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p-adic appear

§1. Classical Theory

§2. Non-commutative Iwasawa theory for ~~totally~~ real ext.

§3. " for elliptic curves

§4. Relation with non-comm. Tamagawa number conj.

does not appear

very general

(for good ordinary case)

§1. Classical theory

Algebraic obj $\longleftrightarrow$ analytic obj (zeta function)

(1) Alg side

$p \neq 2$

$$F_n = \mathbb{Q}(\zeta_{p^n})^+ \qquad F_\infty = \mathbb{Q}(\zeta_{p^\infty})^+ = \bigcup_n F_n$$

$$G = \mathrm{Gal}(F_\infty/\mathbb{Q}) \xrightarrow[\text{cyclotomic char}]{\cong} \mathbb{Z}_p^\times/\{\pm 1\}$$

$$\Lambda = \mathbb{Z}_p[[G]] = \varprojlim_n \mathbb{Z}_p[\mathrm{Gal}(F_n/\mathbb{Q})]$$

Fix Σ : a finite set of primes $p \in M$

M = the largest pro-p abel. ext. of F_∞ unramified outside Σ

$$X = \mathrm{Gal}(M/F_\infty)$$

$$\left(\cong \mathrm{Hom}\Big(\varinjlim \underset{\substack{\uparrow \\ \text{ideal class gp}}}{Cl(F_n(\zeta_p))^-}, \mathbb{Q}_p/\mathbb{Z}_p(1)\Big) \text{ if } \Sigma = \{p\} \right)$$

... finitely generated torsion Λ-module

$$\Lambda \cong \underbrace{\mathbb{Z}_p[[T]] \times \cdots \times \mathbb{Z}_p[[T]]}_{\frac{(p-1)}{2}}$$

$$\begin{array}{c} M \\ | \\ F_\infty \\ | \\ \mathbb{Q} \end{array} \Big] G$$

(2) Analytic side

ξ_Σ : p-adic zeta function $\in Q(\Lambda)$

$$Q(\Lambda) = \text{total quotient ring of } \Lambda = \left\{ \frac{a}{b} \;\middle|\; a, b \in \Lambda,\ b : \text{non-zero divisor} \right\} = \underbrace{\mathrm{Frac}(\mathbb{Z}_p[[T]]) \times \cdots \times \mathrm{frac}(\mathbb{Z}_p[[T]])}_{\frac{p-1}{2}}$$

satisfying

(i) $a\xi_\Sigma \in \Lambda$ for $a \in \mathrm{Ker}(\underbrace{\Lambda \to \mathbb{Z}_p}_{\sigma \mapsto 1 \quad \forall \sigma \in G})$

(ii) $\forall r > 0$: even

For ring hom $\Lambda \to \mathbb{Z}_p$ induced by $\kappa^r : G \cong \mathbb{Z}_p^\times/\{\pm 1\} \xrightarrow{c \mapsto c^r} \mathbb{Z}_p^\times$

$f \mapsto f(\kappa^r)$

$$\xi_\Sigma(\kappa^r) = \zeta_\Sigma(1-r)$$

p-adic zeta funct.

$$\zeta_\Sigma(s) = \zeta(s) \cdot \prod_{\ell \in \Sigma} (1 - \ell^{-s})$$

$$= \prod_{\ell \notin \Sigma} (1 - \ell^{-s})^{-1}$$

(3) Iwasawa main conj. (proved by Mazur-Wiles)

category $\mathcal{C} = \left\{ \begin{array}{c} \text{finitely generated torsion} \\ \Lambda\text{-module} \end{array} \right\}$

$K_0(\mathcal{C})$: abelian group

$$\begin{cases} \text{generator} : [M] \quad (M \in \mathcal{C}) \\ \text{relation} : [M] = [M'] + [M''] \\ \quad \text{if } 0 \to M' \to M \to M'' \to 0 \text{ exact} \end{cases}$$

$$\begin{tikzcd}
\cdots \arrow[r] & K_1(\mathcal{C}) \arrow[r] & K_1(\Lambda) \arrow[r] \arrow[d, equal] & K_1(Q(\Lambda)) \arrow[r, "\partial"] \arrow[d, equal] & K_0(\mathcal{C}) \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & K_0(\Lambda) \arrow[dd, "\cong"] \\
& & \Lambda^\times & Q(\Lambda)^\times \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & {[\Lambda/(f)]} & \\
& & \Lambda \ni & f \arrow[ur, mapsto] & & K_0(Q(\Lambda))
\end{tikzcd}$$

$$\boxed{\partial(\xi_\Sigma) = [X] - [\mathbb{Z}_p]}$$

$$X \longrightarrow \bigoplus_{i=1}^{n} \Lambda/(f_i) \qquad f_i \in \Lambda \cap Q(\Lambda)^\times$$

kernel cokernel finite

$$\Rightarrow \quad [X] = \sum_{i=1}^{n} [\Lambda/(f_i)] \qquad \partial(f_1 \cdots f_n) = [X]$$

