kappa,\infty} \neq 0 \in \mathbb{C}$$
$$C^+_{\kappa,p} \in \mathbb{C}_p$$

a pair $(C^+_{\kappa,\infty}, C^+_{\kappa,p})$ is well-defined.

<u>Problem 1</u> There are several p-adic Hodge theory (Faltings, Tsuji, Niziol)

Problem 2 $\quad C^+_{\kappa,p}$ can be zero.

(example by Y. André $\quad \exists M_\kappa, \exists \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ s.t. $C^+_{\kappa,p} = 0$)

Conj. B (more precise form)

Assume (Geom), ..., (NV)

(1) $\exists$ a suitable choice of $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$

$$C^+_{\kappa,p} \neq 0 \quad \forall \kappa \in S$$

(2) $\exists! \; L_{p,\mathcal{T}} \in R \otimes \mathcal{O}_{\mathbb{C}_p}$ s.t.

$$\frac{\kappa(L_{p,\mathcal{T}})}{C^+_{\kappa,p}} = P_\kappa \cdot Q_\kappa \cdot \frac{L(\kappa(\mathcal{T}), 0)}{C^+_{\kappa,\infty}}$$

where P_κ "p-Euler factor"

char pol. of $\varphi \curvearrowright D_{crys}(V^+_\kappa)$

$D_{crys}(V^-_\kappa)$

$$Q_\kappa = \prod (\alpha_i^{-1})$$

α_i eigenvalues of

$\varphi \curvearrowright D_{crys}(V^+_\kappa)$

Example By Hida theory, Hida's ordinary Hecke alg.

$\mathbb{I}$: local domain finite flat $/ \mathcal{O}[[X]]$

& $\mathbb{T} \cong \mathbb{I}^{\oplus 2} \curvearrowleft G_{\mathbb{Q}}$

$\mathcal{F} = \{f\} \leftrightarrow \mathbb{I}$ corresponds to a family of ordinary modular forms

$$f \in \mathcal{F} \quad \overset{1:1}{\longleftrightarrow} \quad \exists! \; \kappa_f : \mathbb{I} \longrightarrow \overline{\mathbb{Q}_p}$$

s.t. $\mathbb{T} \underset{\mathbb{I} \nearrow \kappa_f}{\otimes} \overline{\mathbb{Q}_p} \cong V_f$

Put $R \overset{\text{def}}{:=} \mathbb{I} \hat{\otimes}_{\mathbb{Z}_p} \Lambda_{\text{cyc}}$

$\mathcal{T} := \mathbb{T} \hat{\otimes} \Lambda_{\text{cyc}}(\widetilde{\chi})$

$\mathcal{S} = \left\{ \chi^j \phi \circ \kappa_f : R \to \overline{\mathbb{Q}_p} \;\middle|\; 1 \leqslant j \leqslant \mathrm{wt}(f) - 1 \right\}$

<u>Thm</u> (Kitagawa - O.)

Assume

(SL) Image of $G_{\mathbb{Q}} \to GL_2(\mathbb{I})$ contains $SL_2(\mathbb{I})$

$\exists! \; L_{p,\mathcal{T}} \in R \otimes \mathcal{O}_{\mathbb{C}_p}$ p-adic L-funct. in the sense of Conj. B.

$\mathbb{T}$: Hida deformation with complex mult. by K

$\mathbb{T} \cong \mathbb{I}^{\oplus 2}$ (s.t. $G_K \curvearrowright \mathbb{T}$ is abelian for K : imag. quad. field.)

$R \cong \mathcal{O}[[\mathrm{Gal}(K_\infty/K)]]$ $\quad K_\infty/K$: $\mathbb{Z}_p^2$-ext of K

$\mathcal{T} \cong \mathrm{Ind}_{\mathbb{Q}}^{K} \mathcal{O}[[\mathrm{Gal}(K_\infty/K)]](\widetilde{\chi})$

Katz, Yager

$\exists !$ $L_{p,\mathfrak{F}}^{K\text{-}Y} \in R$

which has exactly the same interpolation with the one constructed by modular symbols (except complex & p-adic periods)

<u>Open question</u> $\left(L_{p,\mathfrak{F}}^{K\text{-}Y}\right) = \left(L_{p,\mathfrak{F}}^{Krt}\right]$?