This tells

$$f_1 \cdot \cdots \cdot f_n \equiv \xi_\Sigma \mod \Lambda^\times \qquad \frac{Q(\Lambda)^\times}{\Lambda^\times} \cong K_0(\mathcal{C})$$

$$\mathrm{Char}(X) \longleftrightarrow [X], \qquad \mathrm{Char}(X) = f_1 \cdots f_n$$

$$\left[\ \mathbb{Z}_p[[T]]/(p,T)\ \right] = 0 \quad \text{because}$$

$$0 \to \mathbb{Z}_p[[T]]/(T) \xrightarrow{p} \mathbb{Z}_p[[T]]/(T) \to \mathbb{Z}_p[[T]]/(p,T) \to 0$$

exact

§2. Non-comm. Iwasawa theory for totally real ext.

Assume $\quad \underbrace{F_\infty \supset \mathbb{Q}(\zeta_{p^\infty})^+ \supset \mathbb{Q}}_{\text{Galois}}$

- $G = \mathrm{Gal}(F_\infty/\mathbb{Q})$ is a p-adic Lie group
- F_∞ is totally real
- Σ a finite set of primes $\qquad p \in \Sigma$
- G has no p-torsion (for simplicity) $\qquad$ ramified primes $\in \Sigma$

in $F_\infty/\mathbb{Q}$

$\Lambda = \mathbb{Z}_p[[G]]$

$X = \mathrm{Gal}(M/F_\infty)$ $\quad$ M : the largest pro-p abelian ext. of F_∞ unramified outside Σ

f.g. torsion Λ-module

$$X \text{ “}=\text{” } \mathrm{Hom}\left(\varinjlim Cl(F_n(\zeta_p))^-,\ \mathbb{Q}_p/\mathbb{Z}_p(1)\right)$$

$[F_n(\zeta_p) : F_n] = 2$

$[\underbrace{F_n : \mathbb{Q}}_{\text{Galois}}] < \infty, \quad F_\infty = \bigcup_n F_n$

(2) Analytic side --- hard

$\xi_\Sigma \in Q(\Lambda)$? Impossible !!

In §1 : $\xi_{\Sigma'} = \xi_\Sigma \cdot \prod_{\ell \in \Sigma' - \Sigma} \left(1 - \frac{\sigma_\ell}{\ell}\right)$

$\Sigma' \subset \Sigma$

σ_ℓ : Frobenius of $\ell \in G$

$$\left(1 - \frac{\sigma_\ell}{\ell}\right)(\kappa^r) = \left(1 - \frac{\ell^r}{\ell}\right) = \left(1 - \frac{1}{\ell^{1-r}}\right)$$

So ξ_Σ in §1 is like $\prod_{\ell \notin \Sigma} \left(1 - \frac{\sigma_\ell}{\ell}\right)$

(has no sense)

If Λ is non-comm.

ξ_Σ should be like $\prod_{\ell \in \Sigma} \left(1 - \frac{\sigma_\ell}{\ell}\right)^{-1}$

σ_ℓ : Frobenius of $\ell \in G$

$$\xi_{\Sigma'} = \xi_\Sigma \cdot \underbrace{\prod_{\ell \in \Sigma' - \Sigma} \left(1 - \frac{\sigma_\ell}{\ell}\right)}_{\text{non-commutative product}}$$

if $\xi_\Sigma , \xi_{\Sigma'} \in Q(\Lambda)$

R : ring $\quad \underset{\substack{\| \\ GL_1(R)}}{R^\times} \longrightarrow \underset{\text{abelian group}}{K_1(R)} := \varinjlim_n GL_n(R) \big/ [GL_n(R), GL_n(R)]$

can be non-comm

$$GL_n(R) \subset GL_{n+1}(R) \qquad g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$$

Rem $Q(\Lambda)$ exists
total quotient ring

$\xi_\Sigma \in K_1(Q(\Lambda))$ good idea

better idea: $\xi_\Sigma \in K_1(Q(\Lambda_{S^*}))$

$$\Lambda_{S^*} = (S^*)^{-1}\Lambda \subset Q(\Lambda)$$

S^* is defined as follows $\qquad H = \mathrm{Gal}(F_\infty / \mathbb{Q}(\zeta_{p^\infty})^+)$

$$S := \{ f \in \Lambda \mid \Lambda/\Lambda f \text{ is f.g. as a } \mathbb{Z}_p[[H]]\text{-mod} \}$$

$$S \subset \{\text{non-zero div. of } \Lambda\}$$

$\Lambda_S = S^{-1}\Lambda$ is defined $\qquad S^* = \bigcup_{n \geq 0} S p^n$

$$\Lambda_{S^*} = (S^*)^{-1}\Lambda = \Lambda_S[\tfrac{1}{p}] \subset Q(\Lambda)$$

Conj: $\exists\, \xi_\Sigma \in K_1(\Lambda_{S^*})$ such that

$\xi_\Sigma(\kappa^r \rho) = L_\Sigma(1-r, \rho)$ for any $r > 0$
even integer

and

$$\begin{tikzcd} G_\infty \arrow[r, "\rho"] \arrow[d] & GL_r(K) \\ \mathrm{Gal}(F_n/\mathbb{Q}) \arrow[ur] & \end{tikzcd} \qquad [K : \mathbb{Q}_p] < \infty$$

$\left(\begin{array}{c} \text{We fix} \\ \overline{\mathbb{Q}} \subset \mathbb{C} \\ \cap \\ \overline{\mathbb{Q}_p} \end{array} \right)$

$$\kappa^r \rho : \Lambda \xrightarrow{\text{ring hom}} M_r(K)$$

induces

$$\begin{array}{ccc} K_1(\Lambda) & \longrightarrow & K_1(M_r(K)) = K_1(K) = K^\times \\ & f \mapsto f(\kappa^r\rho) & \cap \\ \downarrow & & K \cup \{\infty\} \\ K_1(\Lambda_{S^*}) & \xrightarrow{\text{defined}} & \nearrow \\ \Downarrow & & \end{array}$$

$$f \in \Lambda \cap (\Lambda_{S^*})^\times \quad \text{class}(f) \longmapsto \det(\underbrace{(\kappa^r\rho)(f)}_{\in M_r(K)}) \in K$$

(3) Main conj.

$$\mathcal{C} = \{\text{f.g. } S^*\text{-torsion } \Lambda\text{-module}\}$$

$$K_1(\mathcal{C}) \longrightarrow K_1(\Lambda) \longrightarrow K_1(\Lambda_{S^*}) \xrightarrow{\partial} K_0(\mathcal{C}) \to K_0(\Lambda) \xrightarrow{!} K_0(Q(\Lambda))$$

$$\text{class}(f) \longmapsto [\Lambda/\Lambda f]$$

$[K : \mathbb{Q}_p] < \infty$

$$f \in \Lambda \cap (\Lambda_{S^*})^\times$$
$$\cup$$
$$S^*$$

(this)
+ existence of ξ_Σ

<u>Conj.</u> $\partial(\xi_\Sigma) = [X] - [\mathbb{Z}_p]$

<u>Kakde Hara Ritter-Weiss, Kato</u>

<u>Conj.</u> $X \in \mathcal{C}$ $\quad$ (S^*-torsion)

$\Uparrow$

Iwasawa's conj. $\mu = 0$ for all cyclotomic $\mathbb{Z}_p$-ext of number fields

Kakde

OK if $\begin{cases} \mu = 0 \text{ for } F_n \quad \forall n \geq 0 \\ G : \text{pro-}p \\ H = H_0 \supset H_1 \supset \cdots \supset H_m = \{1\} \\ \quad H_i : \text{normal} \quad H_i / H_{i-1} \text{ abelian} \end{cases}$

<u>Results</u> : existence of ξ_Σ and the main conj.

$$\begin{array}{l} F_\infty \\ \mid \\ F_n(\zeta_{p^\infty})^+ \\ \mid \\ F_n \\ \mid \\ \mathbb{Q} \end{array}$$

<u>Rem.</u> Why is $K_1(Q(\Lambda))$ not good ?

- $f(\kappa^r{}_p)$ not defined
- $K_1(\Lambda_{S^*}) \xrightarrow{\text{not inj.}} K_1(Q(\Lambda))$

$$\begin{array}{ccc} K_1(\Lambda_{S^*}) & \longrightarrow & K_1(Q(\Lambda)) \\ \exists\, a & \longmapsto & 0 \\ \downarrow & & \\ \neq 0 \in K_1(\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})^+/\mathbb{Q})]]_{S^*}) & & \end{array}$$

§ 3. Elliptic curves

$E/\mathbb{Q}$ elliptic curve
good red at p

~~(1) $X \neq$ Hom~~

F_∞ F_n Σ as in §2

- We do not need F_∞ totally real
- $F_\infty \supset \mathbb{Q}(\zeta_{p^\infty})^+$ can be replaced by $F_\infty \supset$ cyclotomic $\mathbb{Z}_p$-ext of $\mathbb{Q}$

Σ should contain bad primes for E

(1) $X = \mathrm{Hom}\left(\varinjlim_n \mathrm{Sel}(E/F_n),\ \mathbb{Q}_p/\mathbb{Z}_p \right)$

(2) Conj. $\xi_\Sigma \in K_1(\tilde{\Lambda}_{\tilde{S}^*})$

$$\tilde{\Lambda} = W(\overline{\mathbb{F}}_p)[[G]] \supset \Lambda$$

such that

$$\xi_\Sigma(\rho) = \frac{L(E,1,\rho)}{\text{period}} \times (\text{simple thing})$$

(3) $\partial(\xi_\Sigma) = [W(\overline{\mathbb{F}}_p) \otimes_{\mathbb{Z}_p} X] + \left(\text{simple small thing}\right)$

$\rho : \mathrm{Gal}(F_n/\mathbb{Q}) \to GL_r$

$\mathrm{rk}(\rho\text{-component of } E(F_n))$

$= \operatorname{ord}_{s=1} L(E, s, \rho)$

§4. Relation with non-comm. Tamagawa number conj.

Burns, Flach

Huber-Kings

Perrin-Riou, Kato

commutative

Non-comm. Tam. conj

Λ : pro-p ring

T : f.g. proj Λ-module with Λ-linear conj. action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ unramified outside Σ

Then, we have zeta isom

$$\zeta_\Lambda(T) : \det_\Lambda(0) \xrightarrow{\cong} \det_\Lambda\left(R\Gamma_c\left(\mathbb{Z}[\tfrac{1}{\Sigma}], T\right)\right)$$

related to zeta values

$R\Gamma_c\left(\mathbb{Z}[\frac{1}{\Sigma}], T\right)$ is the mapping fiber of

$$R\Gamma\left(\mathbb{Z}[\tfrac{1}{\Sigma}], T\right) \longrightarrow \bigoplus_{v \in \Sigma \cup \{\infty\}} R\Gamma(\mathbb{Q}_v, T)$$

étale cohomology

Galois cohom.

Category $(\det/R)$ $\quad R$: ring

object : $P - Q$ (symbol) $\quad$ (P, Q : f.g. proj. R-modules) morphism :

$$\mathrm{Mor}\left(P-Q,\ P'-Q'\right)$$
$$= \varinjlim_{n} \underbrace{[\mathrm{Aut}(Y_n), \mathrm{Aut}(Y_n)]}_{\text{commutator group of } \mathrm{Aut}(Y_n)} \backslash \mathrm{Isom}(X_n, Y_n)$$

$$X_n = P \oplus Q' \oplus R^n$$
$$Y_n = P' \oplus Q \oplus R^n$$

$\mathrm{Aut}(Y_n)$ acts on $\mathrm{Isom}(X_n, Y_n)$

$$[P-Q] := [P] - [Q] \in K_0(R) = K_0(\{\text{f.g. proj. } R\text{-modules}\})$$

$[P-Q] = [P'-Q']$
iff $P-Q \cong P'-Q'$ in $(\det/R)$

$$\mathrm{Aut}(P-Q) = K_1(R)$$
$$\cong \varinjlim [GL_n(R), GL_n(R)] \backslash GL_n(R) = K_1(R)$$

$P-Q$ is denoted also as $\dfrac{\det P}{\det Q}$

Λ as before §1 or §2 or §3

M f.g. Λ-modules

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

Λ

P_i : f.g. proj. [redacted]-mod

$$\det M := \frac{\det\left(\bigoplus_{i:\text{even}} P_i\right)}{\det\left(\bigoplus_{i:\text{odd}} P_i\right)} \quad \text{is defined}$$

$\bigcap$

$(\det/\Lambda)$

If $M, N \in \mathcal{C}$

$$\Lambda_{S^*} \otimes_{\Lambda} M = 0, \quad \Lambda_{S^*} \otimes_{\Lambda} N = 0$$

(I apply this to $M = X$, $N = Z_f$ in §1, §2)

naturally

$$\frac{\det_{\Lambda_{S^*}}(\Lambda_{S^*} \otimes_{\Lambda} M)}{\det_{\Lambda_{S^*}}(\Lambda_{S^*} \otimes_{\Lambda} N)} \underset{v}{\cong} \det_{\Lambda_{S^*}}(0)$$

$[M] - [N] = 0$ in $K_0(\Lambda)$

$$\det_{\Lambda}(0) \underset{u}{\cong} \frac{\det_{\Lambda}(M)}{\det_{\Lambda}(N)}$$

$$v \circ u : \det_{\Lambda_{S^*}}(0) \cong \det_{\Lambda_{S^*}}(0)$$

$$K_0(\mathcal{C}) \xrightarrow{0} K_0(\Lambda) \xrightarrow{\subset} K_0(\Lambda_{S^*})$$

$v \circ u \in K_1(\Lambda_{S^*})$

$K_1(R)$

$\partial(v \circ u) = [M] - [N]$ holds

We have

$$\mathrm{Isom}\left(\det_\Lambda(0) \longrightarrow \frac{\det_\Lambda(M)}{\det_\Lambda(N)} \right) \xrightarrow{\text{bij}} \left\{ f \in K_1(\Lambda_{S^*}) \;\middle|\; \partial f = [M]-[N] \right\}$$

$$u \longmapsto v \circ u$$

In §2,
$$H^m_c\left(\mathbb{Z}[\tfrac{1}{\Sigma}], T\right) = \begin{cases} \mathrm{Gal}(M/F_\infty) & \text{if } m=2 \\ \mathbb{Z}_p & \text{if } m=3 \\ 0 & \text{if } m \neq 2,3 \end{cases}$$

$$T = \Lambda(1)$$

here $\underset{\sigma}{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \ni \sigma$ acts by $a \mapsto a\sigma^{-1}$, $a \in \Lambda$

Main conj. saids

$$\begin{array}{ccc} \xi_\Sigma : \det_\Lambda(0) & \xrightarrow{\cong} & \det_\Lambda R\Gamma_c\left(\mathbb{Z}[\tfrac{1}{\Sigma}], T\right) \\ \| & & \| \\ \xi_\Lambda(T) & & \dfrac{\det_\Lambda \bigoplus_{m:\text{even}} H^m_c\left(\mathbb{Z}[\tfrac{1}{\Sigma}], T\right)}{\det_\Lambda \bigoplus_{m:\text{odd}} H^m_c\left(\mathbb{Z}[\tfrac{1}{\Sigma}], T\right)} \end{array}$$

$$= \frac{\det_\Lambda(X)}{\det_\Lambda(\mathbb{Z}_p)}$$

In §3,

NC Tam. num. conj.

$$\det_\Lambda(0) \overset{?}{=} \det_\Lambda R\Gamma_c\left(\mathbb{Z}[\tfrac{1}{\Sigma}], T\right)$$

$$T = \Lambda \underset{\mathbb{Z}_p}{\otimes} T_p E \qquad = \det_\Lambda\left(X \oplus \underline{R\Gamma(\mathbb{R}, T) \oplus R\Gamma(\mathbb{Q}_p, \Lambda \underset{\mathbb{Z}_p}{\otimes} T')} \right)$$

at p, $\quad 0 \to \underset{\substack{\uparrow \\ \text{rk } 1}}{T'} \to T_p E \to \underset{\substack{\uparrow \\ \text{rk } 1}}{T''} \to 0$ over $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$

~~~ : can be killed by <u>local conj. at p</u>
(<u>ε- conj</u>)

$$\underbrace{\det_\Lambda(0) = \det_\Lambda(X)}_{\xi_\Sigma}$$

Iwasawa main conj.

$$\zeta(1-r) = \# \text{ Gal cohom}$$

$r > 0$
r : odd

Lichtenbaum conj.
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